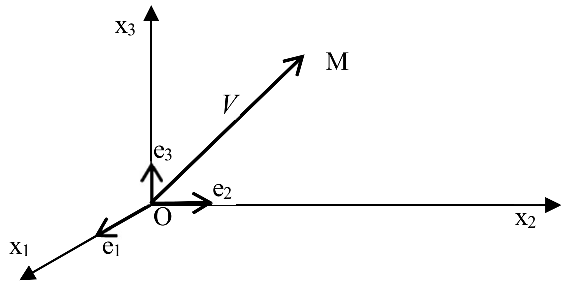
Chapter I

Introduction, Essential Mathematics

I-1 Space and coordinate system

In this course, we always place ourselves in the Euclidean physical space equipped with a direct reference point and the orthonormal basis (e_1, e_2, e_3). A vector V is represented by its components in the following form :



$$\vec{V} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = (v_1\ , v_2, v_3)^T$$

$$= v_1 e_1 + \ v_2 e_2 + \ v_3 e_3$$

I-2 Indicial notation

I-2-1 Partial derivative

$$u,_i = \frac{\partial u}{\partial x_i} \quad ; \quad v,_{ij} = \frac{\partial^2 v}{\partial x_i \partial x_j} \quad ; \quad w,_{iij} = \frac{\partial^3 w}{\partial x_i^2 \partial x_j}$$

Example : $u,_1 = \frac{\partial u}{\partial x_1}$ $\quad v,_{23} = \frac{\partial^2 v}{\partial x_2 \partial x_3}$ $\quad w,_{112} = \frac{\partial^3 w}{\partial x_1^2 \partial x_2}$

I-2-2 Summation Convention

$$a_i b_i = \sum_{i=1}^{n} a_i b_i = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$$

$S_j = a_j^i x_i = a_j^k x_k = a_j^e x_e$ *i,k,e ; summed (or dummy) index*

j : free index

Attention : $a_i + b_i \neq (a_1 + b_1) + (a_2 + b_2) + \cdots + (a_n + b_n)$

Example : the scalar product of two vectors : $\vec{U} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$ *and* $\vec{V} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$

$$\vec{U} \cdot \vec{V} = u_i v_i = u_1 v_1 + \ u_2 v_2 + \ u_3 v_3$$

I-2-3 Kronecker Symbol (Delta)

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Example : The scalar product of the orthonormal basis vectors:

$$\vec{e_i} \cdot \vec{e_j} = \delta_{ij} \qquad i=1,2,3 \qquad j=1,2,3$$

I-2-4 Permutation Symbol

$$\varepsilon_{ijk} = \begin{cases} 0 & if\ i = j, i = k \ \ or \ \ j = k \\ +1 & if\ (i,j,k)\ are\ in\ this\ order\ (1,2,3), (3,1,2)\ or\ (2,3,1)\ (called\ even\ permutation) \\ -1 & if\ (i,j,k)\ are\ in\ this\ order\ (2,1,3), (1,3,2)\ or\ (3,2,1)\ (called\ odd\ permutation) \end{cases}$$

Example : the vectorial product of two vectors : $\vec{U} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$ *and* $\vec{V} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$

$$\vec{W} = \vec{U} \wedge \vec{V} \quad with \quad w_i = \varepsilon_{ijk} \cdot u_j \cdot v_k \qquad i=1,2,3 \qquad j=1,2,3 \qquad k=1,2,3$$

$$\vec{W} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{pmatrix}$$

I-3- Second order tensors :

Let two vectors $\vec{X}$ and $\vec{Y}$ be defined in the vector space R^3 equipped with the basis $\{e_1,e_2,e_3\}$

$$\vec{X} = x^i e_i \qquad \text{et} \qquad \vec{Y} = y^i e_i$$

We define the tensor product of two vectors $\vec{X}$ and $\vec{Y}$ by:

$$U = \vec{X} \otimes \vec{Y} = (x^1y^1, x^1y^2, x^1y^3, x^2y^1, x^2y^2, x^2y^3, x^3y^1, x^3y^2, x^3y^3)$$
$$= x^i y^j \, e_i \otimes e_j$$

<u>Definition</u>: A tensor A is a linear operator (application) on R^3 whose components are defined with respect to a basis.

$$\forall X,Y \in R^3 \qquad A(X+Y) = A(X) + A(Y)$$
$$\forall \lambda \in R\, ; X \in R^3 \qquad A(\lambda X) = \lambda A(X)$$

The relation Y=A(X) can be written :

$$\begin{pmatrix} y^1 \\ y^2 \\ y^3 \end{pmatrix} = \begin{pmatrix} A_1^1 & A_2^1 & A_3^1 \\ A_1^2 & A_2^2 & A_3^2 \\ A_1^3 & A_2^3 & A_3^3 \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} \quad \text{ou} \quad y^j = A_i^j x^i$$

(A) is the matrix associated with the tensor A

A_i^j is the elements (or components) of matrix

Remarks :

Tensor of order zero: Scalar

Tensor of order one: Vector

Hight orderTensor : (tensor of order two, tensor of order tree, …)

Transpose of tensor

The Transpose of tensor is a tensor noted ${}^{t}A$ or A^{t} such that :

$$\forall \vec{X}, \overrightarrow{X'} \qquad \overrightarrow{X'} A \vec{X} = \vec{X} A^{t} \overrightarrow{X'} = \vec{X}\ {}^{t}A \overrightarrow{X'}$$

so $A_{ij} = A^{t}_{ji}$ (permutation of lines and columns).

Example : $A = \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix}$ $\qquad A^{t} = {}^{t}A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$

if $A = {}^{t}A$ then A is symmetric

if $A = -{}^{t}A$ then A is antisymmetric (or skew-symmetric)

Properties :

$A = {}^{t}({}^{t}A)$

${}^{t}[A_1 + A_2 + \cdots + A_n] = {}^{t}[A_1] = {}^{t}A_1 + {}^{t}A_2 + \cdots + {}^{t}A_n$

${}^{t}[A_1 \cdot A_2 \cdot \cdots \cdot A_n] = {}^{t}A_n \cdot \cdots {}^{t}A_2 \cdot {}^{t}A_1$

Particular matrix :

Identity matrix I such as $I_{ij} = \delta_{ij}$

Matrix inverse $(A)^{-1}$ *such as* $(A) \cdot (A)^{-1} = I$

A is orthogonal matrix if $(A)^{t} = (A)^{-1}$

Trace of the matrix $tr(A) = A_{ii}$

Theorem: Every tensor can be decomposed, and in a unique way, into the sum of a symmetric tensor and an antisymmetric tensor.

$$A = \frac{A + {}^{t}A}{2} + \frac{A - {}^{t}A}{2} = A^{s} + A^{a} = \begin{cases} A^{s} = \dfrac{A + {}^{t}A}{2} & \text{symmetric tensor} \\ A^{a} = \dfrac{A - {}^{t}A}{2} & \text{antisymmetric tensor} \end{cases}$$

I-4 Coordinate Transformations (change of basis)

Let $\{e_j\}$ be the initial basis and $\{f_j\}$ a new basis such that: : $f_j = P_j^i \, e_i$

P : Transformation matrix

If $\{f_j\}$ is an orthonormal basis then $P^{-1} = {}^tP$

a) <u>Vector</u>

$$\vec{X} = x^i e_i = \bar{x}^j f_j \qquad \Rightarrow \qquad {}^tP\bar{x} = x$$

$\{f_j\}$ is an orthonormal basis $\Rightarrow$ $\bar{x} = ({}^tP)^{-1}x = (P^{-1})^{-1}x = Px$

b) <u>Tensor</u>

in the initial basis Y=A(X)

in the new basis $\bar{Y} = \bar{A}(\bar{X})$ tel que $\bar{A} = (P)A(P^{-1})$

<u>Example :</u>

Let X be the vector which decomposes in the orthonormal basis $\{e_i\}$ by $X = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}$

1) Perform the tensor product T= X ⊗ X and write the components t_{ij} of T in matrix form.

2) Let $\{f_j\}$ j=1,2,3 be an orthonormal basis linked to the basis $\{e_i\}$ i=1,2,3 by the orthogonal transformation matrix [P] by $\{f_j\} = [P]\{e_i\}$ such that:

$$(P) = \begin{pmatrix} 1/2 & \sqrt{3}/2 & 0 \\ -\sqrt{3}/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

a) Calculate the components t^*_{ij} of the tensor T in the basis $\{f_j\}$.

b) Calculate the components of the vector X in the basis $\{f_j\}$ and calculate the tensor product X ⊗ X in the basis $\{f_j\}$. What do you observe?

Solution :

1°) $t_{11} = x_1\, x_1 =1$ $t_{12} = x_1\, x_2 =3$ $t_{13} = x_1\, x_3 =-2$

$t_{21} = x_2\, x_1 =3$ $t_{22} = x_2\, x_2 =9$ $t_{23} = x_2\, x_3 =--6$

$t_{31} = x_3\, x_1 =-2$ $t_{32} = x_3\, x_2 =-6$ $t_{33} = x_3\, x_3 =-4$

$$T = X \otimes X = \begin{pmatrix} 1 & 3 & -2 \\ 3 & 9 & -6 \\ -2 & -6 & 4 \end{pmatrix}$$

2°) a the basis {f_j } is orthonormal ($|f_i|=1$ et $f_i \perp f_j$ if $i\neq j$)

$$T^* = (P)(T)(P^{-1}) = \frac{1}{4}\begin{pmatrix} 28+6\sqrt{3} & -6+8\sqrt{3} & -4-12\sqrt{3} \\ -6+8\sqrt{3} & 12-6\sqrt{3} & -3+\sqrt{3} \\ -4-12\sqrt{3} & -3+\sqrt{3} & +16 \end{pmatrix}$$

b) $X^* = (P)(X)$ $\Rightarrow$ $\begin{pmatrix} x_1^* \\ x_2^* \\ x_3^* \end{pmatrix} = \begin{pmatrix} 1/2 & \sqrt{3}/2 & 0 \\ -\sqrt{3}/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1+3\sqrt{3} \\ 3-\sqrt{3} \\ -4 \end{pmatrix}$

$t_{11}^* = x_1^* x_1^* = \frac{1}{4}(28+6\sqrt{3})$ $t_{12}^* = x_1^* x_2^* = \frac{1}{4}(-6+8\sqrt{3})$

$t_{13}^* = x_1^* x_3^* = \frac{1}{4}(-4-12\sqrt{3})$ $t_{21}^* = x_2^* x_1^* = \frac{1}{4}(-6+8\sqrt{3})$

$t_{22}^* = x_2^* x_2^* = \frac{1}{4}(12-6\sqrt{3})$ $t_{23}^* = x_2^* x_3^* = \frac{1}{4}(-12+4\sqrt{3})$

$t_{31}^* = x_3^* x_1^* = \frac{1}{4}(-4-12\sqrt{3})$ $t_{32}^* = x_3^* x_2^* = \frac{1}{4}(-12+4\sqrt{3})$

$t_{33}^* = x_3^* x_3^* = \frac{1}{4}(16)$

We observe the compnents of T^* are equal to t^*_{ij} $\Rightarrow$ $T^* = X^* \otimes X^*$

I-5 Vector product operator

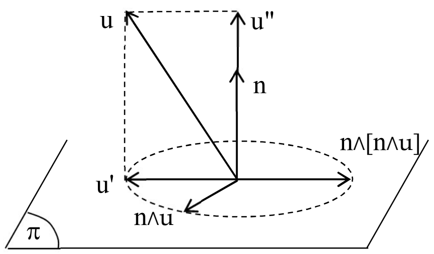
Let two vectors be : $\vec{n} = n_i e_i$ et $\vec{u} = u_i e_i$

The vector product on $\vec{n}$. By $\vec{u}$. Is as follows :

$$\vec{n} \wedge \vec{u} = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} \wedge \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} n_2u_3 - n_3u_2 \\ n_3u_1 - n_1u_3 \\ n_1u_2 - n_2u_1 \end{pmatrix} = (*\,n)\vec{u} \quad \text{with} \quad (*\,n) = \begin{pmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{pmatrix}$$

(*n) is the matrix of the tensor vector product on the left by $\vec{n}$.

I-6 Projection operators



$\vec{n}$ vector normal to the plane π

$$\vec{u} = \overrightarrow{u'} + \overrightarrow{u''}$$

The projection of the vector $\vec{u}$ onto the plane π is :

$$\overrightarrow{u'} = -(*\,n)^2\vec{u}$$

$-(*\,n)$ is the matrix of the orthogonal projection operator on the plane π

Orthogonal projection of the vector $\vec{u}$ onto the axis axis whose unit vector is $\vec{n}$

$$\overrightarrow{u''} = [I + (*\, n)^2]\vec{u}$$

$I + (*\, n)^2$ is the matrix of the orthogonal projection operator onto the axis whose unit vector is $\vec{n}$.

Remark : if the vector $\vec{n}$ is unitary ($|\vec{n}| = 1$) then: $I + (*\, n)^2 = (\vec{n})(\vec{n})^t$

Example :

Consider an orthonormal basis of R^3 defined by the following vectors:

$$f_1 = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)^t \quad f_2 = \left(-\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)^t \quad f_3 = \left(0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)^t$$

Given the following vector: $A = \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix}$

a) Determine the projection of this vector on the axes whose unit vectors are f_1, f_2 and f_3.

b) Determine the projection of this vector onto the plane formed by the vectors f_1 and f_2.

Solution

a) Let $\bar{A}_1$, $\bar{A}_2$, $\bar{A}_3$ be the projections of vector A respectively on axes f_1, f_2 and f_3.

$\{f_i\}$ is orthonormal basis $\Rightarrow$ $\bar{A}_i = [I + (*\, f_i)^2]A = [(\vec{f_i}) \cdot (\vec{f_i})^t]A$

$$\bar{A}_1 = [(\vec{f_1}) \cdot (\vec{f_1})^t]A = \left[\begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}\begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{pmatrix}\right]\begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix} = \left[\frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}\right]\begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 7 \\ 7 \\ 7 \end{pmatrix}$$

$$\bar{A}_2 = [(\vec{f_2}) \cdot (\vec{f_2})^t]A = \left[\begin{pmatrix} -2/\sqrt{6} \\ 1/\sqrt{6} \\ 1/\sqrt{6} \end{pmatrix}\begin{pmatrix} -2/\sqrt{6} & 1/\sqrt{6} & 1/\sqrt{6} \end{pmatrix}\right]\begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix} = \frac{1}{6}\begin{pmatrix} 4 & -2 & -2 \\ -2 & 1 & 1 \\ -2 & 1 & 1 \end{pmatrix}\begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix} = \frac{1}{6}\begin{pmatrix} 10 \\ -5 \\ -5 \end{pmatrix}$$

$$\bar{A}_3 = [(\vec{f_3}) \cdot (\vec{f_3})^t]A = \left[\begin{pmatrix} 0 \\ -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}\begin{pmatrix} -2/\sqrt{6} & 1/\sqrt{6} & 1/\sqrt{6} \end{pmatrix}\right]\begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

c) Let $\bar{A}_{12}$ be the projection of vector A onto the plane formed by the vectors f_1 and f_2.

$\bar{A}_{12} = [-(*f_3)^2]A$

$$\bar{A}_{12} = \left[-(*f_3)^2\right]A = \left[-\begin{pmatrix} 0 & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 \end{pmatrix}^2\right]\begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ 3/2 \\ 3/2 \end{pmatrix}$$

I-7 Eigenvalues and Eigenvectors of Symmetric Second-Order Tensors

Definition : An eigenvector $\vec{X}$ of the tensor A is a non-zero vector that, when a linear transformation represented by the matrix (A) is applied to it, results in a vector that is parallel to the original vector.

This translates to the following mathematical equation : $A(\vec{X}) = \lambda(\vec{X})$

$\lambda \in$ R: eigenvalue associated with the eigenvector $\vec{X}$

This relation can be rewritten as : $[A - \lambda I](\vec{X}) = \vec{0}$

For symmetric second-order tensors, this expression is simply a homogeneous system of three linear algebraic equations in the unknowns $\vec{X}_1, \vec{X}_2, \vec{X}_3$.

The system possesses a nontrivial solution if and only if the determinant of its coefficient matrix vanishes, that is : det(A-λI)=0

Expanding the determinant produces a cubic equation in terms of λ *(called characteristic equation)*:

$$det[A - \lambda I] = -\lambda^3 + I_a\lambda^2 - I_b\lambda + I_c = 0$$

Where :

$$I_a = tr(A) = A_{ii} = A_{11} + A_{22} + A_{33}$$

$$I_b = \tfrac{1}{2}(A_{ii}A_{jj} - A_{ij}A_{ij}) = \begin{vmatrix} A_{11} & A_{12} \\ A_{12} & A_{22} \end{vmatrix} + \begin{vmatrix} A_{22} & A_{23} \\ A_{23} & A_{33} \end{vmatrix} + \begin{vmatrix} A_{11} & A_{13} \\ A_{13} & A_{33} \end{vmatrix}$$

$$I_c = \det(A) = det\left[A_{ij}\right]$$

The scalars I_a, I_b, and I_c are called the *fundamental invariants of the tensor* A.

The roots of the characteristic equation determine the allowable values for λ ($\lambda_1, \lambda_2, \lambda_3$), and each of these may be back-substituted into the relation ($[A - \lambda I](\vec{X}) = \vec{0}$) to solve for the associated principal direction $\vec{X}$ ($\vec{X}_1, \vec{X}_2, \vec{X}_3$).

Remarks :

a) if the tensor A is summetric in R^3, then all three roots λ_1, λ_2, λ_3 of the cubic equation must be real

b) All three principal values $(\lambda_1, \lambda_2, \lambda_3)$ distinct; thus, the three corresponding principal directions $(\vec{X}_1, \vec{X}_2, \vec{X}_3)$ are unique and orthogonal. In this case, these vectors $(\vec{X}_1, \vec{X}_2, \vec{X}_3)$ form an orthogonal basis in which the tensor A is diagonal. In this basis the elements of the tensor A are the eigenvalues $(\lambda_1, \lambda_2, \lambda_3)$. Thus, in this new basis, the tensor A is written in the following form:

$$A = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

Note that the fundamental invariants of the tensor A can be expressed in terms of the principal values as :

$$I_a = tr(A) = A_{ii} = A_{11} + A_{22} + A_{33} = \lambda_1 + \lambda_2 + \lambda_3$$

$$I_b = \tfrac{1}{2}(A_{ii}A_{jj} - A_{ij}A_{ij}) = \begin{vmatrix} A_{11} & A_{12} \\ A_{12} & A_{22} \end{vmatrix} + \begin{vmatrix} A_{22} & A_{23} \\ A_{23} & A_{33} \end{vmatrix} + \begin{vmatrix} A_{11} & A_{13} \\ A_{13} & A_{33} \end{vmatrix} = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_1\lambda_3$$

$$I_c = \det(A) = det[A_{ij}] = \lambda_1 \cdot \lambda_2 \cdot \lambda_3$$

Example :

Consider the tensor A defined in the basis (e_1, e_2, e_3) :

$$A = \begin{pmatrix} 57 & 0 & 24 \\ 0 & 50 & 0 \\ 24 & 0 & 43 \end{pmatrix}$$

Determine the eigenvalues and eigenvectors of the tensor A

Solution :

Eigenvalues

Let λ_1 , λ_2 et λ_3 the eigenvalues of the tensor A and X_1 , X_2 et X_3 the eigenvectors corresponding.

$$det(A - \lambda I) = 0 \quad \Rightarrow \quad \left| \begin{pmatrix} 57 & 0 & 24 \\ 0 & 50 & 0 \\ 24 & 0 & 43 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right| = \begin{vmatrix} 57\text{-}\lambda & 0 & 24 \\ 0 & 50\text{-}\lambda & 0 \\ 24 & 0 & 43\text{-}\lambda \end{vmatrix} = 0$$

$$det(A - \lambda I) = (50 - \lambda)[(57 - \lambda)(43 - \lambda) - 576] = (50 - \lambda)(\lambda^2 - 100\lambda + 1875) = 0$$

$$\Rightarrow \begin{cases} \lambda_1 = 50 \\ \lambda_2 = 75 \text{ et and } \lambda_3 = 25 \end{cases}$$

Eigenvectors :

For $\lambda_1 = 50$

$$(A - \lambda_1 I)\, X_1 = 0 \qquad \Rightarrow \begin{pmatrix} 75-50 & 0 & 24 \\ 0 & 50-50 & 0 \\ 24 & 0 & 43-50 \end{pmatrix} \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} 7a_1 + 24c_1 = 0 \\ 0\, b_1 = 0 \\ 24a_1 - 7c_1 = 0 \end{cases} \qquad \Rightarrow \qquad X_1 = \begin{pmatrix} 0 \\ b_1 \\ 0 \end{pmatrix}$$

the value of b_1 is arbitrary

X_1 unit vector $\Rightarrow$ $a_1^2 + b_1^2 + c_1^2 = 1$ $\qquad b_1^2 = 1$

we take $b_1 = +1$ $\Rightarrow$ $X_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

For $\lambda_2 = 75$

$$(A - \lambda_2 I)\, X_1 = 0 \qquad \Rightarrow \begin{pmatrix} 75-75 & 0 & 24 \\ 0 & 50-75 & 0 \\ 24 & 0 & 43-75 \end{pmatrix} \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} -18a_2 + 24\, c_2 = 0 \\ -\, 25\, b_2 = 0 \\ 24\, a_2 - 32\, c_2 = 0 \end{cases} \qquad \Rightarrow \begin{cases} b_2 = 0 \\ c_2 = \frac{3}{4} a_2 \end{cases}$$

X_2 vecteur unitaire $\Rightarrow$ $a_2^2 + b_2^2 + c_2^2 = 1$ $\Rightarrow$ $a_2^2 + \left(\frac{3}{4}\right)^2 a_2^2 = 1$ $\qquad a_2^2 = \frac{16}{25}$

we take $a_2 = \frac{4}{5} = 0.8$ $\Rightarrow$ $c_2 = \frac{3}{5}$ $\Rightarrow$ $X_2 = \begin{pmatrix} 0.8 \\ 0 \\ 0.6 \end{pmatrix}$

For $\lambda_3 = 25$

Same as X_2 we find X_3 .

$(A - \lambda_3 I)\, X_{III} = 0$ $\Rightarrow$ $X_3 = \begin{pmatrix} 0.6 \\ 0 \\ -0.8 \end{pmatrix}$

X_3 can be found by the following relation : $X_3 = X_1 \wedge X_2$

$$X_3 = X_1 \wedge X_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \wedge \begin{pmatrix} 0.8 \\ 0 \\ 0.6 \end{pmatrix} = \begin{pmatrix} 0.6 \\ 0 \\ -0.8 \end{pmatrix}$$

I-8- Differential operators

Let a point P with coordinates x_1, x_2, x_3. $P\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$; $dP\begin{pmatrix} dx_1 \\ dx_2 \\ dx_3 \end{pmatrix}$

$\phi(x_1, x_2, x_3)$: scalar function of the point P coordinates.

$\vec{u}(x_1, x_2, x_3)$: vectorial function of the point P coordinates. $\vec{u}\begin{pmatrix} u_1(x_1, x_2, x_3) \\ u_2(x_1, x_2, x_3) \\ u_3(x_1, x_2, x_3) \end{pmatrix}$

$A(x_1, x_2, x_3)$: tensorial function of the point P coordinates.

$$A(x_1, x_2, x_3) = \begin{pmatrix} A_1^1(x_1, x_2, x_3) & A_2^1(x_1, x_2, x_3) & A_3^1(x_1, x_2, x_3) \\ A_1^2(x_1, x_2, x_3) & A_2^2(x_1, x_2, x_3) & A_3^2(x_1, x_2, x_3) \\ A_1^3(x_1, x_2, x_3) & A_2^3(x_1, x_2, x_3) & A_3^3(x_1, x_2, x_3) \end{pmatrix}$$

I-8-1 Gradient of a scalar function

The derivative operation of the scalar function ϕ :

$$d\phi = \frac{\partial\phi}{\partial x_1}dx_1 + \frac{\partial\phi}{\partial x_2}dx_2 + \frac{\partial\phi}{\partial x_3}dx_3$$

$$d\phi = grad\phi\,.\,\mathrm{dp} \qquad \Rightarrow \qquad \overrightarrow{grad}\,\phi = \begin{pmatrix} \partial\phi/\partial x_1 \\ \partial\phi/\partial x_2 \\ \partial\phi/\partial x_3 \end{pmatrix}$$ it is a vector

I-8-2 Gradient of a vectorial function

The derivative operation of the vectorial function $\vec{u}$:

$$d\vec{u} = \begin{pmatrix} du_1 \\ du_2 \\ du_3 \end{pmatrix} = \begin{pmatrix} \frac{\partial u_1}{\partial x_1}dx_1 + \frac{\partial u_1}{\partial x_2}dx_2 + \frac{\partial u_1}{\partial x_3}dx_3 \\ \frac{\partial u_2}{\partial x_1}dx_1 + \frac{\partial u_2}{\partial x_2}dx_2 + \frac{\partial u_2}{\partial x_3}dx_3 \\ \frac{\partial u_3}{\partial x_1}dx_1 + \frac{\partial u_3}{\partial x_2}dx_2 + \frac{\partial u_3}{\partial x_3}dx_3 \end{pmatrix}$$

$$d\vec{u} = grad\vec{u}\,.\,\mathrm{dp} \qquad \Rightarrow \qquad grad\vec{u} = \begin{bmatrix} \partial u_1/\partial x_1 & \partial u_1/\partial x_2 & \partial u_1/\partial x_3 \\ \partial u_2/\partial x_1 & \partial u_2/\partial x_2 & \partial u_2/\partial x_3 \\ \partial u_3/\partial x_1 & \partial u_3/\partial x_2 & \partial u_3/\partial x_3 \end{bmatrix}$$ it is a tensor

I-8-3 Divergence of a vectorial function

$$div(\vec{u}) = tr(grad\vec{u}) = \frac{\partial u_p}{\partial x_p} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} \qquad \text{it is a scalar}$$

I-8-4 Divergence of tensorial function

$$div(A) = \begin{pmatrix} \frac{\partial A_i^1}{\partial x_i} \\ \frac{\partial A_i^2}{\partial x_i} \\ \frac{\partial A_i^3}{\partial x_i} \end{pmatrix} \qquad \text{it is a vector}$$

I-8-5 Laplacian of a scalar function

$$\Delta\phi = div(grad\phi) = \frac{\partial}{\partial x_p}\left(\frac{\partial\phi}{\partial x_p}\right) = \frac{\partial^2\phi}{\partial x_1^2} + \frac{\partial^2\phi}{\partial x_2^2} + \frac{\partial^2\phi}{\partial x_3^2} \qquad \text{it is a scalar}$$

I-8-6 Laplacaen of a vectorial function

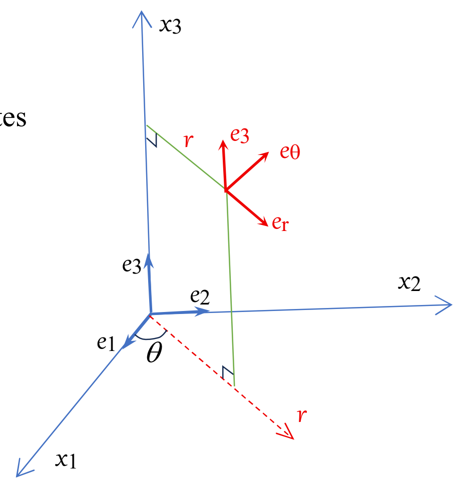
$$\Delta\vec{u} = \begin{pmatrix} \Delta u_1 \\ \Delta u_2 \\ \Delta u_3 \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial x_p}\left(\frac{\partial u_1}{\partial x_p}\right) \\ \frac{\partial}{\partial x_p}\left(\frac{\partial u_2}{\partial x_p}\right) \\ \frac{\partial}{\partial x_p}\left(\frac{\partial u_3}{\partial x_p}\right) \end{pmatrix} = \begin{pmatrix} \frac{\partial^2 u_1}{\partial x_1^2} + \frac{\partial^2 u_1}{\partial x_2^2} + \frac{\partial^2 u_1}{\partial x_3^2} \\ \frac{\partial^2 u_2}{\partial x_1^2} + \frac{\partial^2 u_2}{\partial x_2^2} + \frac{\partial^2 u_2}{\partial x_3^2} \\ \frac{\partial^2 u_3}{\partial x_1^2} + \frac{\partial^2 u_3}{\partial x_2^2} + \frac{\partial^2 u_3}{\partial x_3^2} \end{pmatrix} \qquad \text{it is a vector}$$

I-8-7 Rotationnel of a vectorial function

$$rot(\vec{u}) = \begin{pmatrix} \frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1} \\ \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \end{pmatrix} \qquad \text{it is a vector}$$

$$* \, rot(\vec{u}) = 2\ \text{antisym}(\text{grad}(\vec{u}))$$

I-8-8 Cylindrical coordinates with axis Ox_3



The cylindrical coordinate system with axis Ox_3 is (r, θ, x_3)

The relations between cylindrical coordinates and Cartesian coordinates are given by :

$x_1 = r\cos(\theta)$

$x_2 = r\sin(\theta)$

$x_3 = x_3$

The basis in cylindrical coordinates is given by :

$e_r = \cos(\theta)\, e_1 + \sin(\theta)\, e_2$

$e_\theta = -\sin(\theta)\, e_1 + \cos(\theta)\, e_2$

$e_3 = e_3$

Let a point P with coordinates (r, θ, x_3) $\quad P\begin{pmatrix} r \\ \theta \\ x_3 \end{pmatrix}$

$$dP\begin{pmatrix} dr \\ r d\theta \\ dx_3 \end{pmatrix}$$

a) Gradient of a scalar function

$$d\phi = grad\phi \,.\, \mathrm{dp} \qquad \Rightarrow \qquad \overrightarrow{grad}\,\phi = \begin{pmatrix} \partial\phi/\partial r \\ \frac{1}{r}\partial\phi/\partial\theta \\ \partial\phi/\partial x_3 \end{pmatrix}$$

b) Gradient of a vectorial function

$$d\vec{u} = grad\vec{u} \,.\, \mathrm{dp} \qquad \Rightarrow \qquad grad\vec{u} = \begin{bmatrix} \frac{\partial u_1}{\partial r} & \frac{1}{r}\left(\frac{\partial u_1}{\partial\theta} - u_2\right) & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial r} & \frac{1}{r}\left(\frac{\partial u_2}{\partial\theta} + u_1\right) & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial r} & \frac{1}{r}\left(\frac{\partial u_3}{\partial\theta}\right) & \frac{\partial u_3}{\partial x_3} \end{bmatrix}$$

c) Divergence of a vectorial function

$$div(\vec{u}) = tr(grad\vec{u}) = \frac{\partial u_1}{\partial r} + \frac{1}{r}\left(\frac{\partial u_2}{\partial\theta} + u_1\right) + \frac{\partial u_3}{\partial x_3}$$

d) Laplacian of a scalar function

$$\Delta\phi = div(grad\phi) = \frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r}\frac{\partial^2\phi}{\partial\theta^2} + \frac{\partial^2\phi}{\partial x_3^2}$$