# Chapter 0: Mathematical reminders

## *I.1. Generalities on Physical quantities* (المقادير الفيزيائية)

- A physical quantity [A] is a quantity wich can be measured, with instruments or even by using our senses, and wich that reports a physical property.

- **For example**: length, mass, time, temperature, electric current, light intensity, volume…. etc

- Physical quantities [A] have **numerical magnitude "a"** and **unit** $\{A\}$

$$\underbrace{a}_{\text{numerical magnitude}} = \frac{[A] \longrightarrow \textbf{Physical quantity}}{\{A\} \longrightarrow \textbf{Unit}}$$

**Example: The Velocity V = 10 m/s**

## ➢ There are two types of measurable quantities

- **Scalar quantities:**

  **Length, mass, time, energy……….**

- **Vector quantities:**

  **Velocity, Acceleration, Electric and magnetic field……**

# ***International System of Units(Called « SI » System)***

- This system is composed of the following fundamental units:

| Unit | Physical quantuty |
|---|---|
| Meter (m) | Length |
| Kilogram (Kg) | Mass |
| Second (S) | Time |
| Ampere (A) | Electric current intensity |
| Kelvin (K) | Temperature |
| Candela (Cd) | Luminous intensity |
| Mole | Quantity of matter |

- The First Fourth units form the system **MKSA**

## ***Derived quantities***

- These quantities are expressed as a combination of fundamental quantities.
- The units of all quantities other than fundamental units is called **derived unit**.
- Derived units are obtained in terms of fundamental quantities.

**Exemple :**

**Area :** $m^2$

**velocity :** $m.s^{-1}$ .

**Force :** Newton (N) = $Kg.m.s^{-2}$ .

**Energy :** Joule (J) =$Kg\ m^2.s^{-2}$

# I.2. Equation for dimensions (Dimensional Equations)

- Determine derived units based on fundamental units  nombres réels

$$[A] = M^{\alpha} L^{\beta} T^{\gamma} I^{\lambda}$$

$\boldsymbol{\alpha, \beta, \gamma, \lambda}$: $\boldsymbol{real\ numberd}$

- This equation consists of the equation for dimensions of a quantity A, with:

  **M : Mass, L : Length, T : Time, I : Current intensity**

## Examples:

- **Velocity :** $[V] = L.T^{-1}(m/s)$

- **Acceleration:** $[a] = L.T^{-2}(m/s^2)$

- **Force :** $\vec{F} = m\vec{a} \quad \Rightarrow [F] = ML.T^{-2}(kg.m.s^{-2} = Newton)$

- **Work :** $W = \int \vec{F}.\vec{dl} \quad \Rightarrow [W] = [F][dl] = MLT^{-2}L = ML^2T^{-2}(Kg.m^2/s^2 = Joule)$

## ***Remark :***

- The dimensional equation is used to check the homogeneity of the physical formulas.

## ***Example :***

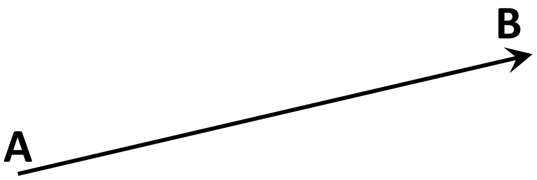
The period of oscillation of a simple pendulum of length L is it given by:

$T = 2\pi\sqrt{\frac{g}{L}}\ldots\ldots\ldots\ldots\ldots\ldots\ldots(I)$ **Ou par** $T = 2\pi\sqrt{\frac{L}{g}}\ldots\ldots\ldots\ldots\ldots\ldots\ldots(II)$

- $(I) \Rightarrow T = 2\pi g^{1/2}L^{-1/2} \quad \Rightarrow [T] = (LT^{-2})^{1/2}L^{-1/2} = T^{-1} \Rightarrow [T] = T^{-1} \quad False$
- $(II) \Rightarrow T = 2\pi L^{1/2}g^{-1/2} \quad \Rightarrow [T] = L^{1/2}(LT^{-2})^{-1/2} = T \Rightarrow [T] = T \quad right$

# II. <u>Reminder on vectors</u>

- Mathematical entity defined by multiple numeric values.
- These values describe the magnitude and orientation of the vector.
- A vector $\overrightarrow{AB}$ is characterized by:



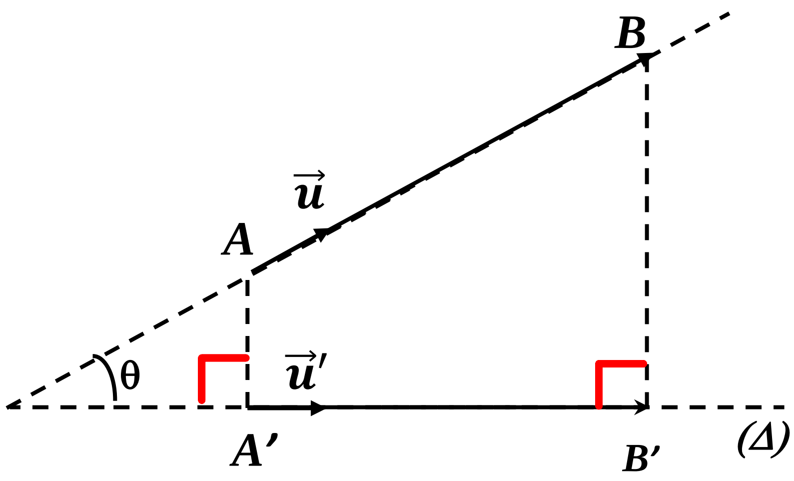

  - Its origin or point of application.
  - Its direction, which is the direction of movement of a mobile having from point A to point B.
  - Its magnitude which presents the length AB. It is noted $\|\overrightarrow{AB}\|$

- **Unit vector** or **orth** is a vector whose length is equal to one.

## *II.1- Projecting a vector onto an axis:*

$$\overrightarrow{AB} = \left\|\overrightarrow{AB}\right\| \vec{u}$$

$\vec{u}$ represent unit vector, with $\|\vec{u}\| = 1$




A' and B' are perpendicular projections of A and B on the axis *(Δ)*

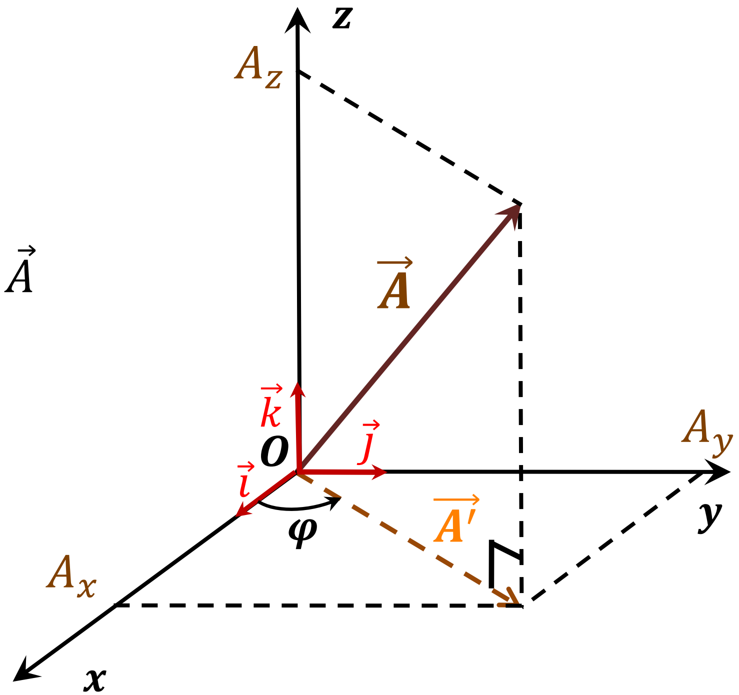
$$\overrightarrow{A'B'} = \left\|\overrightarrow{A'B'}\right\| \overrightarrow{u'}$$

$$\boxed{\overrightarrow{\|A'B'\|} = \overrightarrow{\|AB\|} \cos\theta}$$

## II.2- The components of a vector:

$$\vec{A} = A_x\vec{\imath} + A_y\vec{\jmath} + A_z\vec{k} \qquad \text{Or} \qquad \vec{A}\begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}$$

Tel that $\|\vec{A}\| = \sqrt{A_x^2 + A_y^2 + A_z^2}$ is the magnitude of $\vec{A}$




- If a vector $\overrightarrow{AB}$ set by the coordinates of the points $A(Ax\,; A_y; A_z)$ and $A(B_x; B_y; B_z)$ can be found using the following formula:

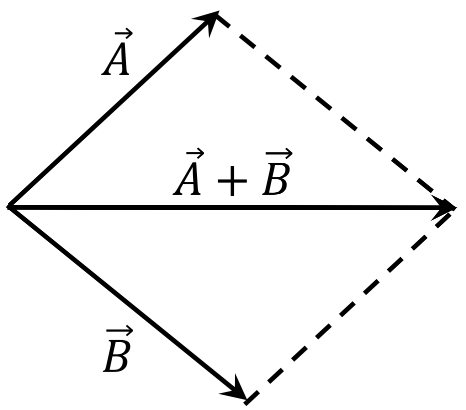
$$\overrightarrow{\boldsymbol{AB}} = (\boldsymbol{B_x} - \boldsymbol{A_x})\vec{\boldsymbol{\imath}} + (\boldsymbol{B_y} - \boldsymbol{A_y})\vec{\boldsymbol{\jmath}} + (\boldsymbol{B_z} - \boldsymbol{A_z})\vec{\boldsymbol{k}}$$

# *Vector operations:*

## *I. Addition*

$$\vec{A}\begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} \text{ et } \vec{B}\begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix}$$

**Geometrically**



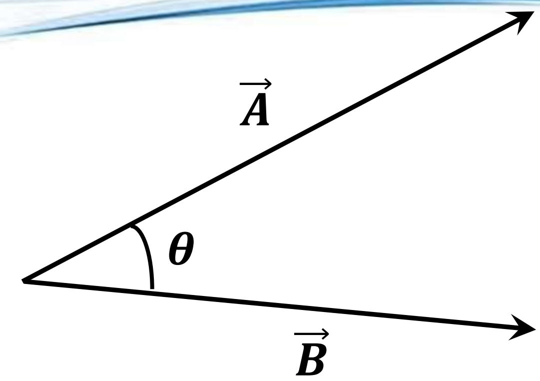

**Analytically :** $(\vec{A} \pm \vec{B})\begin{pmatrix} A_x \pm B_x \\ A_y \pm B_y \\ A_z \pm B_z \end{pmatrix}$

## *Properties :*

- $\sum_{i=1}^{n} \overrightarrow{A_i} = \sum_{i=1}^{n} A_{x_i}\vec{\imath} + \sum_{i=1}^{n} A_{y_i}\vec{\jmath} + \sum_{i=1}^{n} A_{z_i}\vec{k}$
- $\|\vec{A}+\vec{B}\| \neq \|\vec{A}\| + \|\vec{B}\|$
- $(\vec{A}+\vec{B}) = (\vec{B}+\vec{A})$
- $\vec{A}\begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} \Rightarrow -\vec{A}\begin{pmatrix} -A_x \\ -A_y \\ -A_z \end{pmatrix}$
- $(\vec{A}+\vec{B}) + \vec{C} = \vec{A} + (\vec{B}+\vec{C})$

## ***II. multiplication of two vectors:***

### ***II.1 Scalar multiplication:***




$$\boxed{\vec{A}.\vec{B} = \|\vec{A}\|\|\vec{B}\| \cos\left(\widehat{\vec{A},\vec{B}}\right)}$$

- **In Cartesian coordinates:**

$$\boxed{\vec{A}.\vec{B} = A_xB_x + A_yB_y + A_zB_z}$$

- **The angle θ between $\vec{A}$ and $\vec{B}$ is given by:**

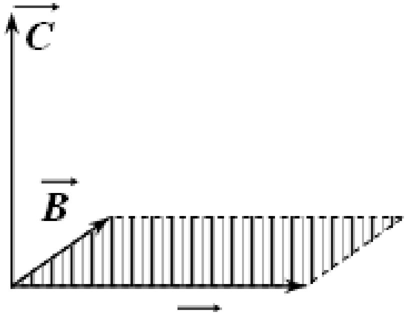
$$\boxed{\cos\theta = \frac{A_xB_x + A_yB_y + A_zB_z}{\sqrt{{A_x}^2 + {A_y}^2 + {A_z}^2}.\sqrt{{B_x}^2 + {B_y}^2 + {B_z}^2}}}$$

# Properties :

- $\vec{A}.\vec{B} = \vec{B}.\vec{A}$
- $\vec{A}.(\vec{B} + \vec{C}) = \vec{A}.\vec{B} + \vec{A}.\vec{C}$
- $\vec{A}.(\vec{B}.\vec{C}) = (\vec{A}.\vec{B}).\vec{C}$
- $\vec{A}.\vec{A} = \|\vec{A}\|^2$
- $(\lambda\vec{A}).\vec{B} = \lambda(\vec{A}.\vec{B}) = \vec{A}.(\lambda\vec{B})$
- $\vec{A} \perp \vec{B} \Rightarrow \vec{A}.\vec{B} = 0(\vec{A}$ and $\vec{B}$ are ortogonal $)$

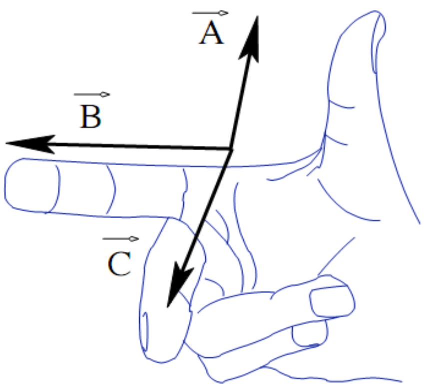
## II.2. Vector multiplication :

**The vector *multiplication* of vectors $\vec{A}$ and $\vec{B}$ , denoted $\vec{A} \wedge \vec{B}$ , is a vector $\vec{C}$ with:**



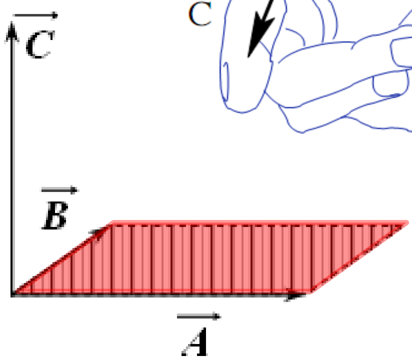

- $\vec{C}$ is perpendicular to the plane formed by the vectors $\vec{A}$ and $\vec{B}$

- The direction is given by using the right-hand rule.




$\Rightarrow$ $(\vec{A}\vec{B}\vec{C})$ **make a direct trihedron (ثلاثية مباشرة).**

- the magnitude of $\vec{C}$ corresponds to the area of the parallelogram constructed on $\vec{A}$ and $\vec{B}$




***Analytically :***

$$\boxed{\|\vec{C}\| = \|\vec{A} \wedge \vec{B}\| = \|\vec{A}\|.\|\vec{B}\|.\left|\sin\left(\widehat{\vec{A},\vec{B}}\right)\right|}$$

## ❑ Cartesian coordinates of $\vec{C}$ :

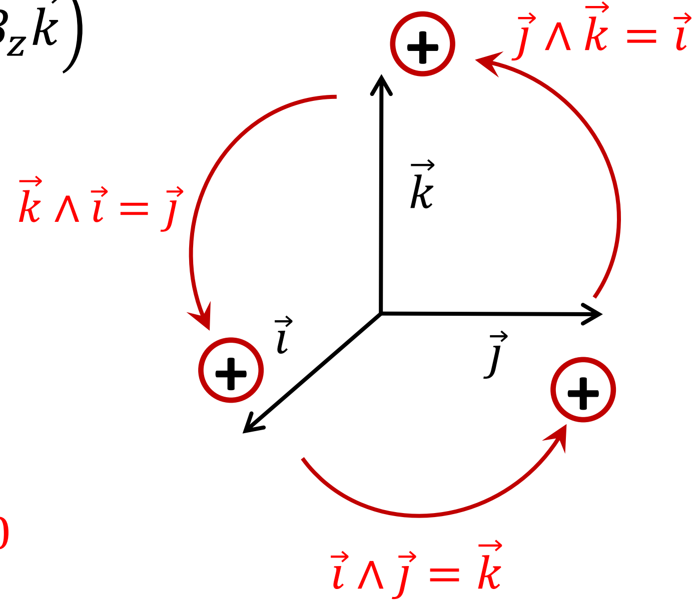
$$\vec{A}\begin{pmatrix}A_x\\A_y\\A_z\end{pmatrix} \text{ et } \vec{B}\begin{pmatrix}B_x\\B_y\\B_z\end{pmatrix}$$

$$\vec{C}=\vec{A}\wedge\vec{B}=\left(A_x\vec{\imath}+A_y\vec{\jmath}+A_z\vec{k}\right)\wedge\left(B_x\vec{\imath}+B_y\vec{\jmath}+B_z\vec{k}\right)$$

$$=A_xB_x\vec{\imath}\wedge\vec{\imath}+A_xB_y\vec{\imath}\wedge\vec{\jmath}+A_xB_z\vec{\imath}\wedge\vec{k}$$

$$+A_yB_x\vec{\jmath}\wedge\vec{\imath}+A_yB_y\vec{\jmath}\wedge\vec{\jmath}+A_yB_z\vec{\jmath}\wedge\vec{k}$$

$$+A_zB_x\vec{k}\wedge\vec{\imath}+A_zB_y\vec{k}\wedge\vec{\jmath}+A_zB_z\vec{k}\wedge\vec{k}$$




- $\vec{\imath}\wedge\vec{\imath}=\|\vec{\imath}\|\|\vec{\imath}\|\sin(\vec{\imath},\vec{\imath})=\vec{0}$ Also $\vec{\jmath}\wedge\vec{\jmath}=\vec{k}\wedge\vec{k}=0$

$$\boxed{\Rightarrow\vec{A}\wedge\vec{B}=\left(A_yB_z-A_zB_y\right)\vec{\imath}-\left(A_xB_z-A_zB_x\right)\vec{\jmath}+\left(A_xB_y-A_yB_x\right)\vec{k}}$$

## ❖ Determinant method:

$$\vec{A} \wedge \vec{B} = \begin{array}{ccc} + & - & + \\ \end{array}$$

$$\vec{A} \wedge \vec{B} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = +\vec{i}\left(A_y B_z - A_z B_y\right) - \vec{j}(A_x B_z - A_z B_x) + \vec{k}\left(A_x B_y - A_y B_x\right)$$

$$\Rightarrow \vec{A} \wedge \vec{B} = \left(A_y B_z - A_z B_y\right)\vec{i} - (A_x B_z - A_z B_x)\vec{j} + \left(A_x B_y - A_y B_x\right)\vec{k}$$

### ❑ Properties:

$1.\ \vec{A} \wedge \vec{B} = -\left(\vec{B} \wedge \vec{A}\right)$

$2.\ \vec{A}//\vec{B} \Rightarrow \vec{A} \wedge \vec{B} = \vec{0}$

## *II.3. Mixed product:*

Mixed product is a triple vector product that combines the concept of *scalar* and *vectorial* products to yield a scalar value: $m = \vec{A}.(\vec{B} \wedge \vec{C})$

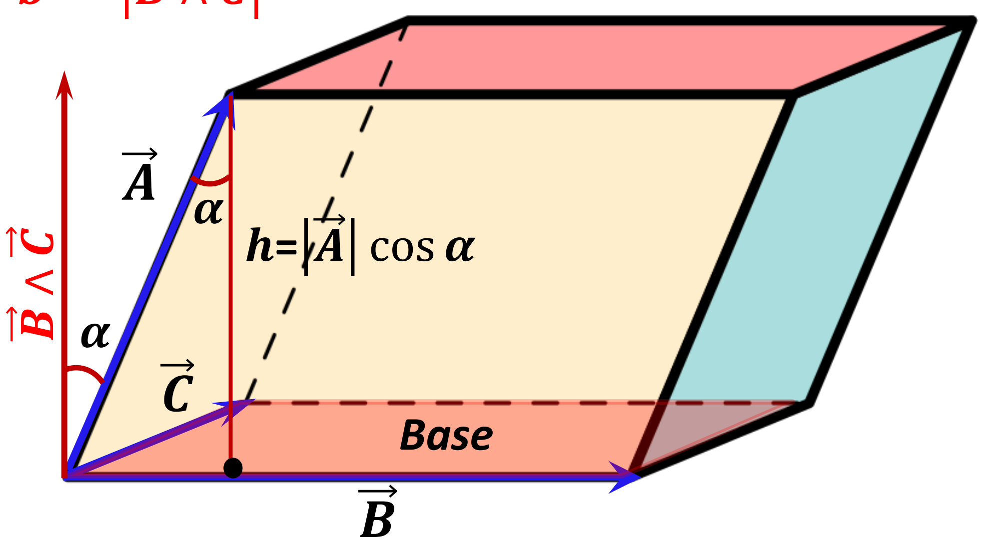
### *Geometric interpretation:*

- ❑ The absolute value ***m*** of the mixed product is the volume of the parallelepiped formed by the vectors $\vec{A}, \vec{B}$ *and* $\vec{C}$.
- ❑ The vector $\vec{B} \wedge \vec{C}$ is perpendicular of the base of The parallelepiped and its magnitude equal the area of the base: $\boldsymbol{b} = |\vec{\boldsymbol{B}} \wedge \vec{\boldsymbol{C}}|$
- ❑ The altitude of the parallelepiped $h$ is given by: $\boldsymbol{h} = |\vec{A}| \cos \boldsymbol{\alpha}$

➢ **Therefor, the volume is given by :**

$$\boldsymbol{V = Base(b) \times h}$$

$$\vec{A}.(\vec{B} \wedge \vec{C}) = \underbrace{|\vec{B} \wedge \vec{C}|}_{b} . \underbrace{|\vec{A}| \cos \alpha}_{h}$$



## ***Mixed product properties :***

- $\vec{A}.(\vec{B}\wedge\vec{C}) = \vec{B}.(\vec{C}\wedge\vec{A}) = \vec{C}.(\vec{A}\wedge\vec{B}) = (\vec{B}\wedge\vec{C}).\vec{A} = (\vec{C}\wedge\vec{A}).\vec{B} = (\vec{A}\wedge\vec{B}).\vec{C}$

- $\vec{A}.(\vec{B}\wedge\vec{C}) = -\vec{A}.(\vec{C}\wedge\vec{B}) = -\vec{B}.(\vec{A}\wedge\vec{C}) = -\vec{C}.(\vec{B}\wedge\vec{A})$

- If any two of vectors $\vec{A}$, $\vec{B}$ and $\overrightarrow{C}$ are parallel, or if $\vec{A}$, $\vec{B}$ and $\overrightarrow{C}$ are **Coplanar**, then:

$$\vec{A}.(\vec{B}\wedge\vec{C}) = 0$$

- **Analytically, if**: $\vec{A}\begin{Bmatrix} a_x \\ a_y \\ a_z \end{Bmatrix}$, $\vec{B}\begin{Bmatrix} b_x \\ b_y \\ b_z \end{Bmatrix}$ and $\vec{C}\begin{Bmatrix} c_x \\ c_y \\ c_z \end{Bmatrix}$:

$$\vec{A}.(\vec{B}\wedge\vec{C}) = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}$$

$$\boxed{= a_x(b_y c_z - b_z c_y) - a_y(b_x c_z - b_z c_x) + a_z(b_x c_y - b_y c_x)}$$

## *II.4. Vector triple product*

$$\boxed{\vec{A} \wedge (\vec{B} \wedge \vec{C}) = (\vec{A}.\vec{C})\vec{B} - (\vec{A}.\vec{B})\vec{C} = \vec{B}(\vec{A}.\vec{C}) - \vec{C}(\vec{A}.\vec{B})}$$

***Properties:***

- **Non-Associativity:** $\vec{A} \wedge (\vec{B} \wedge \vec{C}) \neq (\vec{A} \wedge \vec{B}) \wedge \vec{C}$
  - $\vec{A} \wedge (\vec{B} \wedge \vec{C}) = (\vec{A}.\vec{C})\vec{B} - (\vec{A}.\vec{B})\vec{C}$
  - $(\vec{A} \wedge \vec{B}) \wedge \vec{C} = (\vec{A}.\vec{C})\vec{B} - (\vec{B}.\vec{C})\vec{A}$
- The vector $\vec{A} \wedge (\vec{B} \wedge \vec{C})$ is in the plane defined by $\vec{B}$ and $\vec{C}$
- The vector $(\vec{A} \wedge \vec{B}) \wedge \vec{C}$ is in the plane defined by $\vec{A}$ and $\vec{B}$

## *II.5. Differential Operators :*

- ***Operator Nabla :*** $\vec{\nabla} = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}$ **Ou** $\vec{\nabla} = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix}$

- ***Gradient operator :***

The gradient operator is a differential operator that applies to a scalar function dependent on space and time and transforms it into a vector dependent on space and time. It is read “gradient f” or “nabla f” and is noted :

$$\overrightarrow{\boldsymbol{grad}}\boldsymbol{f} \quad \textbf{or} \quad \vec{\boldsymbol{\nabla}}\boldsymbol{f}$$

In the Cartesian coordinate system the gradient is expressed as follows:

$$\vec{\nabla}f(x,y,z,t) = \frac{\partial f(x,y,z,t)}{\partial x}\vec{i} + \frac{\partial f(x,y,z,t)}{\partial y}\vec{j} + \frac{\partial f(x,y,z,t)}{\partial z}\vec{k}$$

**Properties:**

- $\vec{\nabla}(\alpha f + \beta g) = \alpha\vec{\nabla}f + \beta\vec{\nabla}g$ (with $(\alpha, \beta) \in \mathbb{R}^2$
- $\vec{\nabla}(f.g) = f\vec{\nabla}g + g\vec{\nabla}f$

**Example:**

$$f(x, y, z) = 3x^2y + z$$

- Calculate $\overrightarrow{grad} f(x, y, z)$ in point M(1, 2, -2)

**Sol:**

$$\overrightarrow{grad} f(x, y, z) = \left.\frac{\partial f}{\partial x}\right|_{(y,z)=Cts} \vec{\imath} + \left.\frac{\partial f}{\partial y}\right|_{(x,z)=Cts} \vec{\jmath} + \left.\frac{\partial f}{\partial z}\right|_{(x,y)=Cts} \vec{k}$$

$$\Longrightarrow \overrightarrow{grad} f(x, y, z) = \vec{\nabla} f(x, y, z) = 6xy\,\vec{\imath} + 3x^2\,\vec{\jmath} + \vec{k}$$

$$\Longrightarrow \overrightarrow{grad} f(1, 2, -2) = \vec{\nabla} f(1, 2, -2) = \mathbf{2\vec{\imath} + 3\vec{\jmath} + \vec{k}}$$

## ❑ ***Divergence operator:***

The divergence operator is a differential operator that applies to a vector field and returns a scalar field. It reads divergence and is noted:

$$\boxed{div\vec{A} \text{ or } \vec{\nabla}.\vec{A}}$$

In the Cartesian coordinate system the Divergence of $\vec{A}$ is expressed as follows:

$$\boxed{div\vec{A} = \vec{\nabla}.\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}}$$

**Properties:**

- ❑ $div(\vec{A} + \vec{B}) = div\vec{A} + div\vec{B}$
- ❑ $div(\alpha\vec{A}) = \alpha\, div\vec{A}$
- ❑ $div(f\vec{A}) = f\, div\vec{A} + \overrightarrow{grad} f.\vec{A}$ (with $f$ is scalar function)

**Demonstration : Home work (واجب منزلي)**

## ❑ *Rotational operator:*

The rotational operator is a differential operator that transforms a vector field into another vector field. It reads rotational of $\vec{A}$ and is noted: $\boxed{\overrightarrow{rot}\vec{A} \text{ Or } \vec{\nabla} \wedge \vec{A}}$

➢ In the Cartesian coordinate system the Rotational of $\vec{A}$ is expressed as follows:

$$\boxed{\vec{\nabla} \wedge \vec{A} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\vec{i} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\vec{j} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\vec{k}}$$

➢ ***Properties:***

❑ $div\left(\overrightarrow{rot}\vec{A}\right) = \vec{\nabla}.\left(\vec{\nabla} \wedge \vec{A}\right) = 0$

❑ $\overrightarrow{rot}\left(\alpha\vec{A} + \beta\vec{B}\right) = \alpha\overrightarrow{rot}\vec{A} + \beta\overrightarrow{rot}\vec{B}$

❑ $\overrightarrow{rot}\,\overrightarrow{grad}f = \vec{\nabla} \wedge \vec{\nabla}f = 0$

❑ $\overrightarrow{rot}\left(f\vec{A}\right) = \vec{\nabla} \wedge \left(f\vec{A}\right) = \vec{\nabla}f \wedge \vec{A} + f\vec{\nabla}f \wedge \vec{A}$

**Demonstration : Home work (واجب منزلي)**

- ***Laplacian operator***

**Pierre-Simon Laplace**
(1749 - 1827)

## *1. The Scalar Laplacian*

The scalar Laplacian operator is a differential operator of order two that transforms a scalar function into another scalar function. The scalar Laplacian is obtained by taking the divergence of the gradient and denoted: $\boxed{\Delta f = div\left(\overrightarrow{grad}f\right) = \nabla^2 f}$

- In the Cartesian coordinate: $\Delta f = \dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} + \dfrac{\partial^2 f}{\partial z^2}$

- ***Properties :***

  - $\Delta(\alpha f + \beta g) = \alpha\Delta f + \beta\Delta g$

  - $\Delta(fg) = (\Delta f)g + 2(\nabla f).(\nabla g) + f(\Delta g)$

## 2. ***The Vector Laplacian:***

Laplacian also applies to a vector field. In this case it returns another vector field

and denotes: $\triangle \vec{A}$

By definition, the vector Laplacian is obtained using the identity (***Vector triple product )***:

$$\overrightarrow{rot}\,\overrightarrow{rot}\,\vec{A} = \vec{\nabla} \wedge (\vec{\nabla} \wedge \vec{A}) = \vec{\nabla}(\vec{\nabla}.\vec{A}) - \nabla^2\vec{A} = \overrightarrow{grad}(div\vec{A}) - \triangle \vec{A}$$

- ***Properties :***

I. $\overrightarrow{rot}\left(\overrightarrow{grad} f\right) = \vec{0}$

II. $div\left(rot\vec{A}\right) = 0$

# Chapter 01: Kinematics of a material point

## I.1. Objective:

The aim of kinematics is to describe in qualitative terms the motion of a body without looking at the causes that produce it.

The study of the motion of a body is based on the study of its successive positions relative to a reference frame, as well as its velocity and acceleration and the relationships between these three quantities as a function of time.

## I.2. Definitions:

- ***Material point (particle) :*** An object with negligible dimensions on a macroscopic scale, which is assimilated to a geometric point.

**In reality, the study of the motion of an object can be described by:**

- **Motion around its center of mass.**
- **Motion of its own center of mass**

## Trajectory, Curvilinear Coordinates, Equations of motion

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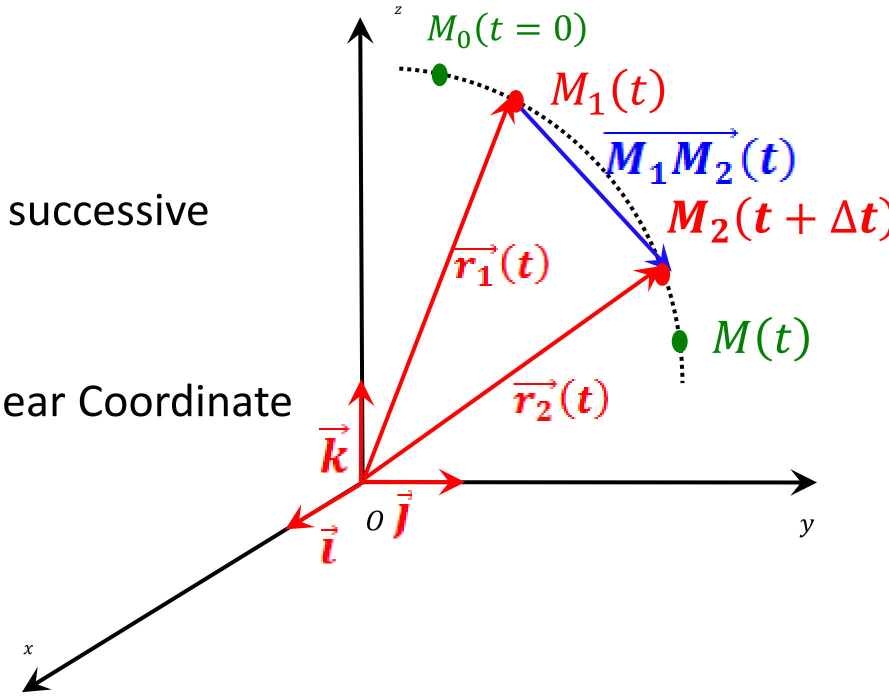



Consider "M" as a moving particle in space

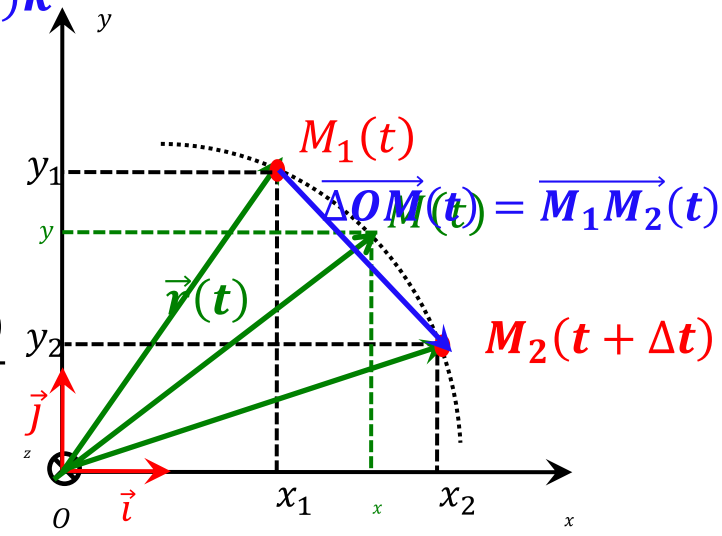
- The trajectory "M" is the geometric locus of the successive positions occupied by "M" over time.
- The algebraic value $\widehat{M_0M(t)}$ is called the Curvilinear Coordinate
- $M_0M(t) = S(t)$ : Equation of motion of « M »
- $\overrightarrow{OM_1}$ et $\overrightarrow{OM_2}$ : is a Position Vectors of « M » /« O »
- $\overrightarrow{M_1M_2}(t)$ : Displacement vector of « M » from position $M_1$ (t) to position $M_2$(t+ $\Delta$t)

$$\left|\overrightarrow{M_1M_2}(t)\right| = \left|\overrightarrow{\Delta OM}(t)\right| = \left|\overrightarrow{OM_2}(t) - \overrightarrow{OM_1}(t)\right| = \left|\overrightarrow{r_2}(t) - \overrightarrow{r_1}(t)\right|$$

# II. Curvilinear Motion

The motion of "M" is defined by its position vector at each time "t" with:

$$\overrightarrow{OM}(t) = \vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$$




## II.1. Velocity :

### ➢ Average Velocity:

$$\vec{v}_{ave} = \frac{\overrightarrow{\Delta OM}(t)}{\Delta t} = \frac{\overrightarrow{M_1M_2}(t)}{\Delta t} = \frac{\overrightarrow{OM}_2(t+\Delta t) - \overrightarrow{OM}_1(t)}{\Delta t}$$

**In Cartesian coordinates:**

$$\left.\begin{array}{l} \overrightarrow{OM}_1(t) = x_1\vec{i} + y_1\vec{j} + z_1\vec{k} \\ \overrightarrow{OM}_2(t) = x_2\vec{i} + y_2\vec{j} + z_2\vec{k} \end{array}\right\} \Rightarrow \vec{v}_{ave} = \frac{x_2 - x_1}{\Delta t}\vec{i} + \frac{y_2 - y_1}{\Delta t}\vec{j} + \frac{z_2 - z_1}{\Delta t}\vec{k}$$

$$\vec{v}_{ave} = \frac{\Delta x}{\Delta t}\vec{i} + \frac{\Delta y}{\Delta t}\vec{j} + \frac{\Delta z}{\Delta t}\vec{k}$$

## ➢ Instantaneous velocity

We obtain it by computing the average velocity for a smaller time interval.

$$\vec{V}_{inst} = \lim_{\Delta t \to 0} \vec{V}_{moy} = \lim_{\Delta t \to 0} \frac{\overrightarrow{\Delta OM}(t)}{\Delta t} = \frac{\overrightarrow{dOM}(t)}{dt} = \frac{\overrightarrow{dr}(t)}{dt}$$

***Operationally, the instantaneous velocity is found by observing the moving body at two very close positions separated by the small distance $dx$ and measuring the small time interval $dt$ required to go from one position to the other position.***

**En coordonnées cartésiennes :**

$$\vec{V}_{inst} = \vec{V}(t) = \frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j} + \frac{dz}{dt}\vec{k}$$

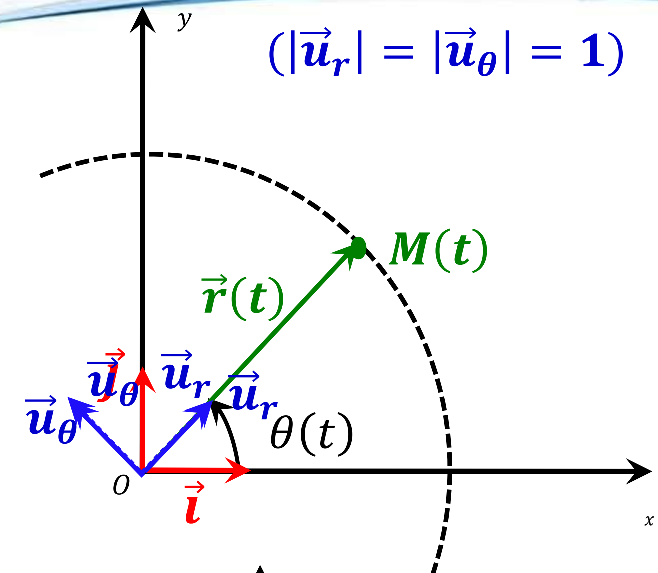
$$\begin{cases} V_x = \dfrac{dx}{dt} \\ V_y = \dfrac{dy}{dt} \\ V_z = \dfrac{dz}{dt} \end{cases}$$

**With** $|\vec{V}| = V = \sqrt{{V_x}^2 + {V_y}^2 + {V_z}^2}$

## ***In polar coordinates (2d):***

$(|\vec{u}_r| = |\vec{u}_\theta| = 1)$

$$\overrightarrow{OM}(t) = \vec{r}(t) = r(t)\vec{u}_r \quad \big(r(t), \theta(t)\big): \textbf{Polar coordinates}$$



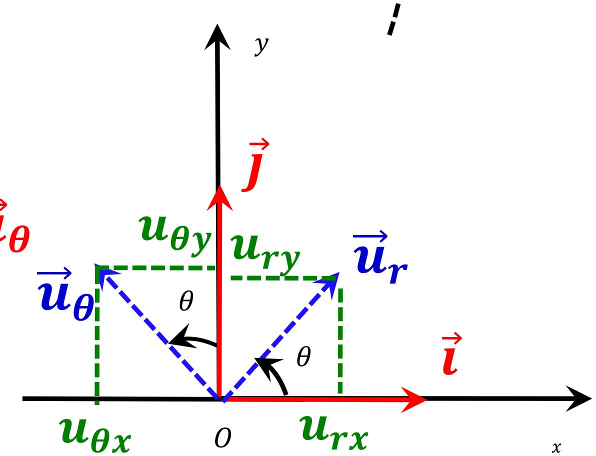

$$\vec{V}(t) = \frac{d\overrightarrow{OM}(t)}{dt} = \frac{d(r(t)\vec{u}_r)}{dt} = \frac{dr(t)}{dt}\vec{u}_r + r(t)\frac{d\vec{u}_r}{dt}$$

**Calcul of $\dfrac{d\vec{u}_r}{dt}$ :**

$$\vec{u}_r = u_{rx}\vec{\imath} + u_{ry}\vec{\jmath} = u_r cos\theta\vec{\imath} + u_r sin\theta\vec{\jmath} = cos\theta\vec{\imath} + sin\theta\vec{\jmath}$$

$$\frac{d\vec{u}_r}{dt} = -sin\theta\frac{d\theta}{dt}\vec{\imath} + cos\theta\frac{d\theta}{dt}\vec{\jmath} = \frac{d\theta}{dt}(-sin\theta\vec{\imath} + cos\theta\vec{\jmath})$$

$$\vec{u}_\theta = -u_{\theta x}\vec{\imath} + u_{\theta y}\vec{\jmath} = -sin\theta\vec{\imath} + cos\theta\vec{\jmath} \Rightarrow \frac{d\vec{u}_r}{dt} = \frac{d\theta(t)}{dt}\vec{u}_\theta$$




$$\boxed{\Rightarrow \vec{V}(t) = \frac{dr(t)}{dt}\vec{u}_r + r(t)\frac{d\theta(t)}{dt}\vec{u}_\theta}$$

$$\begin{cases} V_r(t) = \dfrac{dr(t)}{dt} & \textit{Radial component} \\ V_\theta(t) = r(t)\dfrac{d\theta(t)}{dt} & \textit{Transverse component} \end{cases}$$

Avec $|\vec{V}| = \sqrt{{V_r}^2 + {V_\theta}^2}$

## En coordonnées cylindrique :

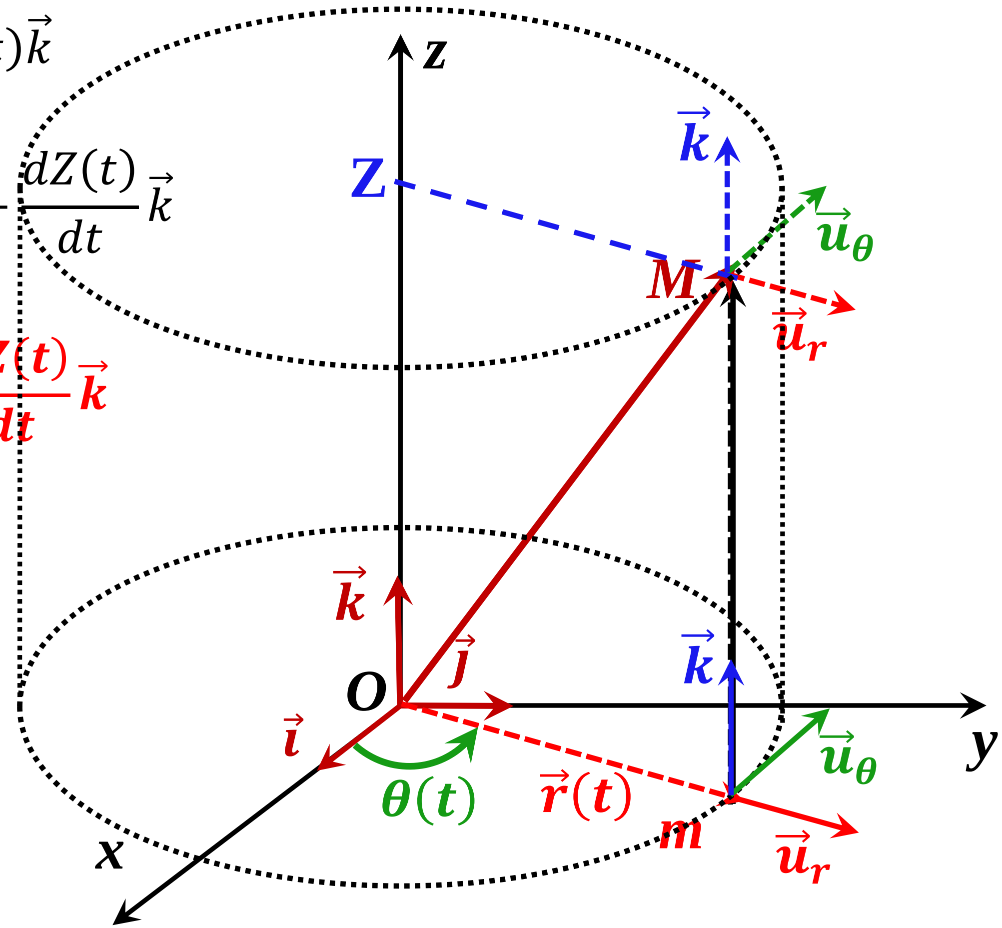
$$\overrightarrow{OM}(t) = \overrightarrow{Om}(t) + \overrightarrow{mM}(t) = r(t)\overrightarrow{u_r} + Z(t)\vec{k}$$

$$\vec{V}(t) = \frac{d\overrightarrow{OM}(t)}{dt} = \frac{dr(t)}{dt}\vec{u}_r + \vec{r}(t)\frac{d\vec{u}_r}{dt} + \frac{dZ(t)}{dt}\vec{k}$$

$$\Rightarrow \boldsymbol{\vec{V}(t) = \frac{dr(t)}{dt}\vec{u}_r + r(t)\frac{d\theta(t)}{dt}\vec{u}_\theta + \frac{dZ(t)}{dt}\vec{k}}$$

$$\begin{cases} \boldsymbol{V_r(t) = \dfrac{dr(t)}{dt}} \\ \boldsymbol{V_\theta(t) = r(t)\dfrac{d\theta(t)}{dt}} \\ \boldsymbol{V_z(t) = \dfrac{dZ(t)}{dt}} \end{cases}$$

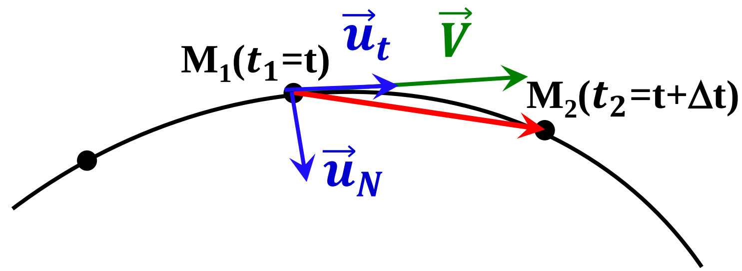
$$\boldsymbol{\|\vec{V}\| = \sqrt{{V_r}^2 + {V_\theta}^2 + {V_Z}^2}}$$




$$\begin{cases} r(t)\text{: Rayonpolaire}(r: 0 \to \infty) \\ \theta(t)\text{: Anglepolaire}(\theta: 0 \to 2\pi) \\ Z(t)\text{: Cote}(z: -\infty \to +\infty) \end{cases}$$

## Intrinsic Coordinates of Velocity : (Tangential and Normal compents)

## المركبات الذاتية للسرعة (المماسية والناظمية)




$$\vec{V}_{inst}(t) = \vec{V}(t) = \lim_{t_2 \to t_1} \frac{\overrightarrow{M_1M_2}(t)}{t_2 - t_1} = \frac{d\overrightarrow{OM}(t)}{dt}$$

**Let's call $\widehat{M_1M_2}(t)$ length of arc having from M1 to M2**

**We have:**

$$\vec{V}(t) = \lim_{t_2 \to t_1} \frac{\overrightarrow{M_1M_2}(t)}{\widehat{M_1M_2}(t)} \frac{\widehat{M_1M_2}(t)}{t_2 - t_1} = \lim_{t_2 \to t_1} \frac{\overrightarrow{M_1M_2}(t)}{\widehat{M_1M_2}(t)} \lim_{t_2 \to t_1} \frac{\widehat{M_1M_2}(t)}{t_2 - t_1}$$

**When:** $t_1 \to t_2 \Rightarrow \widehat{M_1M_2}(t) \to \left\|\overrightarrow{M_1M_2}(t)\right\| \quad \Rightarrow \frac{\overrightarrow{M_1M_2}(t)}{\widehat{M_1M_2}(t)} = \vec{u}_t$

$$\Longrightarrow \vec{V}(t) = \frac{d\widehat{M_1M_2}(t)}{dt}\vec{u}_t = \frac{dS(t)}{dt}\vec{u}_t$$

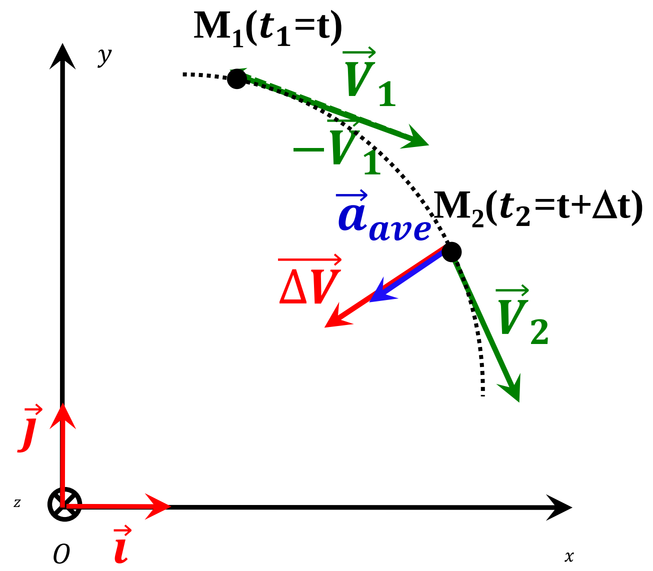
The **intrinsic coordinate system** for each point of the trajectory is defined as a system of reference formed by ***two axes***:

- ***Tangent axis***: its direction is *tangent* to the trajectory and is positive in the same direction than the velocity at that point. It is defined by the unit vector $\vec{u}_t$
- ***Normal axis***: it is *perpendicular* to the trajectory and is positive toward the center of curvature of the trajectory. It is defined by the unit vector $\vec{u}_N$

**This reference system is used to "observe" the changes in the magnitude and *direction* of the *velocity vector*.**

## ➢ Average Acceleration:

$$\vec{a}_{ave} = \frac{\vec{V}_2(t) - \vec{V}_1(t)}{t_2 - t_1} = \frac{\overrightarrow{\Delta V}(t)}{\Delta t}$$




### In Cartesian coordinates:

$$\vec{V}_1\begin{pmatrix} V_{x1} \\ V_{y1} \\ V_{z1} \end{pmatrix}, \vec{V}_2\begin{pmatrix} V_{x2} \\ V_{y2} \\ V_{z3} \end{pmatrix}$$

$$\vec{a}_{ave} = \frac{V_{x_2} - V_{x_1}}{t_2 - t_1}\vec{i} + \frac{V_{y_2} - V_{y_1}}{t_2 - t_1}\vec{j} + \frac{V_{z_2} - V_{z_1}}{t_2 - t_1}\vec{k}$$

$$\vec{a}_{ave} = \frac{\Delta V_x}{\Delta t}\vec{i} + \frac{\Delta V_y}{\Delta t}\vec{j} + \frac{\Delta V_Z}{\Delta t}\vec{k}$$

$$\begin{cases} \Delta V_x = V_{x_2} - V_{x_1} \\ \Delta V_y = V_{y_2} - V_{y_1} \\ \Delta V_z = V_{z_2} - V_{z_1} \end{cases}$$

## ➢ Instantaneous acceleration

$$\vec{a}_{inst} = \vec{a} = \lim_{\Delta t \to 0} \vec{a}_{ave} = \lim_{\Delta t \to 0} \frac{\overrightarrow{\Delta V}(t)}{\Delta t} = \lim_{t_2 \to t_1} \frac{\vec{V}_2(t) - \vec{V}_1(t)}{t_2 - t_1} = \frac{\overrightarrow{dV}(t)}{dt}$$

In Cartesian coordinates:

$$\vec{a}_{inst} = \frac{dV_x}{dt}\vec{\imath} + \frac{dV_y}{dt}\vec{\jmath} + \frac{dV_z}{dt}\vec{k}$$

$$\begin{cases} V_x = \dfrac{dx}{dt} \\ V_y = \dfrac{dy}{dt} \\ V_z = \dfrac{dz}{dt} \end{cases} \Longrightarrow \vec{a} = \frac{d^2x}{dt^2}\vec{\imath} + \frac{d^2y}{dt^2}\vec{\jmath} + \frac{d^2z}{dt^2}\vec{k}$$

$$\begin{cases} a_x = \dfrac{dV_x}{dt} = \dfrac{d^2x}{dt^2} \\ a_y = \dfrac{dV_y}{dt} = \dfrac{d^2y}{dt^2} \\ a_z = \dfrac{dV_z}{dt} = \dfrac{d^2z}{dt^2} \end{cases}$$

$$\|\vec{a}\| = a = \sqrt{a_x{}^2 + a_y{}^2 + a_z{}^2}$$

***In Polar coordinates:***

$$\vec{V}(t)=\frac{dr(t)}{dt}\vec{u}_r+r(t)\frac{d\theta(t)}{dt}\vec{u}_\theta \quad \Rightarrow \vec{a}(t)=\frac{\overrightarrow{dV}}{dt}=\frac{d}{dt}\left(\frac{\boldsymbol{dr(t)}}{\boldsymbol{dt}}\vec{\boldsymbol{u}}_{\boldsymbol{r}}\right)+\frac{d}{dt}\left(\boldsymbol{r(t)}\frac{\boldsymbol{d\theta(t)}}{\boldsymbol{dt}}\vec{\boldsymbol{u}}_{\boldsymbol{\theta}}\right)$$

$$=\frac{d^2r(t)}{dt^2}\vec{u}_r+\frac{dr(t)}{dt}\underbrace{\frac{d\vec{u}_r}{dt}}_{\frac{d\theta(t)}{dt}\vec{u}_\theta}+\frac{dr(t)}{dt}\frac{d\theta(t)}{dt}\vec{u}_\theta+r(t)\frac{d^2\theta(t)}{dt^2}\vec{u}_\theta+r(t)\frac{d\theta(t)}{dt}\underbrace{\frac{d\vec{u}_\theta}{dt}}_{-\frac{d\theta(t)}{dt}\vec{u}_r}$$

$$\Rightarrow \vec{a}=\frac{d^2r(t)}{dt^2}\vec{u}_r+\frac{dr(t)}{dt}\frac{d\theta(t)}{dt}\vec{u}_\theta+\frac{dr(t)}{dt}\frac{d\theta(t)}{dt}\vec{u}_\theta+r(t)\frac{d^2\theta(t)}{dt^2}\vec{u}_\theta-r(t)\frac{d\theta(t)}{dt}\frac{d\theta(t)}{dt}\vec{u}_r$$

$$\Rightarrow \vec{\boldsymbol{a}}=\underbrace{\left(\frac{\boldsymbol{d^2r(t)}}{\boldsymbol{dt^2}}-\boldsymbol{r(t)}\left(\frac{\boldsymbol{d\theta(t)}}{\boldsymbol{dt}}\right)^{\boldsymbol{2}}\right)}_{a_r}\vec{\boldsymbol{u}}_{\boldsymbol{r}}+\underbrace{\left(\boldsymbol{2}\frac{\boldsymbol{dr(t)}}{\boldsymbol{dt}}\frac{\boldsymbol{d\theta(t)}}{\boldsymbol{dt}}+\boldsymbol{r(t)}\frac{\boldsymbol{d^2\theta(t)}}{\boldsymbol{dt^2}}\right)}_{a_\theta}\vec{\boldsymbol{u}}_{\boldsymbol{\theta}}$$

## In Cylindrical coordinates:

$$\vec{V}(t) = \frac{dr(t)}{dt}\vec{u}_r + r(t)\frac{d\theta(t)}{dt}\vec{u}_\theta + \frac{dZ}{dt}\vec{k}$$

$$\Rightarrow \vec{a}(t) = \frac{d}{dt}\left(\frac{dr(t)}{dt}\vec{u}_r\right) + \frac{d}{dt}\left(r(t)\frac{d\theta(t)}{dt}\vec{u}_\theta\right) + \frac{d^2Z}{dt^2}\vec{k}$$

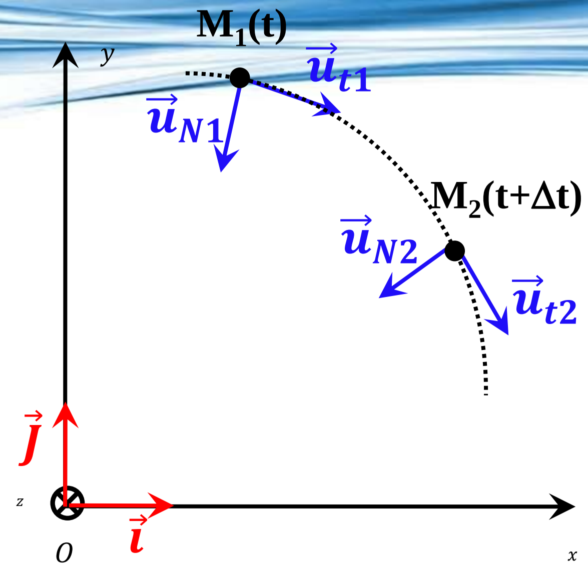
Using the same method, we find:

$$\Rightarrow \vec{a} = \underbrace{\left(\frac{d^2r(t)}{dt^2} - r(t)\left(\frac{d\theta(t)}{dt}\right)^2\right)}_{a_r}\vec{u}_r + \underbrace{\left(2\frac{dr(t)}{dt}\frac{d\theta(t)}{dt} + r(t)\frac{d^2\theta(t)}{dt^2}\right)}_{a_\theta}\vec{u}_\theta + \underbrace{\frac{d^2Z}{dt^2}}_{a_z}\vec{k}$$

$$\|\vec{a}\| = a = \sqrt{a_r{}^2 + a_\theta{}^2 + a_z{}^2}$$

## Intrinsic compents of acceleration:

We'v : $\vec{V}(t) = \frac{dS(t)}{dt}\vec{u}_t$




$$\vec{a} = \frac{d\vec{V}}{dt} = \frac{d}{dt}\left(\frac{dS(t)}{dt}\vec{u}_t\right) = \frac{d^2S(t)}{dt^2}\vec{u}_t + \frac{dS(t)}{dt}\frac{d\vec{u}_t}{dt}$$

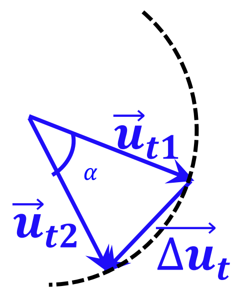
Calculation of $\frac{d\vec{u}_t}{dt}$ :

❑ **Property :**

**The derivative of a unit vector is a vector orthogonal to that vector**

$$\|\vec{u}_t\| = 1 \Rightarrow \frac{d\vec{u}_t}{dt} \perp \vec{u}_t \quad \Rightarrow \frac{d\vec{u}_t}{dt} = \frac{du_t}{dt}\vec{u}_N$$

$\|\vec{u}_{t1}\| = \|\vec{u}_{t2}\|$




**On the other hand, we have:**

$$\begin{cases} \overrightarrow{\Delta u_t} = \vec{u}_{t2} - \vec{u}_{t1} \\ \|\overrightarrow{\Delta u_t}\| = \|\vec{u}_{t1}\| sin\alpha \end{cases} \quad t_1 \to t_2 \Rightarrow \begin{cases} \overrightarrow{\Delta u_t} \to d\vec{u}_t \\ \alpha \to d\alpha \ (sind\alpha \approx d\alpha) \end{cases} \quad \Rightarrow \|d\vec{u}_t\| = \|\vec{u}_t\| d\alpha = d\alpha$$

$$\Rightarrow \frac{d\vec{u}_t}{dt} = \frac{du_t}{dt}\vec{u}_N = \frac{d\alpha}{dt}\vec{u}_N$$

$$\vec{a} = \frac{d^2S(t)}{dt^2}\vec{u}_t + \frac{dS(t)}{dt}\frac{d\alpha}{dt}\vec{u}_N = \frac{d^2S}{dt^2}\vec{u}_t + \frac{dS}{dt}\frac{d\alpha}{\boldsymbol{dS}}\frac{\boldsymbol{dS}}{dt}\vec{u}_N$$

$\frac{dS}{d\alpha} = \rho$: Trajectory Radius

$$\vec{a} = \underbrace{\frac{d^2S(t)}{dt^2}}_{\vec{a}_t}\vec{u}_t + \underbrace{\frac{1}{\rho}\left(\frac{dS(t)}{dt}\right)^2}_{\vec{a}_N}\vec{u}_N = \frac{dV(t)}{dt}\vec{u}_t + \frac{1}{\rho}V(t)^2\vec{u}_N$$

$$\begin{cases} \frac{d^2S}{dt^2} = \frac{dV}{dt} \\ \frac{1}{\rho}\frac{dS}{dt} = \frac{1}{\rho}V \end{cases}$$

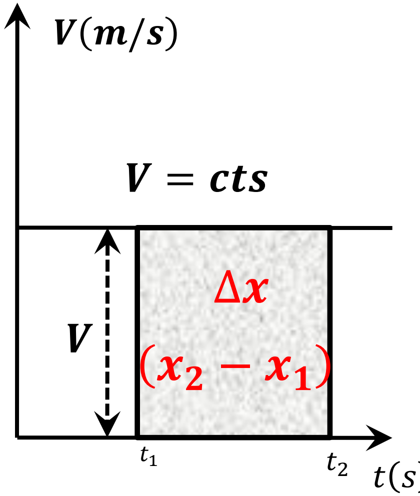
$\frac{d^2S}{dt^2} = \frac{dV}{dt}$: ***Tangential component of $\vec{a}$ related to the change in modulus of $\vec{V}$***

$\frac{1}{\rho}\frac{dS}{dt} = \frac{1}{\rho}V$: ***Normal component of $\vec{a}$ related to the change in direction of $\vec{V}$***

$$\boldsymbol{\vec{a} = a_t\vec{u}_t + a_N\vec{u}_N} \qquad \|\vec{a}\| = a = \sqrt{a_t^2 + a_N^2}$$

## II.3. Transition from speed to distance travelled – Integral calculation :

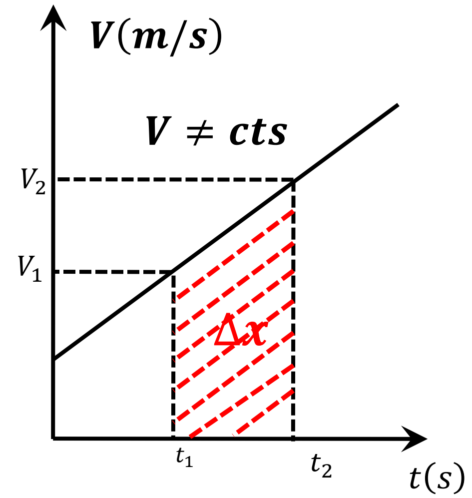
❖ *Let be a mobile "**M**" moving with a constant velocity in rectilinear motion*




$$\Rightarrow V_{moy} = V_{inst} = V = \frac{\Delta x}{\Delta t} = \frac{x_2 - x_1}{t_2 - t_1}$$

**Knowing** $V$ **and** $x_1$ **at** $t = t_1 \quad \Rightarrow x_2 = x_1 + V(t_2 - t_1)$

**The distance** $\Delta x$ **traveled between** $t_1$ **and** $t_2$ **is measured by the area under the curve** $V(t)$**:** $\Delta x = V(t_2 - t_1)$

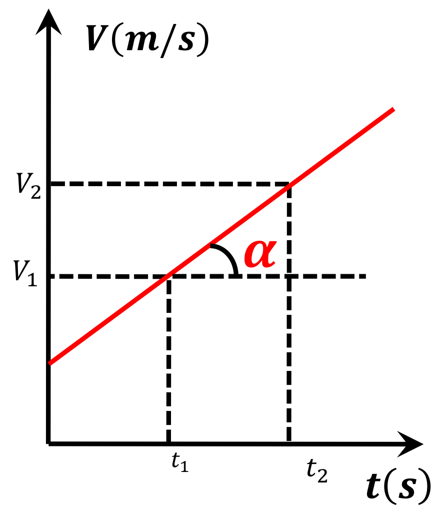



❖ **When the velocity is not constant** $\Delta x$ **is always equal to the area under the curve** $V(t)$ $\left(\Delta x = (V_2 - V_1)(t_2 - t_1)\right)$

## II.4- Transition from acceleration to velocity:

If the motion is defined by the given acceleration, the velocity is equal to the integral of the acceleration (acceleration is the derivative of velocity).

$$a = \frac{dV}{dt} \Rightarrow \int_{V_1}^{V_2} dV = \int_{t_1}^{t_2} a dt$$




**Geometrically:**

- **The acceleration will be the curve tangent of V(t)**

$$a = tg\alpha = \frac{V_2 - V_1}{t_2 - t_1}$$

## ***Exemple :***

An object moves in an oriented straight line with a velocity that obeys the law:

$$a = 4 - t^2 (m/s^2)$$

- Find, as a function of time, the expressions for velocity and position.

We give: $t = 3s \Rightarrow V = 2m/s, x = 9m.$

- Represent the velocity and acceleration vectors at **t = 1s.**

## ***Solution :***

**1.** $V = \int a dt \quad = \int (4 - t^2) dt \quad = 4t - \frac{t^3}{3} \quad +C$

$t = 3s \Rightarrow V = 2m/s \quad \Rightarrow 2 = 4.3 - \frac{3^3}{3} + C \quad \Rightarrow C = -1 \quad \boxed{\Rightarrow V = 4t - \frac{t^3}{3} - 1}$

$x = \int V dt \quad = \int \left(4t - \frac{t^3}{3} - 1\right) dt \quad = 2t^2 - \frac{1}{12} t^4 - t + C'$

$t = 3s \Rightarrow x = 9m \Rightarrow C' = 3/4 \quad \boxed{\Rightarrow x = -\frac{1}{12} t^4 + 2t^2 - t + 3/4}$

**2.**

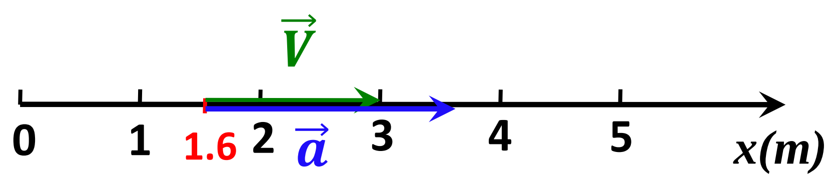
$$\begin{cases} x = -\frac{1}{12}t^4 + 2t^2 - t + 3/4 \\ V = -\frac{t^3}{3} + 4t - 1 \\ a = 4 - t^2 \end{cases}$$

$$\Rightarrow t = 1s: \begin{cases} x = -\frac{1}{12} + 2 - 1 + 3/4 = 1.6m \\ V = -\frac{1}{3} + 4 - 1 = 2.6m/s \\ a = 4 - 1 = 3m/s^2 \end{cases}$$

**Echelle :** $x: 1cm \rightarrow 1m$

$V: 1cm \rightarrow 2m/s$

$a: 1cm \rightarrow 1.5m/s^2$

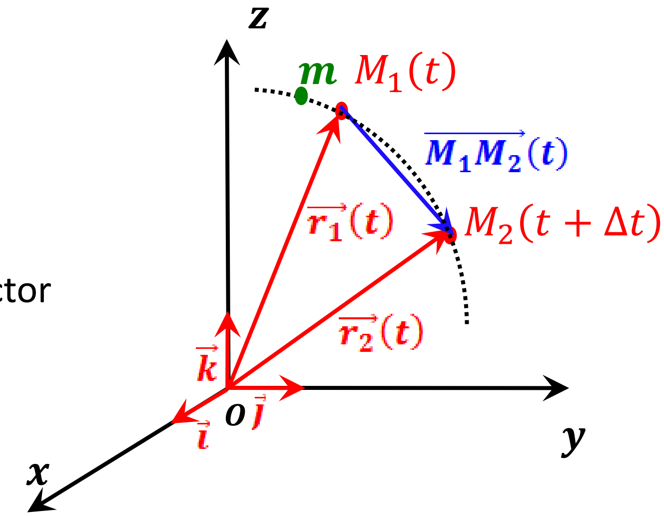


## Summary

- $\widehat{M_1M_2}(t) = S(t)$ :Curvilinear Coordinate

- $\overrightarrow{OM_1}$ *et* $\overrightarrow{OM_2}$ : is a Position Vectors of « M » /« O »

- $\left|\overrightarrow{M_1M_2}(t)\right| = \left|\Delta\overrightarrow{OM}(t)\right| = \left|\overrightarrow{OM_2}(t) - \overrightarrow{OM_1}(t)\right|$ : Displacement vector




### Velocity $(m/s)$

| Average Velocity $(\vec{V}_{ave}(t))$ | Instantaneous velocity $\vec{V}_{inst} = \vec{V}$ |
|---|---|
| $\dfrac{\Delta\overrightarrow{OM}(t)}{\Delta t} = \dfrac{\overrightarrow{M_1M_2}(t)}{\Delta t} = \dfrac{\Delta x}{\Delta t}\vec{i} + \dfrac{\Delta y}{\Delta t}\vec{j} + \dfrac{\Delta z}{\Delta t}\vec{k}$ | $\lim\limits_{\Delta t\to 0}\vec{V}_{ave} = \lim\limits_{\Delta t\to 0}\dfrac{\Delta\overrightarrow{OM}(t)}{\Delta t} = \dfrac{d\overrightarrow{OM}(t)}{dt}$ |

### Acceleration

| Average Acceleration$(\vec{a}_{ave}(t))$ | Instantaneous Acceleration $\vec{a}_{inst} = \vec{a}$ |
|---|---|
| $\dfrac{\Delta\vec{V}(t)}{\Delta t} = \dfrac{\Delta V_x}{\Delta t}\vec{i} + \dfrac{\Delta V_y}{\Delta t}\vec{j} + \dfrac{\Delta V_z}{\Delta t}\vec{k}$ | $\lim\limits_{\Delta t\to 0}\vec{a}_{ave} = \lim\limits_{\Delta t\to 0}\dfrac{\Delta\vec{V}(t)}{\Delta t} = \dfrac{d\vec{V}(t)}{dt}$ |

| | Cartesian Coordinates $(\vec{i},\vec{j},\vec{k})$ | Polar Coordinates $(\vec{u}_r,\ \vec{u}_\theta)$ | Cylindric Coordinates $(\vec{u}_r,\ \vec{u}_\theta,\ \vec{k})$ |
|---|---|---|---|
| $\overrightarrow{OM}(t)$ | $x(t)\vec{i}+y(t)\vec{j}+z(t)\vec{k}$ | $r(t)\vec{u}_r$ | $r(t)\vec{u}_r+z(t)\vec{k}$ |
| $\vec{V}(t)$ | $\frac{dx}{dt}\vec{i}+\frac{dy}{dt}\vec{j}+\frac{dz}{dt}\vec{k}$ | $\frac{dr(t)}{dt}\vec{u}_r+r(t)\frac{d\theta(t)}{dt}\vec{u}_\theta$ | $\frac{dr(t)}{dt}\vec{u}_r+r(t)\frac{d\theta(t)}{dt}\vec{u}_\theta+\frac{dz(t)}{dt}\vec{k}$ |
| $\vec{a}$ | $\frac{d^2x}{dt^2}\vec{i}+\frac{d^2y}{dt^2}\vec{j}+\frac{d^2z}{dt^2}\vec{k}$ | $\left(\frac{d^2r(t)}{dt^2}-r(t)\left(\frac{d\theta(t)}{dt}\right)^2\right)\vec{u}_r+\left(2\frac{dr(t)}{dt}\frac{d\theta(t)}{dt}+r(t)\frac{d^2\theta(t)}{dt^2}\right)\vec{u}_\theta$ | $\left(\frac{d^2r(t)}{dt^2}-r(t)\left(\frac{d\theta(t)}{dt}\right)^2\right)\vec{u}_r+\left(2\frac{dr(t)}{dt}\frac{d\theta(t)}{dt}+r(t)\frac{d^2\theta(t)}{dt^2}\right)\vec{u}_\theta+\frac{d^2Z}{dt^2}\vec{k}$ |

| Intrinsec coordinates $(\vec{u}_t,\ \vec{u}_N)$ | |
|---|---|
| Velocity | $\vec{V}(t)=\frac{dS(t)}{dt}\vec{u}_t=V(t)\vec{u}_t$ |
| Acceleration | $\vec{a}(t)=\frac{dV(t)}{dt}\vec{u}_t+\frac{1}{\rho}V(t)^2\vec{u}_N=a_t\vec{u}_t+a_N\vec{u}_N$ |

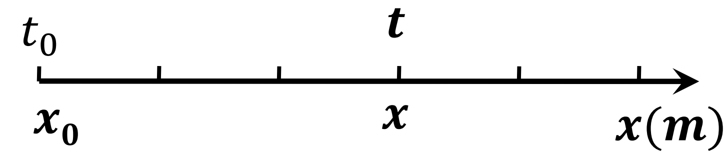
# III. Some specific movements

## III.1. Rectilinear motion:

In this type of motion, the trajectories are straight lines and the position of the mobile is described by a single coordinate $x(t)$ equivalent to the path traveled $S(t)$.

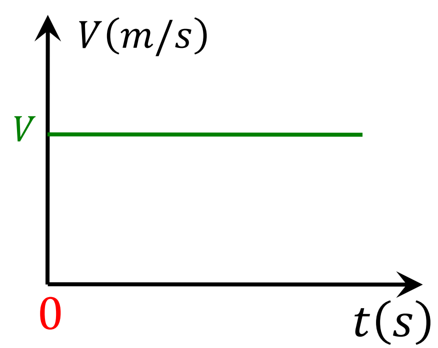
### III.1.1. Uniform Rectilinear Motion (URM): الحركة المستقيمة المنتظمة

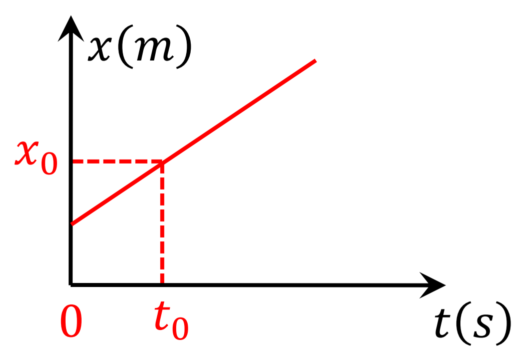
**Characterized by *V(t) = Cts = V***




$$V = \frac{dx}{dt} \Rightarrow dx = Vdt \quad \Rightarrow \int_{x_0}^{x} dx = \int_{t_0}^{t} Vdt \quad \Rightarrow x - x_0 = V(t - t_0)$$

$\Rightarrow$ ***Equation of Motion :*** $\boxed{x(t) = V(t - t_0) + x_0}$



## III.1.2. Uniformly Varied Rectilinear Motion(UVRM):

الحركة المستقيمة المتغيرة بإنتظام

Characterize by $\boldsymbol{a(t) = Cts = a}$

$$t = t_0 : \begin{cases} x = x_0 \\ V = V_0 \end{cases}$$

$$a = \frac{dV}{dt} \Rightarrow dV = adt \quad \Rightarrow \int_{V_0}^{V} dV = a\int_{t_0}^{t} dt \Rightarrow V - V_0 = a(t - t_0)$$

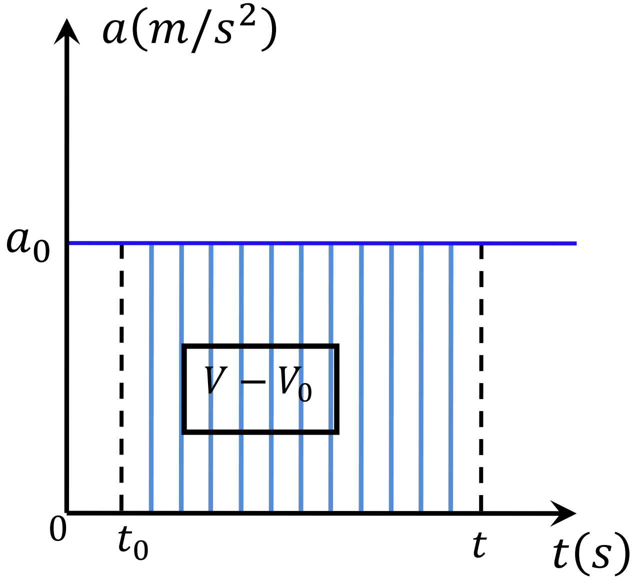
$$\boxed{\Rightarrow \boldsymbol{V(t) = a(t - t_0) + V_0}}$$

- ***Equation of Motion***

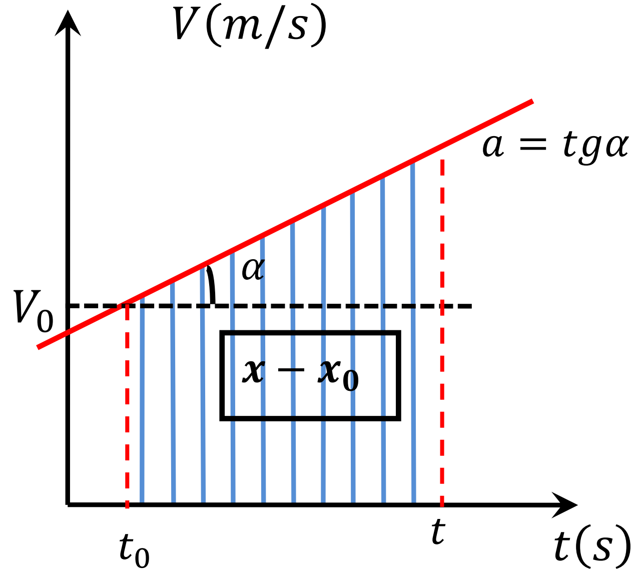
$$V(t) = \frac{dx}{dt} \Longrightarrow \int_{x_0}^{x} dx = \int_{t_0}^{t} V dt \quad = \int_{t_0}^{t} (a(t - t_0) + V_0)dt$$

$$\Rightarrow x - x_0 = \frac{1}{2}a(t - t_0)^2 + V_0(t - t_0)$$

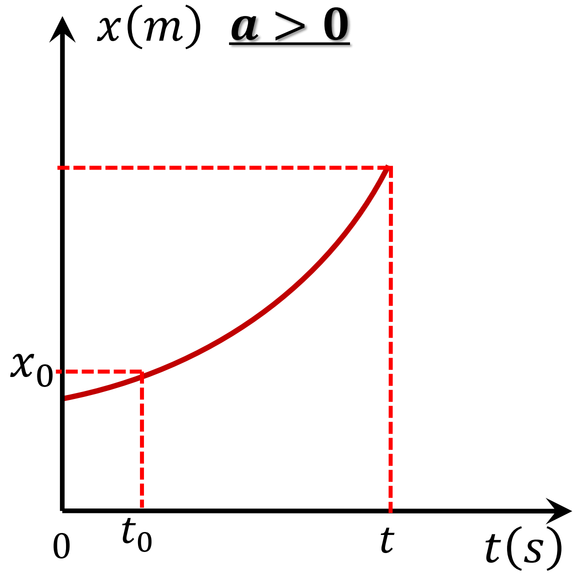
$$\boxed{\Rightarrow \boldsymbol{x = \frac{1}{2}a(t - t_0)^2 + V_0(t - t_0) + x_0}}$$ **(Equation of Motion)**



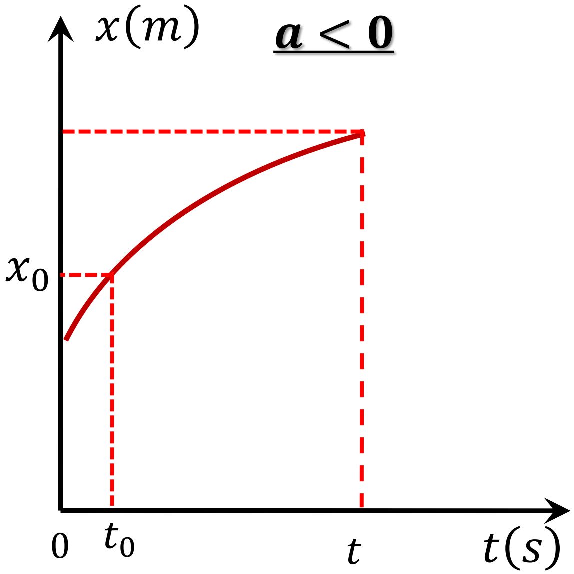
$a(m/s^2)$
$a_0$
$V - V_0$
0
$t_0$
$t$
$t(s)$

$V(m/s)$
$a = tg\alpha$
$\alpha$
$V_0$
$\boldsymbol{x - x_0}$
$t_0$
$t$
$t(s)$

$x(m)$
$\boldsymbol{a > 0}$
$x_0$
0
$t_0$
$t$
$t(s)$

$x(m)$
$\boldsymbol{a < 0}$
$x_0$
0
$t_0$
$t$
$t(s)$

***Remark:***

The acceleration or deceleration of a uniformly varying motion is defined by the sign of the dot product $\vec{a}.\vec{V}$ :

$\vec{a}.\vec{V} > 0$ : **Two possible cases:**

$\vec{a} > 0 \; et \; \vec{V} > 0$: M. Accelerated ***Uniformly*** in the positive direction of motion

$\vec{a} < 0 \; et \; \vec{V} < 0$: M. Accelerated ***Uniformly*** in the negative direction of motion

$\vec{a}.\vec{V} < 0$ : **Two possible cases :**

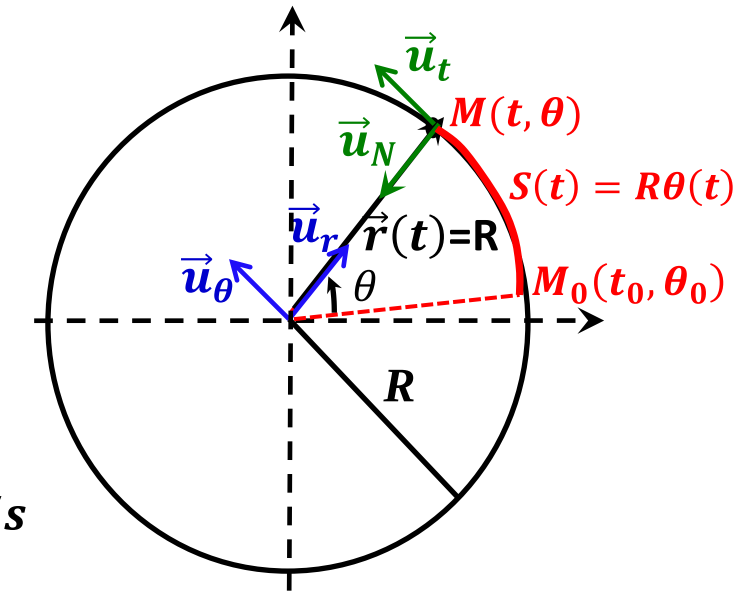
$\vec{a} < 0 \; et \; \vec{V} > 0$: M. decelerated ***Uniformly*** in the positive direction of motion

$\vec{a} > 0 \; et \; \vec{V} < 0$: M. decelerated ***Uniformly*** in the negative direction of motion

## III.2. Circular Motion:

This type of motion is characterized by a circular trajectory with a constant radius :

$$\boldsymbol{r(t) = cte = R}$$




<u>In polar coordinates:</u> $\vec{r}(t) = R\vec{u}_r$

$$\vec{V}(t) = \frac{dr(t)}{dt}\vec{u}_r + r(t)\frac{d\theta(t)}{dt}\vec{u}_\theta = \overset{0}{\cancel{\frac{dR}{dt}}}\vec{u}_r + R\frac{d\theta(t)}{dt}\vec{u}_\theta$$

$$\Rightarrow \boldsymbol{\vec{V}(t) = R\frac{d\theta(t)}{dt}\vec{u}_\theta}$$

$$\boldsymbol{\frac{d\theta(t)}{dt} = \omega(t)}$$ : ***Called Angular Velocity***, $\boldsymbol{[\omega] = rad/s}$

$$\boxed{\Rightarrow \boldsymbol{\vec{V}(t) = R\omega(t)\vec{u}_\theta}}$$

<u>En coordonnées intrinsèques :</u> $\vec{V}(t) = \frac{dS(t)}{dt}\vec{u}_t$

$$S(t) = R\theta(t) \Rightarrow \vec{V}(t) = R\frac{d\theta(t)}{dt}\vec{u}_t = R\omega(t)\vec{u}_t$$

$$\boxed{\Rightarrow \boldsymbol{\vec{V}(t) = R\omega(t)\vec{u}_\theta = R\omega(t)\vec{u}_t}}$$

## Acceleration Expression:

- **In intrinsic coordinates:** $\vec{a} = a_t\vec{u}_t + a_N\vec{u}_N = \dfrac{dV(t)}{dt}\vec{u}_t + \dfrac{1}{\rho}V^2\vec{u}_N$

$$\begin{cases} \rho = R \\ V = R\omega \\ \omega = \dfrac{d\theta}{dt} \end{cases} \Rightarrow \vec{a}(t) = R\frac{d\omega(t)}{dt}\vec{u}_t + R\omega^2(t)\vec{u}_N = \boldsymbol{R\frac{d^2\theta(t)}{dt^2}\vec{u}_t + R\left(\frac{d\theta(t)}{dt}\right)^2\vec{u}_N}$$

- **In polar coordinates:**

$$\vec{a} = \left(\frac{d^2r(t)}{dt^2} - r(t)\left(\frac{d\theta(t)}{dt}\right)^2\right)\vec{u}_r + \left(2\frac{dr(t)}{dt}\frac{d\theta(t)}{dt} + r(t)\frac{d^2\theta(t)}{dt^2}\right)\vec{u}_\theta$$

$$r(t) = R \Rightarrow \vec{a} = \left(\cancelto{0}{\frac{d^2R}{dt^2}} - R\left(\frac{d\theta(t)}{dt}\right)^2\right)\vec{u}_r + \left(2\cancelto{0}{\frac{dR}{dt}}\frac{d\theta(t)}{dt} + R\frac{d^2\theta(t)}{dt^2}\right)\vec{u}_\theta$$

$$\boxed{\boldsymbol{\vec{a} = -R\left(\frac{d\theta(t)}{dt}\right)^2\vec{u}_r + R\frac{d^2\theta(t)}{dt^2}\vec{u}_\theta}} \quad \boxed{\boldsymbol{= -R\omega^2(t)\vec{u}_r + R(t)\frac{d\omega(t)}{dt}\vec{u}_\theta}}$$

$\boldsymbol{\dfrac{d\omega(t)}{dt}} = \alpha$: **Angular acceleration,** $[\alpha] = \boldsymbol{rad/s^2}$

## III.2.1. Uniform Circular Motion(UCM): الحركة الدائرية المنتظمة

This type of motion is characterized by a constant angular velocity:

$$V(t) = Cst = R\omega(t) \Rightarrow \omega(t) = Cst$$

$$\vec{a} = \begin{cases} a_t = R\dfrac{d\omega(t)}{dt} = 0 \\ a_N = R\omega^2 \end{cases} \quad \textbf{Or} \quad \vec{a} = \begin{cases} a_r = -R\omega^2(t) \\ a_\theta = R\dfrac{d\omega(t)}{dt} = 0 \end{cases}$$

- ***Equation of motion:***

$$\omega = \frac{d\theta(t)}{dt} \Rightarrow d\theta(t) = \omega dt \Rightarrow \int_{\theta_0}^{\theta(t)} d\theta(t) = \int_{t_0}^{t} \omega\, dt \Longrightarrow \theta(t) - \theta_0 = \omega\,(t - t_0)$$

So the equation of this motion is given by:

$$\boxed{\boldsymbol{\theta(t) = \omega(t - t_0) + \theta_0}}$$

## *III.2.2. Uniformly Variable Circular Motion (UVCM):* الحركة الدائرية المتغيرة بانتظام

This type of motion is characterized by constant tangential acceleration: $a_t(t) = Cst$

$$a_t(t) = R\frac{d\omega(t)}{dt} = Cst \quad \Rightarrow \frac{d\omega(t)}{dt} = \alpha = Cst \quad \Rightarrow d\omega(t) = \alpha dt \quad \left(\boldsymbol{t = t_0}: \begin{cases} \boldsymbol{\omega = \omega_0} \\ \boldsymbol{\theta = \theta_0} \end{cases}\right)$$

$$\Rightarrow \int_{\omega_0}^{\omega(t)} d\omega(t) = \int_{t_0}^{t} \alpha dt \qquad \boxed{\Rightarrow \boldsymbol{\omega(t) = \alpha(t - t_0) + \omega_0}}$$

**On the other hand, we have:**

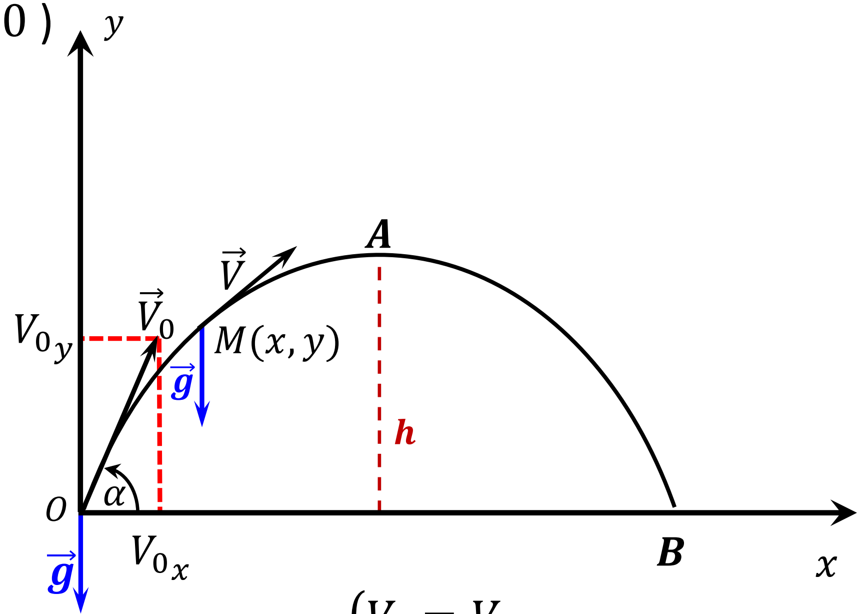
$$\omega(t) = \frac{d\theta(t)}{dt} \Rightarrow \int_{\theta_0}^{\theta(t)} d\theta(t) = \int_{t_0}^{t} \omega(t)dt = \int_{t_0}^{t} (\alpha(t - t_0) + \omega_0)dt$$

$$\Rightarrow \theta(t) - \theta_0 = \frac{1}{2}\alpha(t - t_0)^2 + \omega_0(t - t_0)$$

$$\boxed{\boldsymbol{\theta(t) = \frac{1}{2}\alpha(t - t_0)^2 + \omega_0(t - t_0) + \theta_0}}$$ **(Equation of UVCM)**

## ***Application : Motion of a projectile***

- $\vec{V}_0$: The initial velocity of the projectile ($t_0 = 0$ )




- In this case: $\vec{a} = \vec{g} = -g\vec{j}$

$$\vec{V}_0 = V_{0_x}\vec{\imath} + V_{0_y}\vec{j} \quad \text{Where} \begin{cases} V_{0_x} = V_0 \cos\alpha \\ V_{0_x} = V_0 \sin\alpha \end{cases}$$

- On the other hand, at a given point $M(x, y)$,

  we have: $\vec{V} = \vec{a}(t - t_0) + \vec{V}_0$

- At $t_0 = 0$ :

$$V_x\vec{\imath} + V_y\vec{j} = -gt\vec{j} + V_{0_x}\vec{\imath} + V_{0_y}\vec{j} = V_{0_x}\vec{\imath} + \left(V_{0_y} - gt\right)\vec{j} \quad \Longrightarrow \begin{cases} V_x = V_{0_x} \\ V_y = V_{0_y} - gt \end{cases}$$

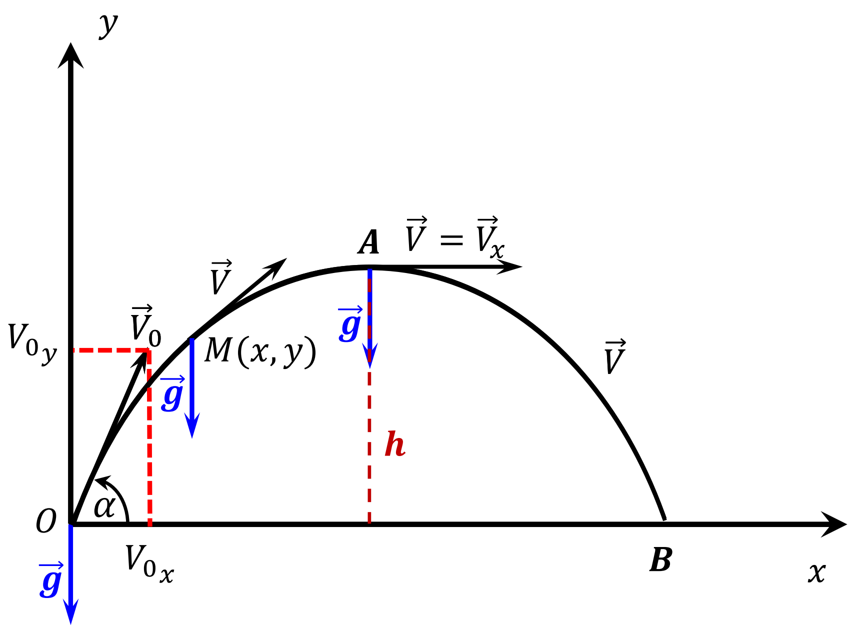
- Also, we have:

$$\overrightarrow{OM} = \vec{r} = x\vec{\imath} + y\vec{j} \quad = \frac{1}{2}\vec{a}(t - t_0)^2 + \vec{V}_0(t - t_0) + \vec{r}_0$$

$$\text{At } t_0 = 0 : \begin{cases} r_0 = 0 \\ a = -g \end{cases} \Longrightarrow x\vec{\imath} + y\vec{j} = -\frac{1}{2}\mathrm{g}t^2\vec{j} + V_{0_x}t\vec{\imath} + V_{0_y}t\vec{j} \Longrightarrow \begin{cases} x = V_{0_x}t \\ y = -\frac{1}{2}gt^2 + V_{0_y}t \end{cases}$$

❑ *The time requiered for the projectile to reach the highest Point **A** is obtained by setting*

$V_y = 0$. *in this point the velocity is horizontal*




***Then:*** $V_y = 0 \Rightarrow V_{0y} - gt = 0 \quad \Rightarrow t = \dfrac{V_{0y}}{g}$

$$\boxed{\Rightarrow \boldsymbol{t} = \frac{\boldsymbol{V_0 \sin \alpha}}{\boldsymbol{g}}}$$

❑ $\boldsymbol{h}$ is obtained by substituting this value of $t$ in the equation of $y$: $y = -\dfrac{1}{2}gt^2 + V_{0y}t$

$$\Rightarrow h = -\frac{1}{2}g\frac{V_0{}^2\, sin^2\, \alpha}{g^2} + V_{0y}\frac{V_0 \sin \alpha}{g}$$

$$= -\frac{1}{2}\frac{V_0{}^2\, sin^2\, \alpha}{g} + V_0 \sin \alpha \frac{V_0 \sin \alpha}{g} \qquad \boxed{\Rightarrow \boldsymbol{h} = \frac{\boldsymbol{1}}{\boldsymbol{2}}\frac{\boldsymbol{V_0{}^2\, sin^2\, \alpha}}{\boldsymbol{g}}}$$

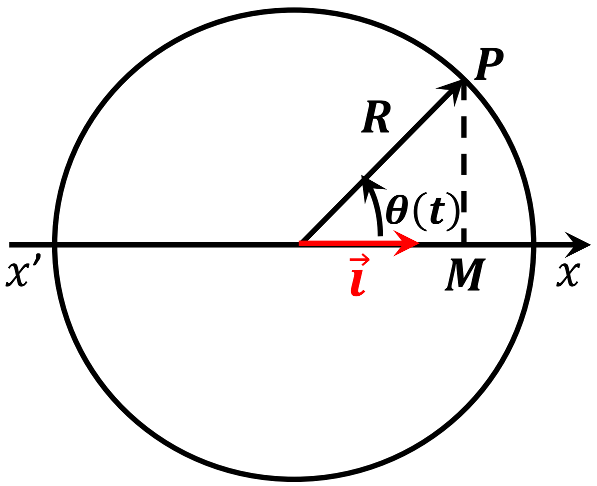
❑ *The time requiered for the projectile to return to ground level at point **B** can be obtainde by making* $y = 0$

$$\Rightarrow -\frac{1}{2}gt^2 + V_{0y}t = 0 \quad \Rightarrow -\frac{1}{2}gt + V_0 \sin \alpha = 0 \qquad \boxed{\Rightarrow \boldsymbol{t} = \frac{\boldsymbol{2V_0\, sin\, \alpha}}{\boldsymbol{g}}}$$

## III.3. Harmonic Motion (Sinusoidal Rectilinear Motion ): الحركة الجيبية

is consider as the projection, on a diameter, of an uniform circular motion of a point "*P*" of an angular velocity $\omega$ on a circle of radius *R*, With $\theta(t) = \omega\, t + \theta_0$

- Let "**M**" be the projection of "***P***" on $(x'x)$:



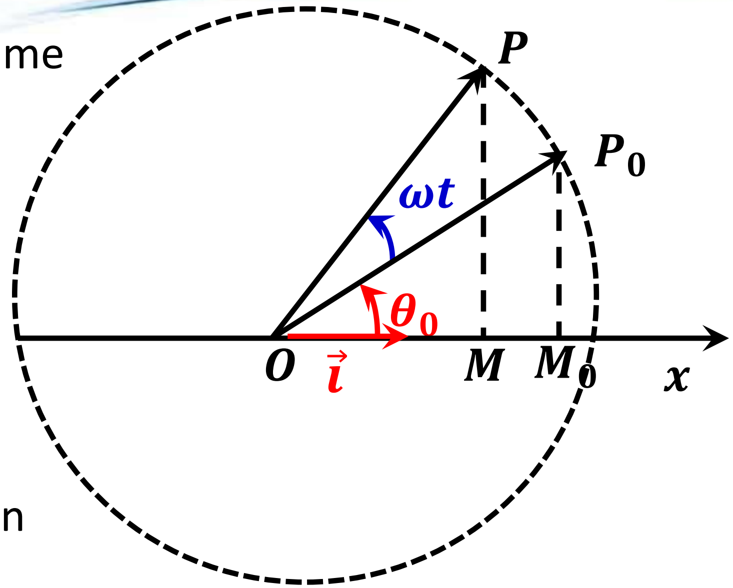

$$\overrightarrow{OM}(t) = x\vec{\imath} = R\cos\theta(t)\vec{\imath} = R\cos(\omega\, t + \theta_0)\vec{\imath}$$

$\omega\, t + \theta_0$: Motion Phase

$\theta_0$ Initial phase or phase at the origin of time

$-R < x < +R$: ***is called amplitude***

- The motion of "P" reproduces itself identically each time that the angle $\omega t$ increases by $\boldsymbol{2\pi}$




$$\boldsymbol{T = \frac{2\pi}{\omega}}$$ : Presents the period of motion

- $\boldsymbol{\omega = 2\pi f}$: Pulsation or angular frequency **(rad/s)**

- $\boldsymbol{f = \frac{1}{T} = \frac{\omega}{2\pi}}$: is the frequency of motion ≡ Oscillation numbers per unit of time related to the period **(1/s, Hz)**

$$\vec{V} = V_x\vec{\imath} = \frac{dx}{dt}\vec{\imath} = \frac{d}{dx}\left(Rcos(\omega\, t + \theta_0)\right)\vec{\imath}$$

$$\boxed{\Longrightarrow \boldsymbol{\vec{V}(t) = -R\omega sin(\omega\, t + \theta_0)\vec{\imath}}}$$

$$\vec{a} = a_x\vec{\imath} = \frac{dV_x}{dt}\vec{\imath} = \frac{d}{dx}\left(-R\omega sin(\omega\, t + \theta_0)\right)\vec{\imath} = -R\omega^2 cos(\omega\, t + \theta_0)\vec{\imath}$$

$$\boxed{\Longrightarrow \boldsymbol{\vec{a}(t) = -\omega^2 x\vec{\imath}}}$$

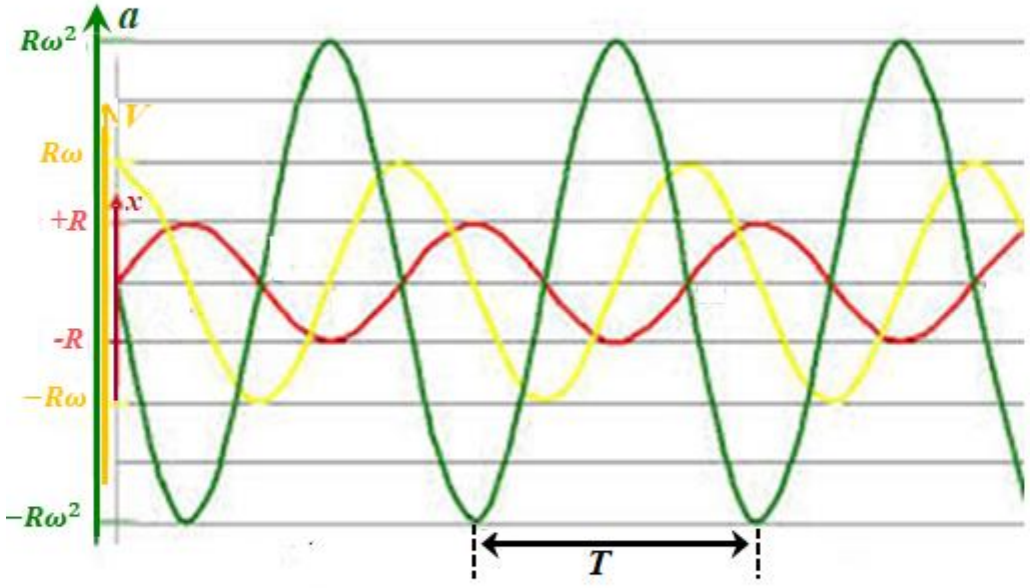
**This indicates that the acceleration in harmonic motion is opposite to the position vector**

$$\Rightarrow \boldsymbol{\vec{a} = -\omega^2 \overrightarrow{OM}}$$

$$\begin{cases} \overrightarrow{OM}(t) = R\cos(\omega t + \theta_0)\vec{i} \\ \vec{V} = -R\omega \sin(\omega t + \theta_0)\vec{i} \\ \vec{a} = -R\omega^2 \cos(\omega t + \theta_0)\vec{i} = -\omega^2 \overrightarrow{OM} \end{cases},$$

- $\boldsymbol{\theta_0 = 0}$ **and** $\boldsymbol{t = 0}$: $\quad \boldsymbol{\cos(\omega t + \theta_0) = 0 \Rightarrow \begin{cases} OM = x = 0 \\ a = 0 \end{cases}}$

$$\boldsymbol{\sin(\omega t + \theta_0) = \pm 1 \Rightarrow V = \pm R\omega}$$



# IV. Relative Motion

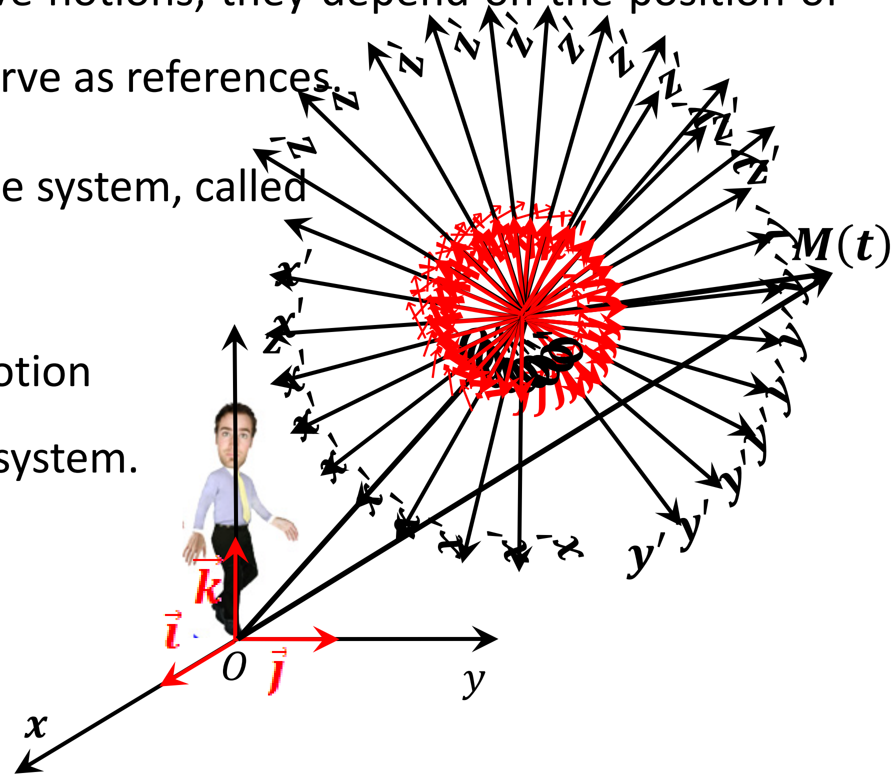
## IV.1. Change of reference system:

In relative physics, rest, like motion, are relative notions, they depend on the position of the mobile in relation to other bodies which serve as references.

- Let $R(O, xyz)$ be a supposedly fixed coordinate system, called an absolute coordinate system.
- Let $R'(O', x'y'z')$ be a coordinate system in motion with respect to R, called a relative coordinate system.

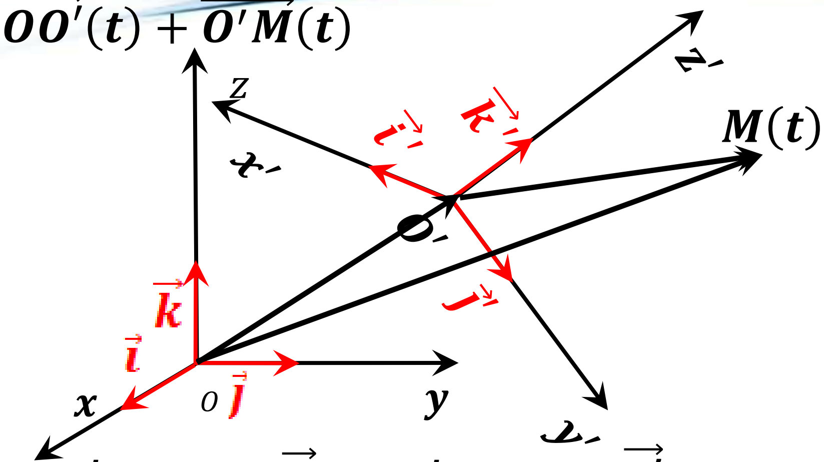
$$\overrightarrow{OM}(t)\Big/_{R} = x\vec{\imath} + y\vec{\jmath} + z\vec{k}$$

$$\overrightarrow{O'M}(t)\Big/_{R'} = x'\vec{\imath'} + y'\vec{\jmath'} + z'\vec{k'}$$




To an observer bound to the R, the motion of R'(O'x'y'z') is known via the motion of O’/O, and the ways in which the axes Ox', Oy', and Oz' rotate around O'

- **Relationship between positions:** $\overrightarrow{\boldsymbol{OM}}(\boldsymbol{t}) = \overrightarrow{\boldsymbol{OO'}}(\boldsymbol{t}) + \overrightarrow{\boldsymbol{O'M}}(\boldsymbol{t})$




$$x\vec{\imath} + y\vec{\jmath} + z\vec{k} = \overrightarrow{OO'} + x'\vec{i'} + y'\vec{j'} + z'\vec{k'}$$

**Relation entre les vitesses:**

$$\vec{V}(t) = \frac{d\overrightarrow{OM}(t)}{dt} = \frac{d\overrightarrow{OO'}(t)}{dt} + \frac{d\overrightarrow{O'M}(t)}{dt}$$

$$\Longrightarrow \frac{dx}{dt}\vec{\imath} + \frac{dy}{dt}\vec{\jmath} + \frac{dz}{dt}\vec{k} = \frac{d\overrightarrow{OO'}}{dt} + \frac{dx'}{dt}\vec{i'} + x'\frac{d\vec{i'}}{dt} + \frac{dy'}{dt}\vec{j'} + y'\frac{d\vec{j'}}{dt} + \frac{dz'}{dt}\vec{k'} + z'\frac{d\vec{k'}}{dt}$$

$$\Longrightarrow \underbrace{\frac{dx}{dt}\vec{\imath} + \frac{dy}{dt}\vec{\jmath} + \frac{dz}{dt}\vec{k}}_{\vec{V}_a(t)} = \underbrace{\frac{dx'}{dt}\vec{i'} + \frac{dy'}{dt}\vec{j'} + \frac{dz'}{dt}\vec{k'}}_{\vec{V}_r(t)} + \underbrace{\frac{d\overrightarrow{OO'}}{dt} + x'\frac{d\vec{i'}}{dt} + y'\frac{d\vec{j'}}{dt} + z'\frac{d\vec{k'}}{dt}}_{\vec{V}_e(t)}$$

$\vec{\boldsymbol{V}}_{\boldsymbol{a}}(\boldsymbol{t})$: **Absolute Velocity** $\vec{\boldsymbol{V}}_{\boldsymbol{r}}(\boldsymbol{t})$ **: Relative Velocity** $\vec{\boldsymbol{V}}_{\boldsymbol{e}}(\boldsymbol{t})$ **: Training Velocity**

$$\boxed{\Rightarrow \vec{\boldsymbol{V}}_{\boldsymbol{a}}(\boldsymbol{t}) = \vec{\boldsymbol{V}}_{\boldsymbol{r}}(\boldsymbol{t}) + \vec{\boldsymbol{V}}_{\boldsymbol{e}}(\boldsymbol{t})}$$

**Remark:**

If the coordinate system $R'$ is translational only with respect to $R$: $\vec{\boldsymbol{i'}}, \vec{\boldsymbol{j'}}, \vec{\boldsymbol{k'}} = \boldsymbol{Cst}$

$$\frac{d\vec{i'}}{dt} = \frac{d\vec{j'}}{dt} = \frac{d\vec{k'}}{dt} = 0 \Longrightarrow \vec{\boldsymbol{V}}_{\boldsymbol{e}}(\boldsymbol{t}) = \frac{\boldsymbol{d}\overrightarrow{\boldsymbol{OO'}}}{\boldsymbol{dt}}$$

**_Relationship between accelerations:_** $\vec{a}(t) = \frac{d\vec{V}}{dt}$

$$\frac{d}{dt}\left(\frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j} + \frac{dz}{dt}\vec{k}\right) = \frac{d}{dt}\left(\frac{dx'}{dt}\vec{i'} + \frac{dy'}{dt}\vec{j'} + \frac{dz'}{dt}\vec{k'}\right) + \frac{d}{dt}\left(\frac{d\overrightarrow{OO'}}{dt} + x'\frac{d\vec{i'}}{dt} + y'\frac{d\vec{j'}}{dt} + z'\frac{d\vec{k'}}{dt}\right)$$

$$\frac{d^2x}{dt^2}\vec{i} + \frac{d^2y}{dt^2}\vec{j} + \frac{d^2z}{dt^2}\vec{k} = \frac{d^2x'}{dt^2}\vec{i'} + \frac{dx'}{dt}\frac{d\vec{i'}}{dt} + \frac{d^2y'}{dt^2}\vec{j'} + \frac{dy'}{dt}\frac{d\vec{j'}}{dt} + \frac{d^2z'}{dt^2}\vec{k'} + \frac{dz'}{dt}\frac{d\vec{k'}}{dt}$$

$$+ \frac{d^2\overrightarrow{OO'}}{dt^2} + \frac{dx'}{dt}\frac{d\vec{i'}}{dt} + x'\frac{d^2\vec{i'}}{dt^2} + \frac{dy'}{dt}\frac{d\vec{j'}}{dt} + y'\frac{d^2\vec{j'}}{dt^2} + \frac{dz'}{dt}\frac{d\vec{k'}}{dt} + z'\frac{d^2\vec{k'}}{dt^2}$$

$$\vec{a}_a = \frac{d^2x}{dt^2}\vec{i} + \frac{d^2y}{dt^2}\vec{j} + \frac{d^2z}{dt^2}\vec{k}$$ **: Absolute Acceleration**

$$\vec{a}_r = \frac{d^2x'}{dt^2}\vec{i'} + \frac{d^2y'}{dt^2}\vec{j'} + \frac{d^2z'}{dt^2}\vec{k'}$$ **: Relative Acceleration**

$$\boxed{\vec{a}_a = \vec{a}_r + \vec{a}_e + \vec{a}_C}$$

$$\vec{a}_e = \frac{d^2\overrightarrow{OO'}}{dt^2} + x'\frac{d^2\vec{i'}}{dt^2} + y'\frac{d^2\vec{j'}}{dt^2} + z'\frac{d^2\vec{k'}}{dt^2}$$ **: Training Acceleration**

$$\vec{a}_C = 2\left(\frac{dx'}{dt}\frac{d\vec{i'}}{dt} + \frac{dy'}{dt}\frac{d\vec{j'}}{dt} + \frac{dz'}{dt}\frac{d\vec{k'}}{dt}\right)$$ **: Coriolis acceleration**

***Remarks:*** It is accepted that:

$$\frac{d\overrightarrow{i'}}{dt} = \vec{\omega} \wedge \overrightarrow{i'}\ , \frac{d\overrightarrow{j'}}{dt} = \vec{\omega} \wedge \overrightarrow{j'}\ , \frac{d\overrightarrow{k'}}{dt} = \vec{\omega} \wedge \overrightarrow{k'}$$

**1-** $$\vec{V}_e = \frac{d\overrightarrow{OO'}}{dt} + x'\frac{d\overrightarrow{i'}}{dt} + y'\frac{d\overrightarrow{j'}}{dt} + z'\frac{d\overrightarrow{k'}}{dt} = \frac{d\overrightarrow{OO'}}{dt} + x'\vec{\omega} \wedge \overrightarrow{i'} + y'\vec{\omega} \wedge \overrightarrow{j'} + z'\vec{\omega} \wedge \overrightarrow{k'}$$

$$= \frac{d\overrightarrow{OO'}}{dt} + \vec{\omega} \wedge \left(x'\overrightarrow{i'} + y'\overrightarrow{j'} + z'\overrightarrow{k'}\right) \quad \boxed{\Rightarrow \vec{V}_e = \frac{d\overrightarrow{OO'}}{dt} + \vec{\omega} \wedge \overrightarrow{O'M}}$$

**2-** $$\vec{a}_C = 2\left(\frac{dx'}{dt}\frac{d\overrightarrow{i'}}{dt} + \frac{dy'}{dt}\frac{d\overrightarrow{j'}}{dt} + \frac{dz'}{dt}\frac{d\overrightarrow{k'}}{dt}\right) = 2\left(\frac{dx'}{dt}\vec{\omega} \wedge \overrightarrow{i'} + \frac{dy'}{dt}\vec{\omega} \wedge \overrightarrow{j'} + \frac{dz'}{dt}\vec{\omega} \wedge \overrightarrow{k'}\right)$$

$$= 2\vec{\omega} \wedge \left(\frac{dx'}{dt}\overrightarrow{i'} + \frac{dy'}{dt}\overrightarrow{j'} + \frac{dz'}{dt}\overrightarrow{k'}\right) \quad \boxed{\Rightarrow \vec{a}_C = 2\vec{\omega} \wedge \vec{V}_r}$$

**3- For $\vec{a}_e$:** $$\frac{d^2\overrightarrow{i'}}{dt^2} = \frac{d}{dt}\left(\frac{d\overrightarrow{i'}}{dt}\right) = \frac{d}{dt}\left(\vec{\omega} \wedge \overrightarrow{i'}\right) = \frac{d\vec{\omega}}{dt} \wedge \overrightarrow{i'} + \vec{\omega} \wedge \frac{d\overrightarrow{i'}}{dt} = \frac{d\vec{\omega}}{dt} \wedge \overrightarrow{i'} + \vec{\omega} \wedge \left(\vec{\omega} \wedge \overrightarrow{i'}\right)$$

**We replace in $\vec{a}_e$ and we find:**

$$\boxed{\vec{a}_e = \frac{d^2\overrightarrow{OO'}}{dt^2} += \frac{d\vec{\omega}}{dt} \wedge \overrightarrow{O'M} + \vec{\omega} \wedge \left(\vec{\omega} \wedge \overrightarrow{O'M}\right)}$$

# Chapter II: Dynamics of a Material Point

## *II.1. Objective :*

The purpose of kinematics is to study the movements of particles as a function of time, without taking into account the causes that cause them.

***Dynamics*** is the science that studies (or determines) the causes of the motions of these particles.

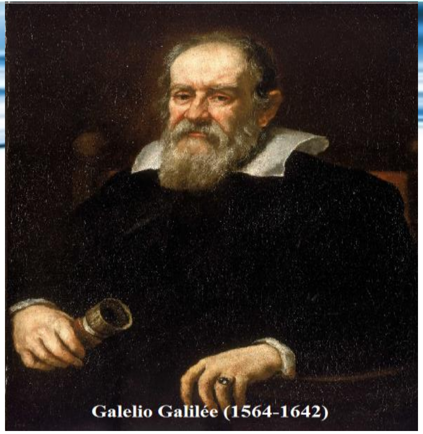
- Why do bodies near the surface of the earth fall with constant acceleration?
- Why does the earth move around the sun in an elliptical orbit ?
- Why do atoms bind together to form molecules?

(Pourquoi les atomes se lient-ils entre eux pour former des molécules ?)

- Why does a spring oscillate when it is stretched?

(Pourquoi un ressort oscille-t-il lorsqu'il est tendu ?)

## *II.2. The Law of Inertia (Galileo's law of Inertia):*




Called Newton's first law, which reads as follows:

***"Every body preservs in its state of rest, or of uniform motion in a right line, unless it is compelled to change that state by impressed forces".***

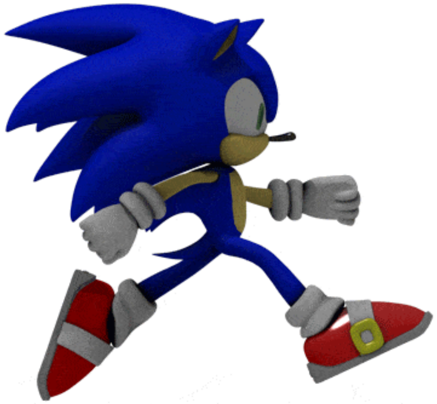
**Or**

***a free particle always moves with constant velocity, or without acceleration.***

- ***In other words**: If no force acts on an object or if the resultant force is zero:*

  - An object at rest remains at rest.

  - A moving object contained to move at a constant velocity.



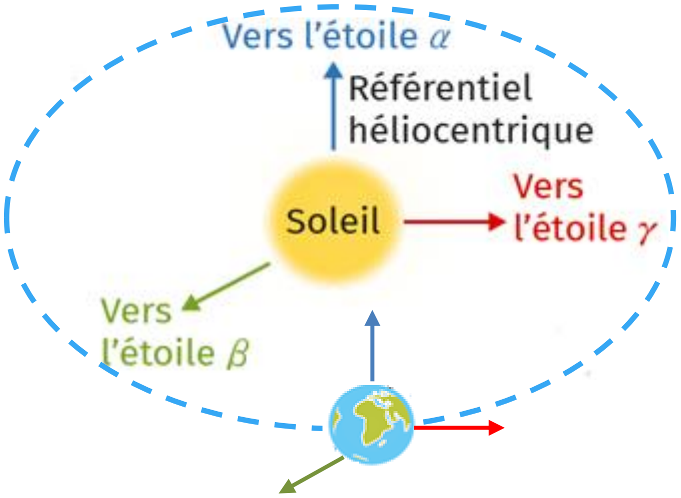
## II.3. Inertial frame of reference (Galilean frame of reference):

Is defined as a frame of reference in which Newton's first law holds.

According to this definition, there is no such thing as an inertial frame of reference; Only approximate frames of reference are available.

***Examples:***

- For most experiments on Earth, the ground-bound frame of reference is a good inertial frame.
- whereas for the motion of the planets, this ground-bound frame of reference is not an inertial frame.
- **Copernican Frame of Reference (Heliocentric):** is the frame of reference centered on the center of mass of the solar system and whose three axes point to three distant stars.
- **Geocentric frame of reference:** is the frame of reference centered on the center of mass of the earth and whose axes are parallel to those of the Copernican frame of reference.



## ***Remarks:***

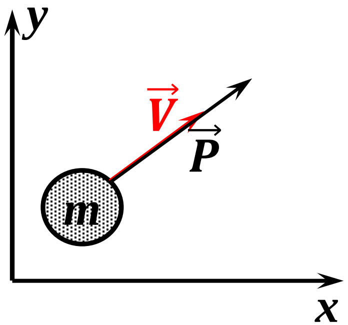
- Any coordinate system that moves at a constant velocity relative to an inertial frame of reference, can it self be considered as an inertial frame of reference.
- The velocities and accelerations of bodies, measured in Galilean reference frames, are said to be absolute, and those measured in non-Galilean reference frames are said to be relative.

## II.4.Momentum (Quantity of motion:

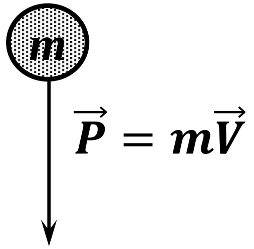
***II.4.1. Definition:*** The momentum of a particle of mas of "$m$" and moving at velocity $\vec{V}$ is defined by :

$$\vec{P} = m\vec{V} \qquad [\vec{P}] = Kg.m/s$$

- **2D Motion:**




***falling mass***




❖ The principle of inertia can then be stated as follows:

***"A free particle moves with a constant momentum in a Galilean frame of reference"***

***Remark:***

$$\frac{d\vec{P}}{dt} = \frac{d(m\vec{V})}{dt} = m\frac{d\vec{V}}{dt} = m\vec{a} = \vec{F}$$

$\Longrightarrow$ The derivative of the momentum vector of a body is equal to the sum of the external forces applied to that body:

$$\sum \vec{F}_{ext} = \frac{d\vec{P}}{dt}$$

## II.4.2. Conservation of momentum:

A system is said to be isolated if it is not subject to any external (interaction) forces.

$$\vec{F} = \vec{0} \Longrightarrow m\frac{d\vec{V}}{dt} = \vec{0} \ \Rightarrow \frac{d\vec{P}}{dt} = \vec{0} \Rightarrow \vec{P} = Cte$$

- **For a system of two particles with $\boldsymbol{m_1}$ and $\boldsymbol{m_2}$ isolated masses:**

The total momentum of the system at time $\boldsymbol{t}$ is:

$$\vec{P} = \vec{P}_1 + \vec{P}_2 = m_1\vec{V}_1 + m_2\vec{V}_2$$

At the moment $\boldsymbol{t}'$ we have: $\overrightarrow{P'} = \overrightarrow{P'}_1 + \overrightarrow{P'}_2 = m_1\overrightarrow{V'}_1 + m_2\overrightarrow{V'}_2$

Isolated System $\Longrightarrow$ Total momentum is retained:

$$\vec{P} = \overrightarrow{P'} \Longrightarrow \vec{P}_1 + \vec{P}_2 = \overrightarrow{P'}_1 + \overrightarrow{P'}_2 \Longrightarrow \overrightarrow{P'}_1 - \vec{P}_1 = \vec{P}_2 - \overrightarrow{P'}_2$$

$$\boxed{\Longrightarrow \overrightarrow{\Delta \boldsymbol{P}_1} = -\overrightarrow{\Delta \boldsymbol{P}_2}}$$

- **For an isolated system of interacting "n" particles:** $\boxed{\vec{\boldsymbol{P}}_T = \sum_{i=1}^{n} \vec{\boldsymbol{P}}_i = \boldsymbol{Cte}}$

***Example:***

A rifle of mass m of 0.8 **kg** fires a bullet of mass of 0.016 kg with a velocity of 700 m/s.

Calculate the recoil velocity of the rifle.

***Solution:***

The system consists of two bodies: Rifle + Bullet

**Principle of conservation of momentum:** $\vec{\boldsymbol{P}}_{\boldsymbol{Before}} = \vec{\boldsymbol{P}}_{\boldsymbol{After}}$

**Before Shooting:** Total momentum is zero

**After Shooting:** Total momentum: $\vec{\boldsymbol{P}}_{\boldsymbol{After}}=\vec{P}_R + \vec{P}_B$

$$\vec{P}_R + \vec{P}_B = \vec{0} \Longrightarrow m_f\vec{V}_F + m_B\vec{V}_B = \vec{0}$$

**By projection:**

$$m_R(-V_R)0 + m_B V_B = 0 \Longrightarrow V_R = \frac{m_B}{m_R}V_B$$

***N.A:***

$$V_R = \frac{0{,}016}{0{,}8}700 = 14\mathrm{m/s}$$

## ***II.5. Newtonian Definition of Force:***

- Any cause capable of modifying the momentum vector of a material point, in a Galilean frame of reference, is called “ ***FORCE*** ".

- So, force is a mathematical notion that, by definition, is equal to the derivative of momentum with respect to time.

- We defined the average force, during a time interval Δt, as:

$$\vec{F}_{ave} = \frac{\overrightarrow{\Delta P}}{\Delta t}$$

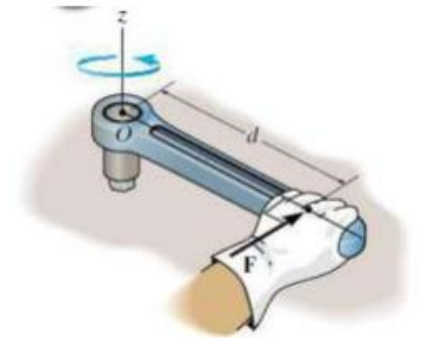
- The instantaneous force is therefore given by:

$$\vec{F}_{inst} = \vec{F} = \lim_{\Delta t \to 0} \frac{\overrightarrow{\Delta P}}{\Delta t} = \frac{d\vec{P}}{dt} = m\frac{d\vec{V}}{dt}$$

$$[\vec{F}] = Kg.ms^{-2} = Newton\ (N)$$

## II.5.1. Moment of a Force about a Point (Torque):

A moment of a force is the tendency of that force to cause a rotation of a body about an axis,



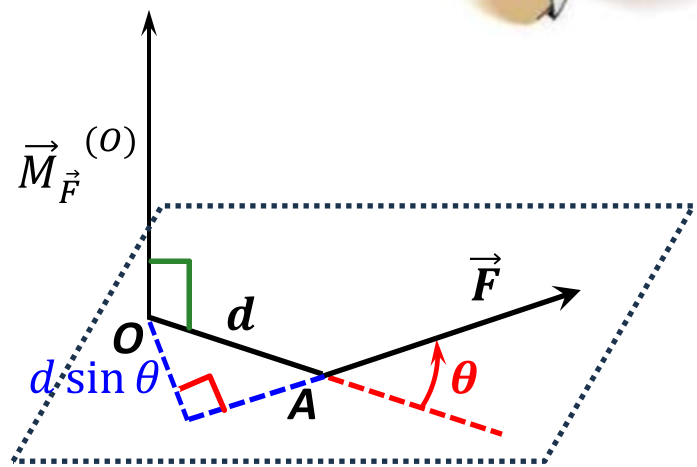
- ***Vector Expression***

The moment of the force $\vec{F}$ about the point ***O***, denoted $\vec{M}_{\vec{F}}^{(O)}$, is:

$$\boxed{\vec{\boldsymbol{M}}_{\vec{F}}^{(\boldsymbol{O})} = \overrightarrow{\boldsymbol{OA}} \wedge \vec{\boldsymbol{F}}}$$

$$\left\| \vec{M}_{\vec{F}}^{(O)} \right\| = \|\overrightarrow{OA}\| \|\vec{F}\| \sin\theta \;= F.d\sin\theta$$

$$\left[\vec{M}_{\vec{F}}^{(O)}\right] = N.m$$




- ***In other words:***

**The magnitude of the moment of a force about a point is (the magnitude of the force) × (the perpendicular distance of the line of action of the force from the point).**

**Example:**

Find the moment of $\vec{F}$ about $P$ when $\theta = 35\circ$,$F = 8N$ and $d = 14m$.

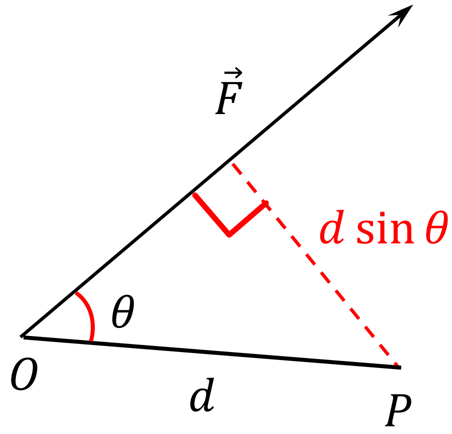
**Solution:**

$$\vec{M}_{\vec{F}}^{(P)} = \overrightarrow{PO} \wedge \vec{F}$$

$$\Longrightarrow \left\| \vec{M}_{\vec{F}}^{(P)} \right\| = \left\| \overrightarrow{PO} \right\| \left\| \vec{F} \right\| \sin\theta \quad ; \quad \left\| \overrightarrow{PO} \right\| = d$$

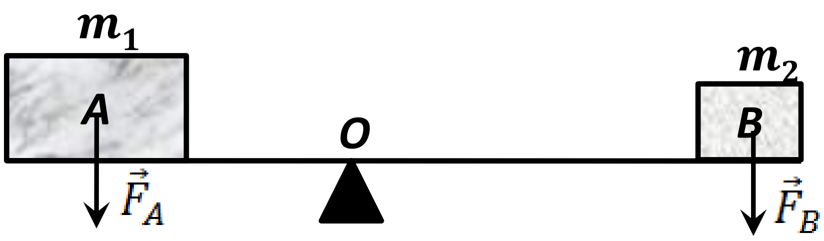
$$\Longrightarrow \left\| \vec{M}_{\vec{F}}^{(P)} \right\| = F . d \sin\theta$$

$$= 8.14. \sin 35° = 64{,}24\ Nm$$



## II.5.2. Center of Inertia or Barycenter: (Center of Gravity)

In equilibrium, the sum of the moments of the forces about "O" equal zero:




(Clockwise moments will equal anticlockwise moments),

$$\sum \vec{M}_{\vec{F}_i}^{(O)} = \vec{0} \Rightarrow \vec{M}_{\vec{F}_A}^{(O)} + \vec{M}_{\vec{F}_B}^{(O)} = \vec{0} \qquad \Rightarrow \overrightarrow{OA} \wedge \vec{F}_A + \overrightarrow{OB} \wedge \vec{F}_B = \vec{0}$$

$$\Rightarrow \overrightarrow{OA} \wedge m_1\vec{g} + \overrightarrow{OB} \wedge m_2\vec{g} = \vec{0} \qquad \Rightarrow \left(m_1\overrightarrow{OA} + m_2\overrightarrow{OB}\right) \wedge \vec{g} = \vec{0}$$

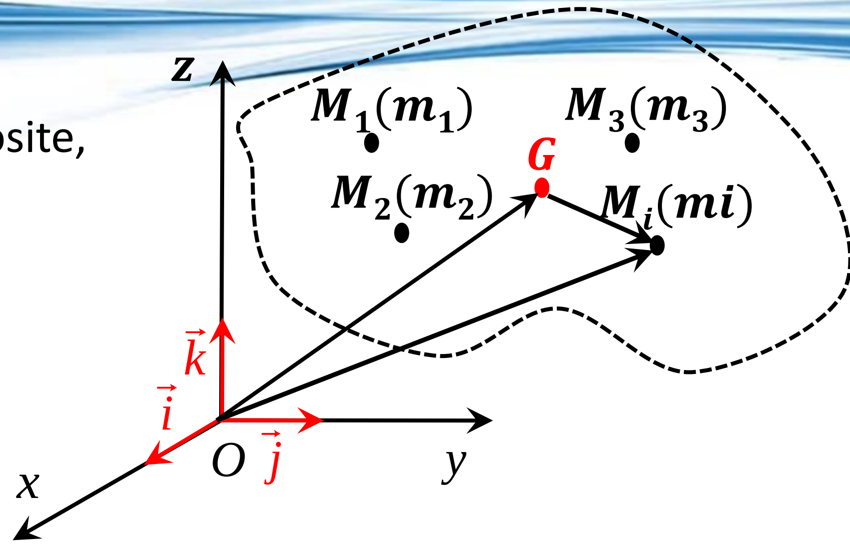
$$\boxed{\Rightarrow \boldsymbol{m_1\overrightarrow{OA} + m_2\overrightarrow{OB} = \vec{0}}}$$

***For a system of m masses (G is a center of gravity):***

$$m_1\overrightarrow{GM_1} + m_2\overrightarrow{GM_2} + \cdots m_n\overrightarrow{GM_n} = \vec{0} \qquad \Rightarrow \sum_i m_i\overrightarrow{GM_i} = \vec{0}$$

On the other hand, according to the diagram opposite,

with ***G*** *is a center of gravity,* we have:




$$\overrightarrow{OG} + \overrightarrow{GM_i} = \overrightarrow{OM_i} \quad \Longrightarrow \overrightarrow{GM_i} = \overrightarrow{OM_i} - \overrightarrow{OG}$$

$$\sum_i m_i \overrightarrow{GM_i} = \vec{0} \quad \Longrightarrow \sum_i m_i\left(\overrightarrow{OM_i} - \overrightarrow{OG}\right) = \vec{0}$$

$$\Longrightarrow \sum_i m_i \overrightarrow{OM_i} = \sum_i m_i \overrightarrow{OG} \quad \Longrightarrow \overrightarrow{OG} = \frac{\sum_i m_i \overrightarrow{OM_i}}{\sum_i m_i}$$

$\sum_i m_i = M$, With M is the total mass of the system.

$$\boxed{\Longrightarrow \overrightarrow{OG} = \frac{1}{M}\sum_i m_i \overrightarrow{OM_i}}$$

This last relation gives the center of inertia of a system consisting of masses $m_i$ located at the points $M_i$

➢ For a continuous environment, the sum becomes integral: $\overrightarrow{OG} = \frac{1}{M}\iiint \overrightarrow{OM}dM$

## ***II.5.3.Newton’s Laws of Motion***

- ***Newton's First Law:***

Newton’s first law states that every object will remain at rest or in uniform motion in a straight line unless compelled to change its state by the action of an external force.

$$\vec{F} = \vec{0}\,, \qquad \vec{V} = Cst$$

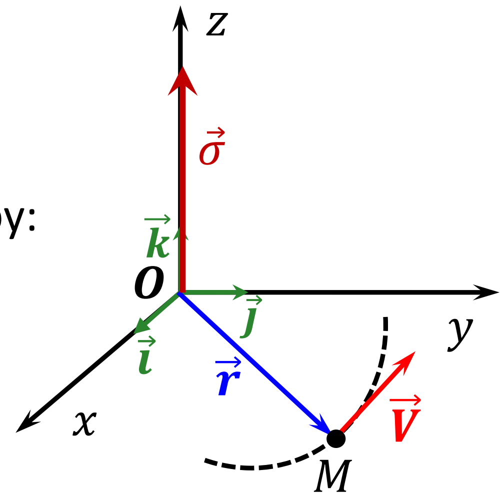
- ***Newton’s Second Law (Fundamental Principle of Dynamics):***

In a Galilean frame of reference, the sum of the external forces applied to a system is equal to the derivative of the momentum vector of the center of inertia of that system.

$$\sum \vec{F}_{ext} = \frac{d\vec{P}}{dt} = \frac{d\left(m\vec{V}\right)}{dt} = m\frac{d\vec{V}}{dt} = m\vec{a} \qquad (\boldsymbol{m = cts})$$

## ➢ ***Angular Momentum Theorem for a particle:***

Consider a particle $M$ of mass $m$, moving in plan $(O,x,y)$ with velocity vector $\vec{V}$ relative to inertial frame $R$.




The particle M has the momentum $\vec{P} = m\vec{V}$ relative to $R$.

The angular momentum $\vec{\sigma}$ (or $\vec{L}$ ) of $M$ with respect to O is given by:

$$\boxed{\vec{\sigma} = \overrightarrow{OM} \wedge \vec{P}}$$

$$\Longrightarrow \vec{\sigma} = \vec{r} \wedge m\vec{V} = m\vec{r} \wedge \vec{V} \qquad \left(\vec{\sigma} \perp (\vec{r}, \vec{V})\right)$$

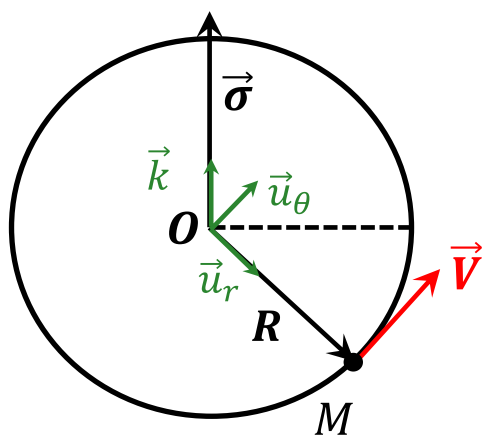
❖ ***In the case of a circular motion with constant velocity angular ω , we have:***

$$\vec{r} = R\vec{u}_r$$

$$\vec{V} = R\omega\vec{u}_\theta$$

$$\Longrightarrow \vec{\sigma} = mR^2\omega(\vec{u}_r \wedge \vec{u}_\theta)$$

$$\Longrightarrow \vec{\sigma} = mR^2\omega\vec{k}$$



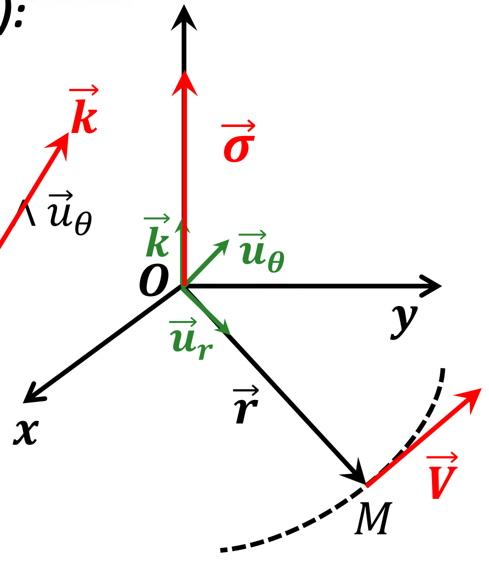
❖ ***In the case of a planar curvilinear motion (Polar coordinates):***

$$\overrightarrow{OM} = \vec{r} = r\vec{u}_r \qquad \vec{V} = V_r\vec{u}_r + V_\theta\vec{u}_\theta$$

$$\vec{\sigma} = m.\vec{r} \wedge \vec{V} = m.r\vec{u}_r \wedge (V_r\vec{u}_r + V_\theta\vec{u}_\theta) = \cancelto{0}{m.rV_r\vec{u}_r \wedge \vec{u}_r} + m.rV_\theta\underbrace{\vec{u}_r \wedge \vec{u}_\theta}_{\vec{k}}$$

$$\boxed{\Rightarrow \vec{\sigma} = m.rV_\theta\vec{k}}$$

$$V_\theta = r\frac{d\theta}{dt} \quad \Rightarrow \vec{\sigma} = mr^2\frac{d\theta}{dt}\vec{k}$$




❑ ***The derivative of $\vec{\sigma}$ with respect to time is given by:***

$$\frac{\overrightarrow{d\sigma}}{dt} = \frac{d(\vec{r} \wedge m\vec{V})}{dt} = \frac{\overrightarrow{dr}}{dt} \wedge m\vec{V} + \vec{r} \wedge m\frac{\overrightarrow{dV}}{dt} = \cancelto{0}{\vec{V} \wedge m\vec{V}} + \vec{r} \wedge \frac{\overrightarrow{dP}}{dt} = \vec{r} \wedge \vec{F}$$

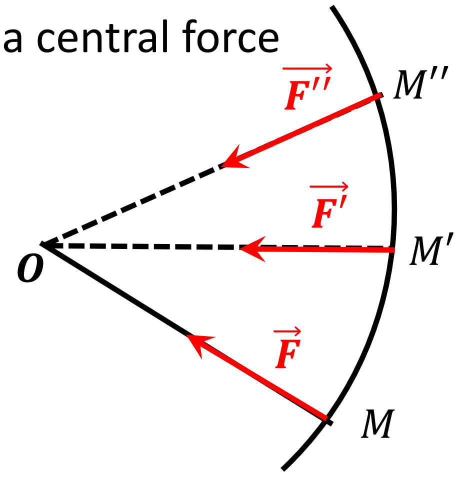
$\vec{F}$: is the resultant force

$$\boxed{\Rightarrow \frac{\overrightarrow{d\sigma}}{dt} = \overrightarrow{M}_{\vec{F}}^{(O)}} \dashrightarrow \textit{(Moment of Force } \vec{F}\textit{)}$$

***Theorem****: the derivative, with respect to time, of the angular momentum of a particle is equal to the moment of the force applied to it when both are measured with respect to the same point.*

## ❖ *In case of central Force:*

A force whose direction always passes through a fixed point is called a central force



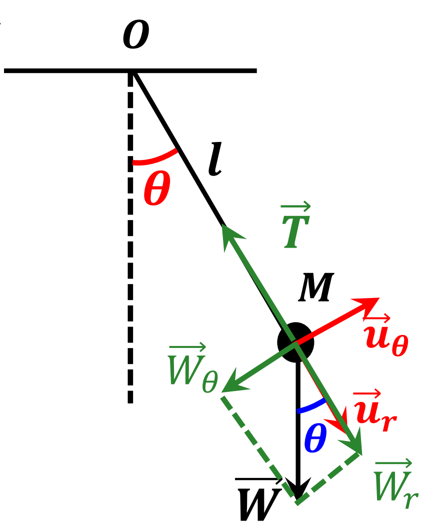

$$\vec{F} \parallel \overrightarrow{OM} \quad \Longrightarrow \frac{d\vec{\sigma}}{dt} = \overrightarrow{OM} \wedge \vec{F} = 0 \quad \Longrightarrow \vec{\sigma} = Cte$$

**Exercise**: *(Simple Pendulum)*

Find the differential equation to write the equation of motion

of a simple pendulum $\theta$(t).

***I- We apply the Newton's second law :*** $\sum \vec{F}_{ext} = m\vec{a} \quad \Rightarrow \vec{W} + \vec{T} = m\vec{a}$




**By projection:**

$$\vec{u}_r: \quad W_r - T = ma_r$$

$$\vec{u}_\theta: \quad -W_\theta = ma_\theta$$

$$\Rightarrow \begin{cases} mgcos\theta - T = -ml\left(\dfrac{d\theta}{dt}\right)^2 \quad \dots\dots\dots..(1) \\ -mg\,sin\theta = ml\dfrac{d^2\theta}{dt^2} \dots\dots\dots\dots\dots(2) \end{cases}$$

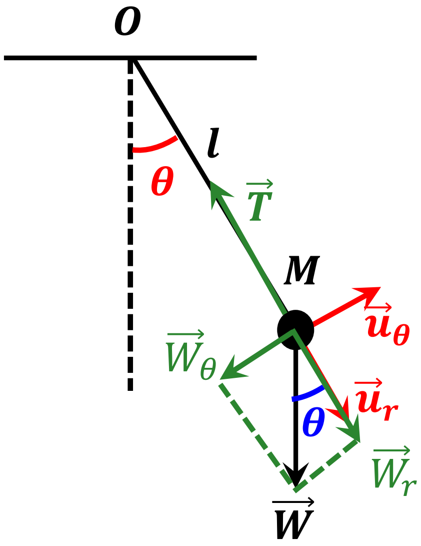
$$(2) \Leftrightarrow ml\frac{d^2\theta}{dt^2} + mgsin\theta = 0 \quad \Rightarrow l\frac{d^2\theta}{dt^2} + \frac{g}{l}sin\theta = 0$$

$$\vec{a} = \left(\frac{d^2r(t)}{dt^2} - r(t)\left(\frac{d\theta(t)}{dt}\right)^2\right)\vec{u}_r + \left(2\frac{dr(t)}{dt}\frac{d\theta(t)}{dt} + r(t)\frac{d^2\theta(t)}{dt^2}\right)\vec{u}_\theta$$

$$(sin\theta \approx \theta) \Longrightarrow \frac{d^2\theta}{dt^2} + \frac{g}{l}\theta = 0$$

## II- Let's apply the angular momentum theorem with respect to O :

$$\frac{\overrightarrow{d\sigma}}{dt} = \vec{M}_{\vec{F}}^{(O)} = \vec{M}_{\vec{W}}^{(O)} + \vec{M}_{\vec{T}}^{(O)}$$




**We have:** $\vec{\sigma} = \overrightarrow{OM} \wedge m\vec{v} = l\,\vec{u}_r \wedge ml\frac{d\theta}{dt}\vec{u}_\theta = ml^2\frac{d\theta}{dt}\vec{k}$ **(circular motion)**

$$\Longrightarrow \frac{\overrightarrow{d\sigma}}{dt} = ml^2\frac{d^2\theta}{dt^2}\vec{k} \;\ldots\ldots\ldots\ldots\ldots..(1)$$

**On the other hand, we have:**

- $\vec{M}_{\vec{T}}^{(O)} = \overrightarrow{OM} \wedge \vec{T} = l\,\vec{u}_r \wedge (-T\vec{u}_r) = \vec{0}$
- $\vec{M}_{\vec{W}}^{(O)} = \overrightarrow{OM} \wedge \vec{W} = l\,\vec{u}_r \wedge (mg\cos\theta\,\vec{u}_r - mg\sin\theta\,\vec{u}_\theta) = -lmg\sin\theta\vec{k}$

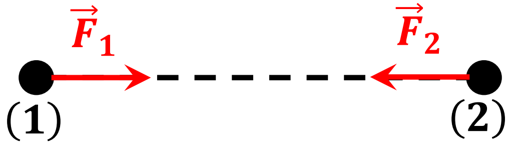
$$\Longrightarrow \vec{M}_{\vec{W}}^{(O)} + \vec{M}_{\vec{T}}^{(O)} = -lmg\sin\theta\vec{k} \;\ldots\ldots\ldots(2)$$

$$(1) = (2) \Leftrightarrow ml^2\frac{d^2\theta}{dt^2}\vec{k} = -lmg\sin\theta\vec{k} \quad \Longrightarrow \frac{d^2\theta}{dt^2} + \frac{g}{l}\sin\theta = 0$$

For small oscillations, we have: $\sin\theta \approx \theta$ $\quad \boxed{\Longrightarrow \frac{d^2\theta}{dt^2} + \frac{g}{l}\theta = 0}$

## ❑ *Newton's Third Law (3rd law of dynamics: Principle of action and reaction):*

Let two particles (1) and (2) interacting with each other, the action of (1) on (2) $(\vec{F}_1)$ is equal and opposite to that exerted by (2) on (1) $(\vec{F}_2)$.



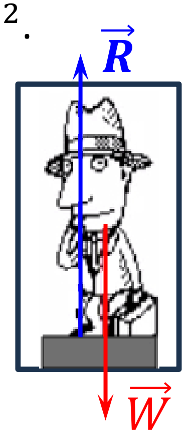

**In the other word:**

*If a particle (1) exerts a force* $(\vec{F}_1)$ *on a particle (2), then (2) exerts a force* $(\vec{F}_2)$ *on (1) in the opposite direction with the same magnitude.*

$$\vec{F}_1 = -\vec{F}_2 \quad \left(\|\vec{F}_1\| = \|\vec{F}_2\|\right)$$

***Example:***

A person of mass 85 kg is standing in a lift which is accelerating downwards at $0.45\ ms^{-2}$. Draw a diagram to show the forces acting on the person and calculate the force the person exerts on the floor of the lift.




**Solution:** using Newton's second law gives: $\sum \vec{F}_{ext} = m\vec{a}$

$$\Rightarrow \vec{R} + \vec{W} = m\vec{a}$$

***By projection:*** $W - R = ma \quad \Rightarrow R = W - Ra \quad = mg - ma$

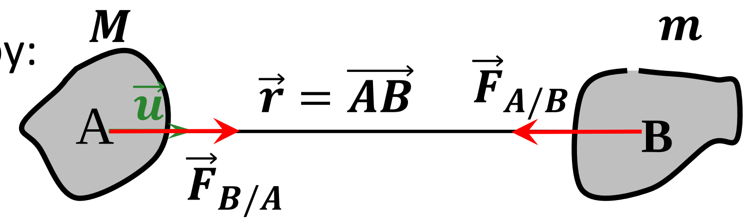
$$R = 795{,}6\ N$$

## *II.6. Some laws of forces:*

### *II.6.1. Newton's Law of Universal Gravitation (1666):*

This law explains the motions of the planets around the sun.

The force of attraction between $\boldsymbol{M}$ and $\boldsymbol{m}$ is given by:



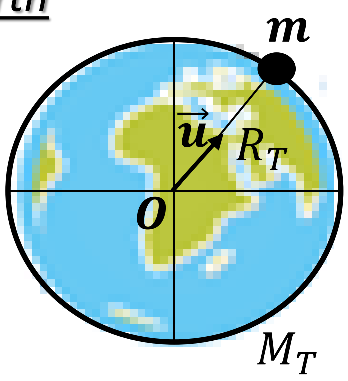

$$\vec{F}_{A/B} = -\frac{GMm}{r^2}\vec{u} \qquad \left(\vec{F}_{A/B} = -\vec{F}_{B/A}\right)$$

**With:**

$G = 6{,}67259.10^{-11}\ m^3Kg^{-1}s^{-2}$ : Universal gravitational constant

$$r = \left\|\overrightarrow{AB}\right\| \qquad \Longrightarrow \vec{F}_{A/B} = -\frac{GMm}{r^2}\frac{\overrightarrow{AB}}{\left\|\overrightarrow{AB}\right\|} = -\frac{GMm}{r^3}\vec{r}$$

**Special case:** *The weight of an object placed on the surface of the earth*




$$\vec{F} = -\frac{GM_T m}{{R_T}^2}\vec{u}$$

***We posit :*** $\vec{g} = -\frac{GM_T}{{R_T}^2}\vec{u} \quad \Longrightarrow \vec{F} = m\vec{g}$

$\overrightarrow{g}$: Gravitational Field of Earth,

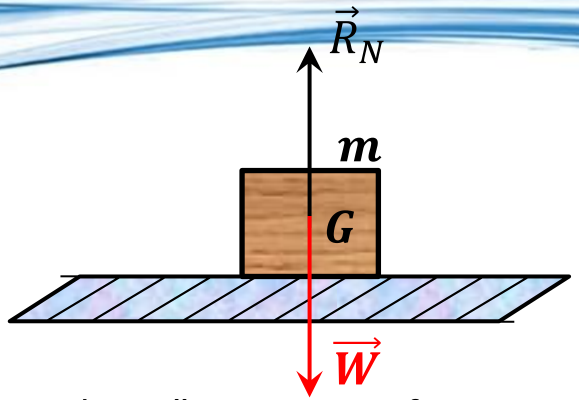
$(M_T = 5{,}9737 \times 10^{24}\ Kg\ ;R_T = 6371\ km\ ;G = 6{,}67259.10^{11}\ m^3Kg^{-1}s^{-2})$

- At the surface level of the earth: $g = g_0 = \frac{GM_T}{{R_T}^2} = 9{,}820251\ m.s^{-2}$
- At an altitude $\boldsymbol{h}$ of the earth's surface: $g = \frac{GM_T}{(R_T+h)^2} = \frac{GM_T}{(R_T+h)^2}\frac{\boldsymbol{{R_T}^2}}{\boldsymbol{{R_T}^2}}$

$$\Longrightarrow \boldsymbol{g} = \frac{\boldsymbol{GM_T}}{\boldsymbol{{R_T}^2}}\left(\frac{\boldsymbol{R_T}}{\boldsymbol{R_T+h}}\right)^2 = \boldsymbol{g_0}\left(\frac{\boldsymbol{R_T}}{\boldsymbol{R_T+h}}\right)^2$$

**(Neglecting the rotational speed of the earth upon itself).**

## II.6.2. Contact forces:




### ❑ Support Reaction:

- The force that a mass $m$, placed on a horizontal support, undergoes from the support is called the ***"support force"***
- The support reaction on $\boldsymbol{m}$ is distributed over the entire "support-object" contact surface

  $\vec{R}_N$ : Represents the resultant of all actions exerted on the contact surface.
- In equilibrium : $\vec{R}_N + \vec{W} = 0 \Longrightarrow \vec{R}_N = -\vec{W}$

### ❑ Frictional forces:

- Frictional forces are forces that appear:

  - Either when an object is moving (Soit lors de mouvement d'un objet),

  - Or that object is subjected to a force that tends to want to move it

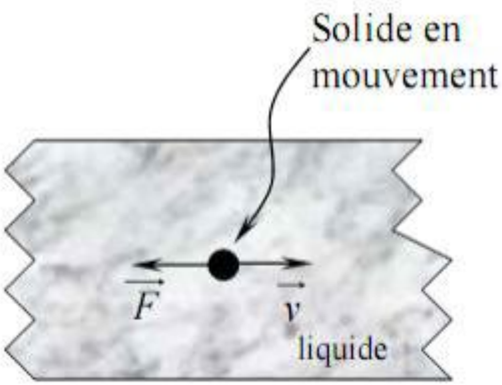
    (Cet objet est soumis à une force qui tend à vouloir de le déplacé).
- We distinguish two types of friction forces:

- Viscous friction (contact: solid – fluid).
- Solid friction (contact: solid-solid).

## ❑ ***Viscous friction:***

Viscous friction is related to the movement of an object $\boldsymbol{M}$ in a fluid medium (air, liquid or other)

At low velocities, the friction ( in magnitude) is proportional to the velocity at which the object is moving.




$$\boxed{\textit{Friction Force}} \leftarrow \vec{F} = -k\vec{V} \rightarrow \boxed{\textit{Object velocity}}$$

$$k \downarrow \boxed{\textit{Positive constant}}$$

***We give:*** $\quad k = -K\eta$

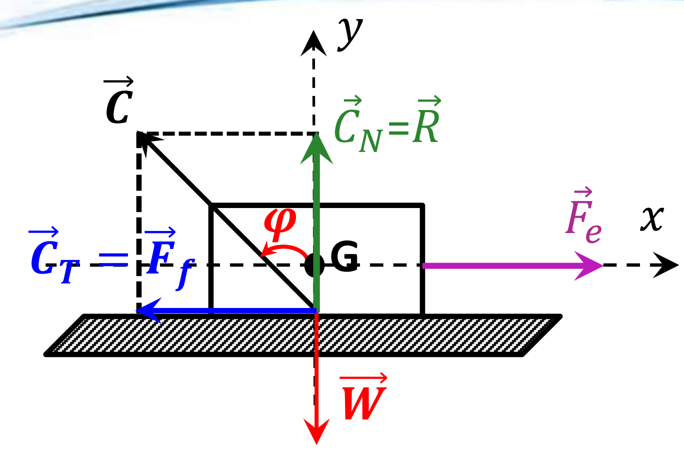
$K$: Depends on the geometric shape of the body

$\eta$: Fluid viscosity coefficient, depends on internal fluid friction,

***Remark:*** For higher speeds, experiments have shown that the frictional forces in this case are given by:

$$\vec{F} = -kV^{n}\vec{u} \qquad \text{with } n \geq 2$$

## ❑ *Solid friction:*




$\vec{F}_e$: Force of entrainment $\vec{C}$: Contact force

$\vec{C}_N = \vec{R}$: Surface reaction force

$\vec{C}_T = \vec{F}_f$: Friction force (Sliding friction)

- The body is initially at rest;
- We increase gradually the value of $\vec{F}_e$
- Each time $\vec{F}_e$ e is increased, the value of the frictional force $\vec{F}_f$ increases until it reaches a maximum value $\vec{F}_{f0} = \vec{C}_{T0}$ which corresponds to the beginning of the object's slippage. $\Longrightarrow$ This position is called: Limit equilibrium state,

Applying the Newton’s second law in this case: $\sum \vec{F}_{ext} = \vec{0} \quad \Longrightarrow \vec{W} + \vec{C} + \vec{F}_e = \vec{0}$

❖ ***By projection on the (Ox) and (Oy) axes:*** $\begin{cases} F_e - C_{T0} = 0 \\ C_{N0} - W = 0 \end{cases} \Longrightarrow \begin{cases} C_{T0} = F_e \\ C_{N0} = W \end{cases}$

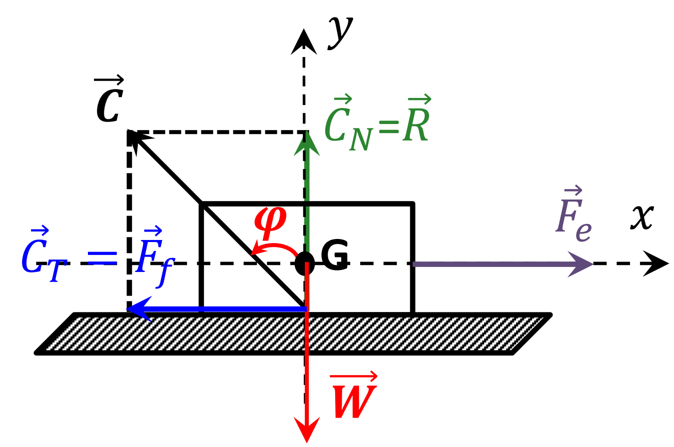
- The static coefficient of friction is defined as:

$$\boldsymbol{\mu_s = tg\varphi = \frac{C_{T0}}{C_{N0}}}$$ : characterizes the limit equilibrium state

➢ When $\vec{F}_e > \vec{F}_{f0}$, the object begins to move from its steady state with uniformly accelerated motion

➢ Applying the Newton's second law in this case: $\sum \vec{F}_{ext} = m\vec{a} \Rightarrow \vec{W} + \vec{C} + \vec{F}_e = m\vec{a}$

***By projection on the (Ox) and (Oy) axes:***




$$\begin{cases} F_e - C_T = ma \\ C_N - W = 0 \end{cases} \Rightarrow \begin{cases} C_T = F_e - ma \\ C_N = W \end{cases}$$

➢ The dynamic coefficient of friction is then defined:

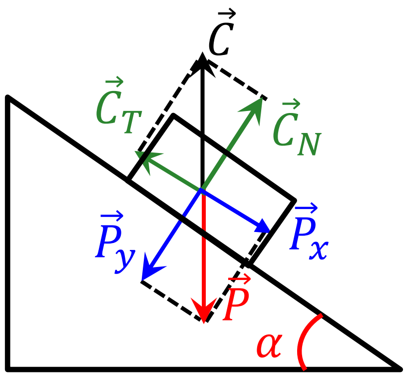
$$\mu_d = tg\varphi = \frac{C_T}{C_N} = \frac{F_e - ma}{mg}$$

***<u>Remarks:</u>***

- ❑ $\mu_s$ and $\mu_d$ depend on the nature of the surfaces in contact,
- ❑ $\mu_d$ is less than $\mu_S$
- ❑ $\mu_d$ is substantially independent of speed
- ❑ $\mu_d$ is substantially independent of the surface area of the surfaces in contact and depends only on their nature

## Application: Inclined Plane

- **_At the limit equilibrium state:_** $\sum \vec{F}_{ext} = \vec{0} \implies \vec{P} + \vec{C}_0 = \vec{0}$




By projection:

$$\begin{cases} P_x - C_{T0} = 0 \\ C_{N0} - P_y = 0 \end{cases} \implies \begin{cases} Psin\alpha_0 = C_T \dots\dots\dots\dots (1) \\ Pcos\alpha_0 = C_N \dots\dots\dots (2) \end{cases}$$

$$\boxed{(1)/(2) \implies tg\alpha_0 = \frac{C_T}{C_N} = \mu_S}$$

- **In the state of motion:** $\alpha_0 \to \alpha \quad (\alpha = \alpha_0 + d\alpha)$

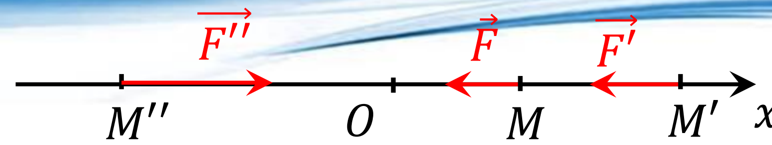
$$\sum \vec{F}_{ext} = m\vec{a} \implies \vec{P} + \vec{C} = m\vec{a}$$

By projection:

$$\begin{cases} P_x - C_T = ma \\ C_N - P_y = 0 \end{cases} \implies \begin{cases} Psin\alpha - ma = C_T \dots (1) \\ Pcos\alpha = C_N \dots\dots\dots (2) \end{cases}$$

$$\boxed{\mu_d = tg\alpha = \frac{C_T}{C_N} = \frac{Psin\alpha - ma}{Pcos\alpha} == \frac{gsin\alpha - a}{gcos\alpha}}$$

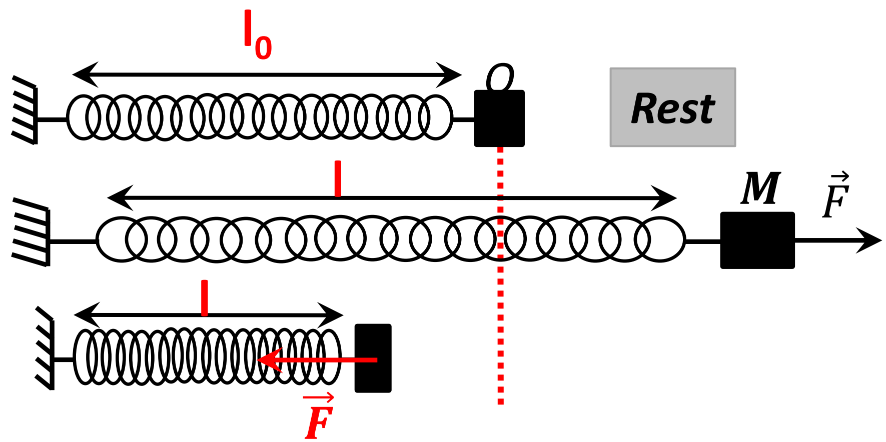
## 3. Elastic Strength:




$\vec{F} = -k\overrightarrow{OM} \Longrightarrow$ proportional and opposite to the position vector $\overrightarrow{OM}$

$k$ : Stiffness Constant

By projection on the axis (Ox): $\vec{F} = -kx\vec{\imath}$

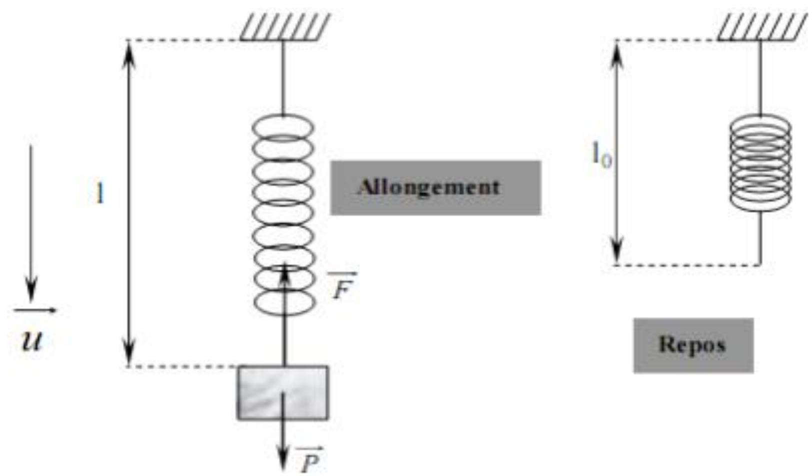



***Example:***

$$\vec{F} = -k\overrightarrow{OM} = -k(l - l_0)\vec{\imath}$$

**Or**

$$\vec{F} = -k(l - l_0)\vec{u}$$

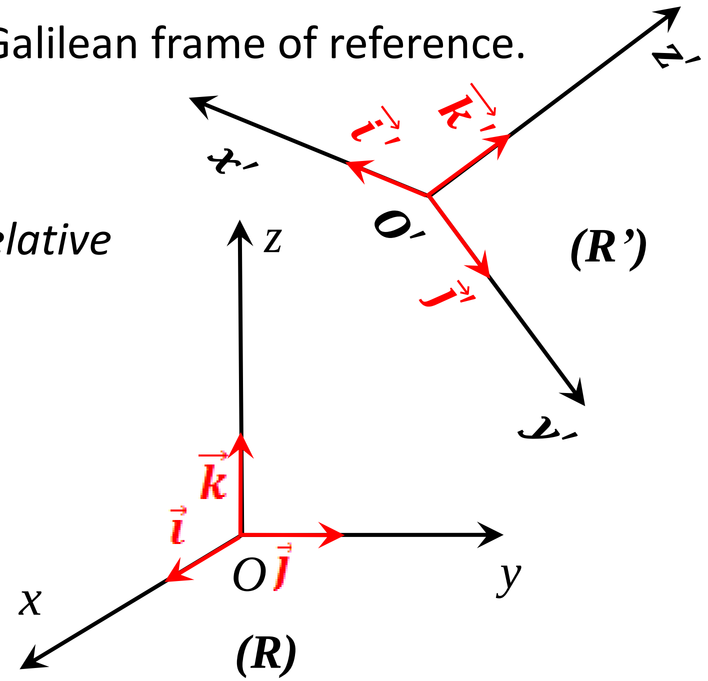


## *II.6. Fundamental Principle of Dynamics in a Non-Galilean Frame of Reference*

- Let (R)a Galilean frame of reference and (R') a non-Galilean frame of reference.

- *R' is in moving relative to R.*

$\Longrightarrow$ *R is the absolute frame of reference and R' is the relative frame of reference*




$$\vec{a}_a = \vec{a}_r + \vec{a}_e + \vec{a}_C$$

$$\Longrightarrow \sum \vec{F}_{ext} = m\vec{a}_a \quad = m\vec{a}_r + m\vec{a}_e + \vec{a}_C$$

<u>In the R' coordinate system, the PFD is:</u>

$$m\vec{a}_r = m\vec{a}_a - m\vec{a}_e - \vec{a}_C \ = \sum \vec{F}_{ext} + \vec{F}_e + \vec{F}_C$$

$\vec{F}_e = -m\vec{a}_e$est la force d'inertie d'entraînement,

$\vec{F}_C = -m\vec{a}_C$ is the Coriolis force of inertia,

$\vec{F}_e$ et $\vec{F}_C$ are non-real forces, they depend on the motion of R’/R.

# Chapter III: Work and Energy

## III.1. Introduction :

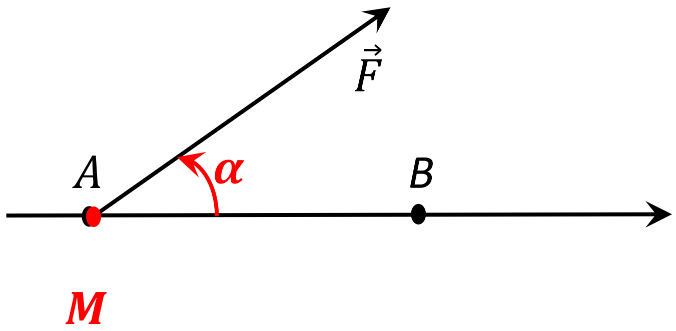
If we know the positions and velocity of the particles of a system and all the forces acting on these particles, we can predict, using Newton's laws, the evolution of this system over time. But in practice, we can't always know all the forces that come into play, and even if we do, the equations to solve are too many or too complicated. For this reason, we appeal to new notions such as “work and energy».

## III.2. Work of a Force:

### III.2.1. Constant Force on a Straight Displacement:

- A force is said to be constant when his magnitude and direction do not change over time.
- A force is said to work when its point of application moves.

- If an object $\boldsymbol{M}$ moves through a rectilinear displacement $\overrightarrow{\boldsymbol{AB}}$ while a constant force $\vec{F}$ is acting on it:

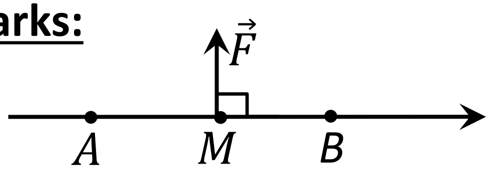



The force does an amount of work equal to:

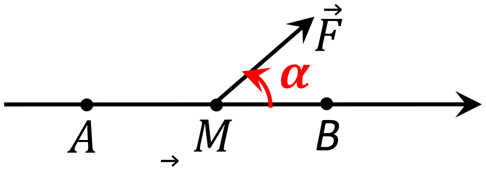
$$\boldsymbol{W_{\vec{F}} = \vec{F}.\overrightarrow{AB} = F.ABcos\alpha} \quad ([W] = Joule)$$

**Remarks:**

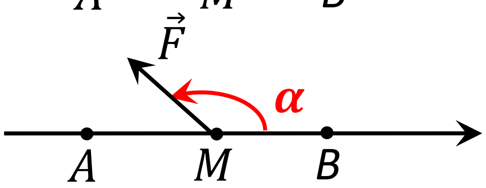
1.




$$\boldsymbol{\alpha = \frac{\pi}{2} \Longrightarrow W_{\vec{F}} = 0}$$ : $\vec{F}$ doesn't work




$$\boldsymbol{0 < \alpha < \frac{\pi}{2} \Longrightarrow W_{\vec{F}} > 0}$$ : Motor work



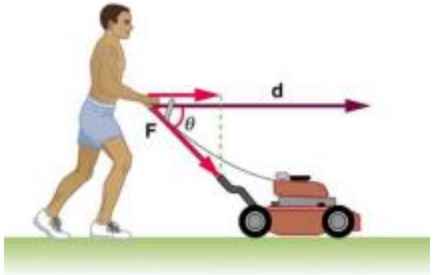

$$\boldsymbol{\frac{\pi}{2} < \alpha < \pi \Longrightarrow W_{\vec{F}} < 0}$$ : Resistance work

2. If several different (constant) forces act on a mass while it moves though a displacement $\overrightarrow{\boldsymbol{AB}}$, then we can talk about the **net work** done by the forces:

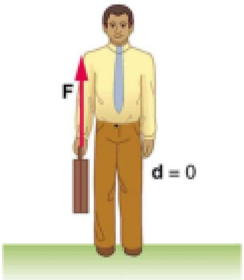
$$W_{net} = \vec{F}_1\overrightarrow{AB} + \vec{F}_2\overrightarrow{AB} + \vec{F}_3\overrightarrow{AB}......+\vec{F}_n\overrightarrow{AB} = \sum_{i=1}^{n} \vec{F}_i.\overrightarrow{AB}$$

## ***Examples of works:***

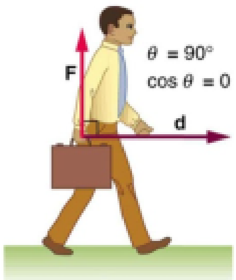
1. The work done bay the force $\vec{F}$ on this lawnmower is $(\boldsymbol{F \times d \times \cos\theta})$



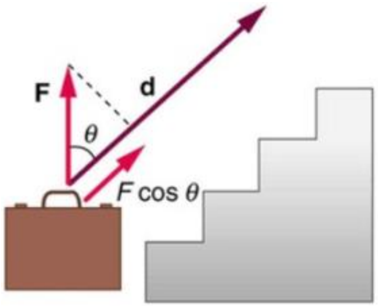

2. A person holding a briefcase does no work on it because there is no motion (d=0)




3. The person moving the briefcase horizontaly at a constant speed deos no work on it.




4. Work is done on the briefcase by carrying it upstairs at a constant speed becasue there is necessarily a component of force F in the direction of the motion.



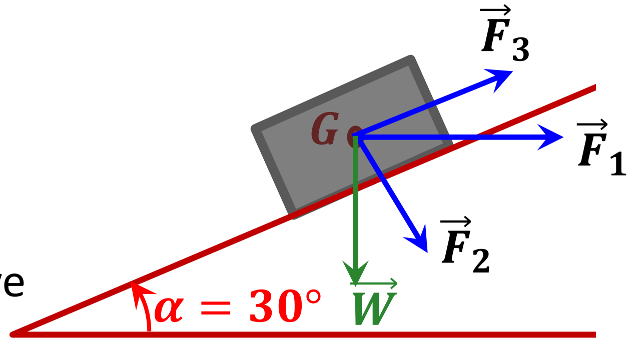

**The work done by a constant force can be calculated as the area under the force-displacement graph**

**Exercise:**

A block of stone moves upwards on a plane inclined at 30° under the action of several forces including: $\boldsymbol{F_1 = 45\ N}$ horizontal.

$\boldsymbol{F_2 = 25\ N}$ Normal to the inclined plane.

$\boldsymbol{F_3 = 35\ N}$ parallel to the inclined plane.




It will be considered that all the forces acting on the block have their point of application at the center of mass $\boldsymbol{G}$ of the block.

Calculate the work of forces $\boldsymbol{F_1}$, $\boldsymbol{F_2}$ and $\boldsymbol{F_3}$ when the block rises 1.5 $m$ on the inclined plane.

**Solution:**

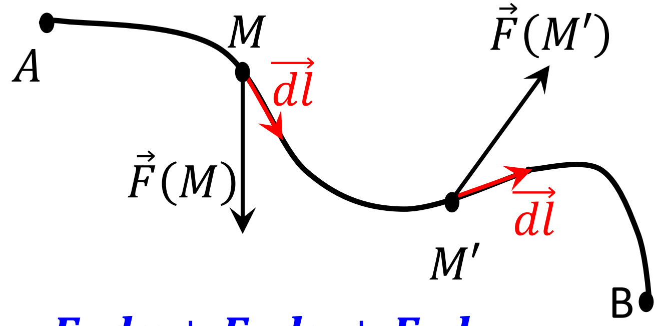
1- $W_1 = \vec{F}_1 . \overrightarrow{AB} = F_1 . AB . cos\alpha = 45.1{,}5 . \cos 30 = 58{,}46J$

2- $W_2 = \vec{F}_2 . \overrightarrow{AB} \quad \left(\vec{F}_2 \perp \overrightarrow{AB}\right) \Rightarrow W_2 = 0$

3- $W_3 = \vec{F}_3 . \overrightarrow{AB} \left(\vec{F}_3 \parallel \overrightarrow{AB}\right) \Rightarrow W_3 = F_3 . AB = 35.1{,}5 = 52{,}5J$

## *III.2.2. Elementary work:*

- When the force $\vec{F}$ which acts on ***M*** is not constant during displacement:



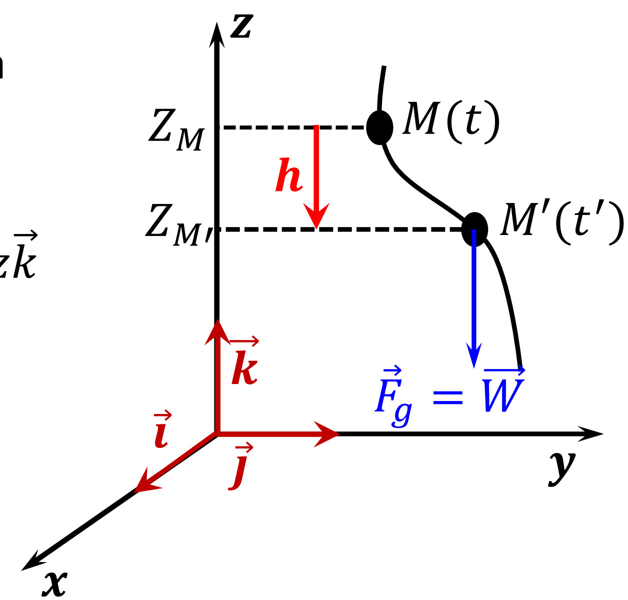

By definition, elementary work is given by:

$$dW_{\vec{F}} = \vec{F}.\overrightarrow{dl} \quad \Longrightarrow W_{\vec{F}} = \int_A^B \vec{F}.\overrightarrow{dl}$$

***In Cartesian coordinates:***
$$\begin{cases} \vec{F} = F_x\vec{\imath} + F_y\vec{\jmath} + F_z\vec{k} \\ \overrightarrow{dl} = dx\vec{\imath} + dy\vec{\jmath} + dz\vec{\jmath} \end{cases} \Longrightarrow \boldsymbol{dW = F_x dx + F_y dy + F_z dz}$$

### *III.2.3. Work Done By Gravitational Force:*

Gravitational force $\vec{F}_g$ is the force that keeps anything with a mass $m$ attracted to the earth.

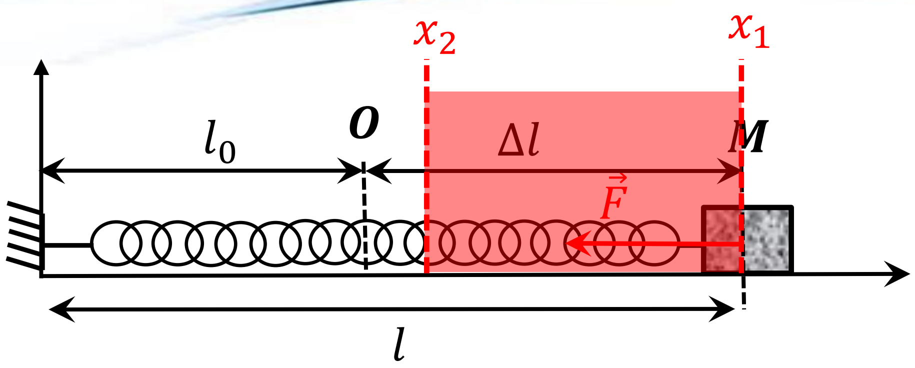



$$W_{\vec{F}_g} = \int_M^{M'} \vec{F}_g.\overrightarrow{dl}\,, \quad \vec{F}_g = -mg\vec{k}\,, \quad \overrightarrow{dl} = dx\vec{\imath} + dy\vec{\jmath} + dz\vec{k}$$

$$\Longrightarrow W_{\vec{F}_g} = \int_M^{M'} -mg.dz \quad = -mg(Z_{M'} - Z_M)$$

Either : $h = Z_M - Z_{M'}$ $\boxed{\Longrightarrow \boldsymbol{W_{\vec{F}_g} = mgh}}$

### *III.2.4. Work done by an elastic force:*




$$W = \int \vec{F}.\overrightarrow{dl}$$

**We have :**

- $\vec{F} = -k\overrightarrow{OM} = -k(l - l_0)\vec{\imath} = -kx\vec{\imath}$
- $\overrightarrow{dl} = dx\vec{\imath}$

$$\Longrightarrow W = \int -kx\vec{\imath}.dx\vec{\imath} = -k\int xdx$$

$$\boldsymbol{\Longrightarrow W = -\frac{1}{2}kx^2 + Cts}$$

When $\vec{F}$ moves from the $x_1$ position to $x_2$ position , We have :

$$\boldsymbol{W = -k\int_{x_1}^{x_2} xdx = -\frac{1}{2}k\left({x_2}^2 - {x_1}^2\right)}$$

The work of this force does not depend on the path followed but only on the initial and final position of the spring

## III.2.5. Power of Force:

Power is the rate at which work is done or energy is transferred in a unit of time.

- Average Power: $P_{ave} = \frac{\Delta W_{\vec{F}}}{\Delta t}$
- Instantaneous Power: $P_{inst} = P(t) = \frac{dW_{\vec{F}}}{dt}$

$([\boldsymbol{P}] = \boldsymbol{Watt})$

The power of a force $\vec{F}$ in a time interval $dt$ manages to move a mobile by a distance $dl$ can be written by:

$$P(t) = \frac{dW_{\vec{F}}}{dt} = \frac{\vec{F}.\overrightarrow{dl}}{dt} = \vec{F}.\frac{\overrightarrow{dl}}{dt} = \vec{F}.\vec{V}$$

## III.3. Energy

Energy, in physics , is the capacity for doing work. Energy can neither be created nor destroyed, and it can only be transformed from one form to another.

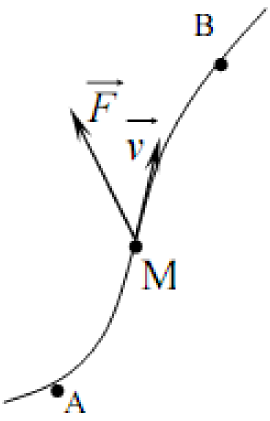
- **Types of Energy**
  - Mechanical energy
  - Chemical energy
  - Electric energy
  - Magnetic energy
  - Radiant energy
  - Nuclear energy
  - Ionization energy
  - Elastic energy
  - Gravitational energy
  - Thermal energy
  - Heat Energy

- All forms of energy are either kinetic or potential:
  - ✓ The energy in motion is known as Kinetic Energy.
  - ✓ Whereas Potential Energy is the energy stored in an object and is measured by the amount of work done.

## *III. 3.1. Kinetic energy*

We define the kinetic energy of a material point M, of mass $m$ and animated with a velocity $V$ , by the quantity $Ec$ , such that :

$$E_C = \frac{1}{2} m V^2$$




- Let a material point $\boldsymbol{M}$, of mass $m$, moves between points A and B under the action of an external force $\vec{F}$.
- According to the fundamental principle of dynamics, we have:

$$\sum \vec{F}_{ext} = m\vec{a} \Longrightarrow \vec{F} = m\frac{\overrightarrow{dV}}{dt}$$

The elementary work of $\vec{F}$ is given by:

$$dW_{\vec{F}} = \underbrace{\vec{F}}_{m\frac{\overrightarrow{dV}}{dt}} . \underbrace{\overrightarrow{dl}}_{\vec{V}dt} = m\frac{\overrightarrow{dV}}{dt}.\vec{V}dt \quad \left(\text{car } \vec{V} = \frac{\overrightarrow{dl}}{dt} \Longrightarrow \overrightarrow{dl} = \vec{V}dt\right)$$

$$\Longrightarrow dW_{\vec{F}} = m\frac{\overrightarrow{dV}}{dt}.\vec{V}dt = mVdV = d\left(\frac{1}{2}mV^2\right) = dE_C$$

So the work done between A and B is given by:

$$W_{\vec{F}} = \int_A^B \vec{F}.\overrightarrow{dl} = \int_A^B dE_C = E_C(B) - E_C(A)$$

***<u>Kinetic energy theorem:</u>***

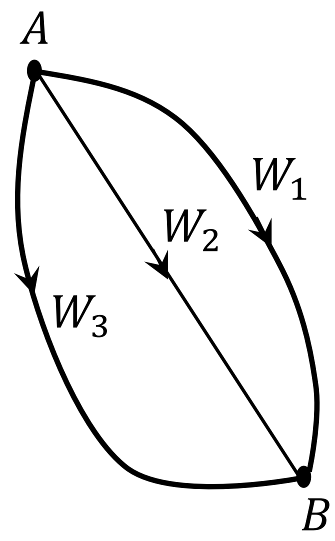
In a Galilean frame of reference, the change in kinetic energy of a material point subjected to a set of external forces between a position A and another position B is equal to the sum of the works of these forces between these two points:

$$\boldsymbol{\Delta E_C = E_C(B) - E_C(A) = \sum W_{A\to B}\left(\overrightarrow{F}_{ext}\right)}$$

## *III.3.2. Conservative and non-conservative forces:*

- Forces are said to be conservative when:

1- Their work does not depend on the path followed but only on the point of departure and the point of arrival.




***For example:***

according to the figure on the right:

$$W_1(A \to B) = W_2(A \to B) = W_3(A \to B)$$

2- The total work on a closed path (i.e. a round trip) is zero.

$$W(A \to A) = W_1(A \to B) + W_3(B \to A) = 0$$

***Examples of conservative forces:***

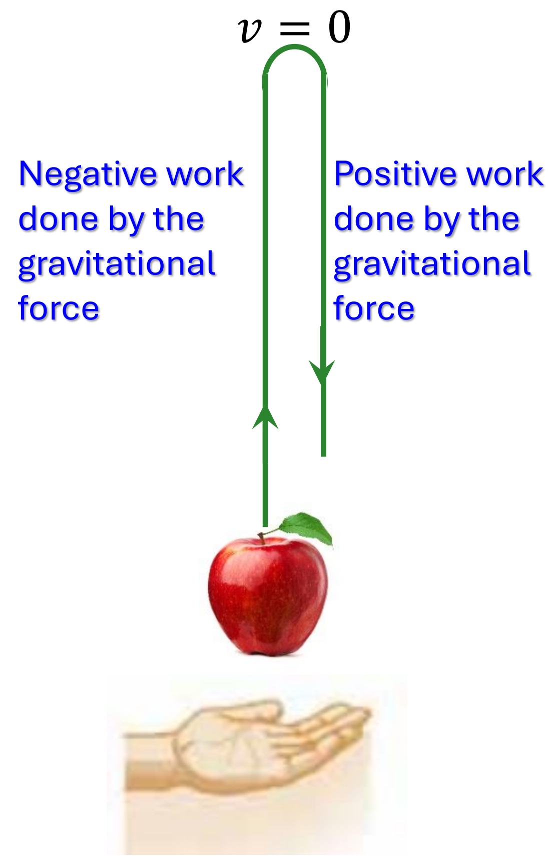
Gravitational forces, elastic forces, gravitational forces......

- Forces are said to be non-conservative when their work depends on the path taken.

***Example of non-conservative forces***: Frictional forces.

## ***III. 3.1. Potential Energy:***

The potential energy of a body or physical system is the energy that is present in it and has the potential to transform into kinetic energy.




- Consider an object near the earth's surface as a system with initially upward velocity.
- Once the object is released, the gravitational force, acting as an external force, does a negative amount of work on the object, and the kinetic energy decreases until the object reaches its highest point, at which its kinetic energy is zero.
- The gravitational force then does a positive job until the object returns to its original starting point with a downward velocity.
- <u>If we ignore the effects of air resistance</u>, then the descending object will have the same kinetic energy as when it was launched.
- All kinetic energy has been completely recovered

$\Longrightarrow$ We define the potential energy $E_P$ as the quantity of energy that must be added to the kinetic energy $E_C$ so that their sum is constant:

$$E_C + E_P = Cte$$

- For a displacement producing a change in kinetic energy $\Delta E_C$, the corresponding change in potential energy $\Delta E_P$ can be given by:

$$\Delta E_P = E_P(B) - E_P(A) = -\Delta E_C \ = -W_{\vec{F}_C}(A \rightarrow B)$$

With $\vec{F}_C$ is a conservative force

$$\Longrightarrow \Delta E_P = -\int_A^B \vec{F}_C . \overrightarrow{dl}$$

Using the notion of elementary work $dW$ of a conservative force $\vec{F}_C$:

$$dW = \vec{F}_C.\overrightarrow{dl} \implies dE_P = -\vec{F}_C.\overrightarrow{dl}$$

We have:

$$1\text{-} \begin{cases} \vec{F}_C = F_x\vec{\imath} + F_y\vec{\jmath} + F_z\vec{k} \\ \overrightarrow{dl} = dx\vec{\imath} + dy\vec{\jmath} + dz\vec{k} \end{cases} \implies \vec{F}_C.\overrightarrow{dl} = F_x dx + F_y dy + F_z dz$$

$$2 - \ dE_P = \frac{\partial E_P}{\partial x}dx + \frac{\partial E_P}{\partial y}dy + \frac{\partial E_P}{\partial z}dz \quad \text{(Total differential of a function)}$$

$$dE_P = -\vec{F}_C.\overrightarrow{dl} \implies \frac{\partial E_P}{\partial x}dx + \frac{\partial E_P}{\partial y}dy + \frac{\partial E_P}{\partial z}dz = -F_x dx - F_y dy - F_z dz$$

$$\implies \begin{cases} F_x = -\dfrac{\partial E_P}{\partial x} \\ F_y = -\dfrac{\partial E_P}{\partial y} \\ F_z = -\dfrac{\partial E_P}{\partial z} \end{cases} \implies F_x\vec{\imath} + F_y\vec{\jmath} + F_z\vec{k} = -\frac{\partial E_P}{\partial x}\vec{\imath} - \frac{\partial E_P}{\partial y}\vec{\jmath} - \frac{\partial E_P}{\partial z}\vec{k} \ = -\vec{\nabla}E_P$$

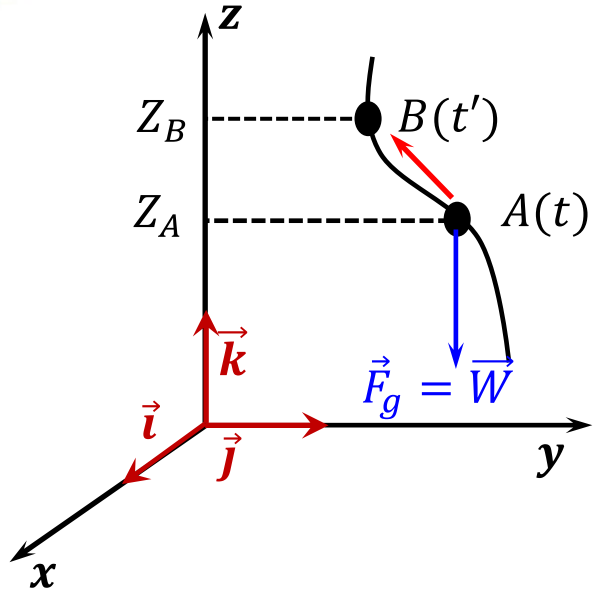
$$\boxed{\implies \vec{F}_C = -\overrightarrow{grad}E_P}$$

## III. 3.1.1. Potential Energy of the Force of Gravity:

$$\Delta E_P = E_P(B) - E_P(A) = -\int_A^B \vec{F}_g . \overrightarrow{dl}$$

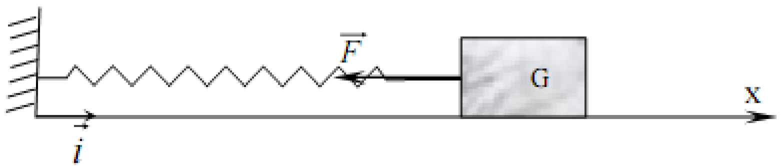
$$\vec{F}_g = \overrightarrow{W} = -mg\overrightarrow{k} ; \overrightarrow{dl} = dx\vec{\imath} + dy\vec{\jmath} + dz\vec{k}$$

$$\Delta E_P = \int_A^B mgdz = mg(Z_B - Z_A) = mgh$$




## III. 3.1.2. Potential Energy of an Elastic Force:

$$\vec{F} = -kx\vec{\imath}$$

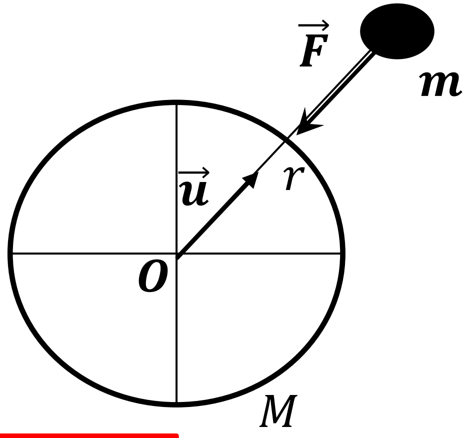



$$\vec{F} = -\overrightarrow{grad}E_P = -\frac{\partial E_P}{\partial x}\vec{\imath} \quad \Longrightarrow \frac{dE_P}{dx} = kx$$

$$E_P = \int kxdx = \frac{1}{2}kx^2 + Cte$$

## ***III. 3.1.3. Potential Energy of a Gravitational Force:***

$$\vec{F}(r) = -\frac{GMm}{r^2}\vec{u} = -\frac{GMm}{r^3}\vec{r} \qquad \left(\vec{u} = \frac{\vec{r}}{r}\right)$$




$$\vec{F}(r) = -\overrightarrow{grad}E_P(r) = -\frac{dE_P(r)}{dr}\vec{u} \Longrightarrow \frac{dE_P(r)}{dr} = \frac{GMm}{r^2}$$

$$\Longrightarrow dE_P(r) = \frac{GMm}{r^2}dr$$

$$\boxed{\Longrightarrow E_P(r) = \int \frac{GMm}{r^2}dr = -\frac{GMm}{r} + Cte}$$

### III.3.2. Mechanical energy

Let be a system moving between points A and B under the effect of conservative and non-conservative forces. According to the kinetic energy theorem, we have:

$$E_C(B) - E_C(A) = \sum W_{A\to B}(\vec{F}_C) + \sum W_{A\to B}(\vec{F}_{NC})$$

With :$\vec{F}_C$: Conservative force and $\vec{F}_{NC}$: non-conservative force

We have: $\sum W_{A\to B}(\vec{F}_C) = -\left(E_P(B) - E_P(A)\right)$

$$\Rightarrow E_C\ (B) - E_C(A) = -\left(E_P(B) - E_P(A)\right) + \sum W_{A\to B}(\vec{F}_{NC})$$

$$\Rightarrow \left(E_C(B) + E_P(B)\right) - \left(E_C(A) + E_P(A)\right) = \sum W_{A\to B}(\vec{F}_{NC})$$

- $\boldsymbol{E_C + E_P = E}$ : **Called « Machanical energy (Totale)**

$$\boxed{\Rightarrow \boldsymbol{E(B) - E(A) = \sum W_{A\to B}(\vec{F}_{NC})}}$$

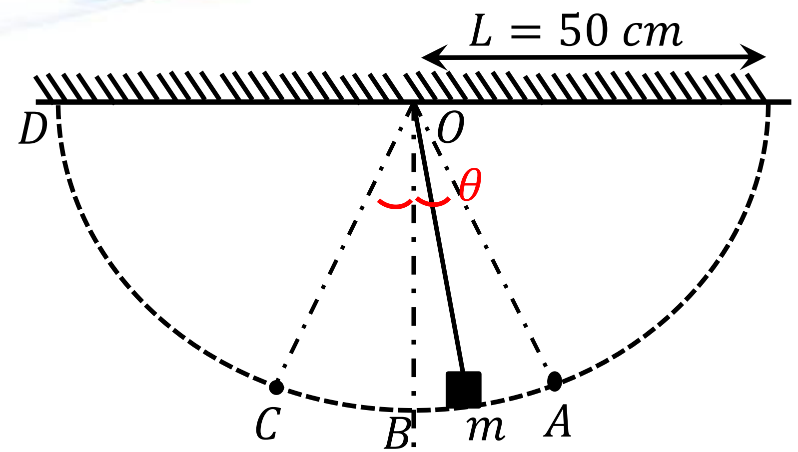
***Mechanical Energy Theorem:***

*The change in the mechanical energy of a system, moving between two points A and B, is equal to the sum of the works of the non-conservative external forces applied to that system :*

$$\boldsymbol{E(B) - E(A) = \sum W_{A \to B}\left(\vec{F}_{NC}\right)}$$

However, when the system is isolated (i.e., it is not subject to any non-conservative external forces) the mechanical energy is conserved $\Longrightarrow \Delta \boldsymbol{E} = \boldsymbol{0}$.

**Exercise:**




A small object of mass m modeled by a point is hung at the end of an inextensible thread of length L. The other end is attached to a bracket(see figure). We do the study in the terrestrial frame of reference. The initial angle is $\theta = 20°$, Length L = 50 cm.

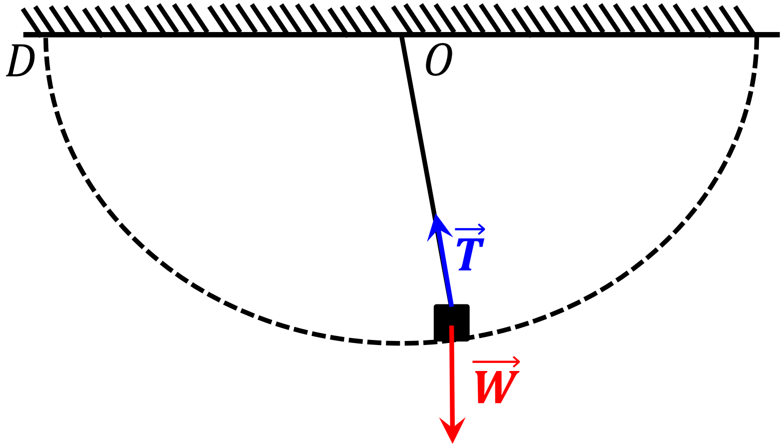
a. Trace the forces acting on the object.

**b.** We let go of the object from point A. Using the kinetic energy theorem, express its velocity $V_B$ at point B as a function of g, L, and $\theta$ , and then calculate it.

**c.** What is its velocity at point C?

**d.** We now throw the object from point A with speed $\vec{V}_A$ tangent to the circle, towards the left. Express the minimum value of the norm of $V_A$ for the object to go to point D as a function of g, L and $\theta$. Calculate it.

**Solution:**




**a.** the forces acting on the object are:
Object Weight $\vec{W}$ and Thread tension $\vec{T}$

**b.** In $A$, the velocity being zero $E_C(A) = 0\ J$.

In $B$ The kinetic energy is $E_c(B) = \frac{1}{2} m {V_B}^2$

Applying the kinetic energy theorem

$$\Delta \boldsymbol{E_C} = \boldsymbol{E_C(B)} - \boldsymbol{E_C(A)} = \sum \boldsymbol{W_{A\to B}}\left(\vec{\boldsymbol{F}}_{\boldsymbol{ext}}\right) = \boldsymbol{W_{A\to B}}\left(\vec{\boldsymbol{T}}\right) + \boldsymbol{W_{A\to B}}\left(\vec{\boldsymbol{W}}\right)$$

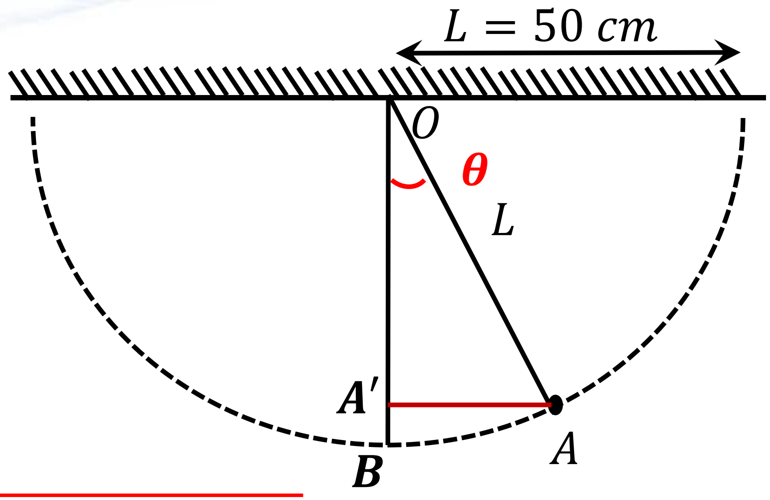
- The tension of the wire $\vec{T}$ is perpendicular to the trajectory, its work is always zero.
- The weight $\vec{W}$ is a conservative force, its work depends only on the start and end positions, and therefore on the difference in altitude $h$ between point A and point B.

$$h = A'B = OB - OA' = L - L\cos\theta$$

$$\Longrightarrow h = L(1 - \cos\theta)$$

$$\Delta E_C = E_C(B) - 0 = W_{A\to B}(\vec{P})$$

$$\Longrightarrow \frac{1}{2}mV_B{}^2 = mgh = mgL(1 - \cos\theta)$$




$$\Longrightarrow V_B = \sqrt{2gL(1 - \cos\theta)} = \sqrt{2.9,8.0,5(1 - \cos 20°)} = 0,77m/s$$

**C.** Point C is at the same height as point A, if no energy is lost, the object

is in C with zero velocity, all the mechanical energy is grouped in the potential energy.

**D.** the object reaches point D with zero velocity

Applying the principle of conservation of mechanical energy between A and D

$$\Delta E_m = 0 \Longrightarrow E_m(D) - E_m(A) = 0$$

$$\Longrightarrow \left(E_C(D) + E_P(D)\right) - \left(E_C(A) + E_P(A)\right) = 0$$

$$\begin{cases} E_C(D) = 0 \\ E_P(D) = mgL \\ E_C(A) = \dfrac{1}{2}mV_A{}^2 \\ E_P(A) = mgL(1 - cos\theta) \end{cases}$$

$$\Longrightarrow mgL = \frac{1}{2}mV_A{}^2 + mgL(1 - cos\theta) = \frac{1}{2}mV_A{}^2 + mgL - mgLcos\theta$$

$$\boxed{\boldsymbol{\Longrightarrow V_A = \sqrt{2gLcos\theta} = 3\ m/s}}$$