Chapter 1

Vector Analysis

This chapter is considered one of the most important sections that form the foundation of linear algebra theories. It serves as a fundamental part for subsequent concepts like matrices, determinants.

1.1 Vector Space

A vector space is a fundamental concept in linear algebra, essential for understanding linear maps, matrices, and determinants.

Definition 1.1

A set $E \neq \emptyset$ is a vector space over a field $\mathbb{K}$ if it has two operations:

- **Vector addition** $(+)$: $E \times E \to E$ mapping $(u, v) \mapsto u + v$
- **Scalar multiplication** $(\cdot)$: $\mathbb{K} \times E \to E$ mapping $(\lambda, u) \mapsto \lambda \cdot u$

These operations must satisfy:

1. $u + v = v + u$ (commutativity)
2. $u + (v + w) = (u + v) + w$ (associativity)
3. There exists $0_E \in E$ such that $u + 0_E = u$ (zero vector)
4. For each u, there exists $-u$ such that $u + (-u) = 0_E$ (additive inverse)
5. $1 \cdot u = u$ (scalar identity)

6. $\lambda \cdot (\mu \cdot u) = (\lambda\mu) \cdot u$ (scalar compatibility)

7. $\lambda \cdot (u + v) = \lambda \cdot u + \lambda \cdot v$ (distributivity)

8. $(\lambda + \mu) \cdot u = \lambda \cdot u + \mu \cdot u$ (distributivity)

Notation:

- Elements of E are called **vectors**
- Elements of $\mathbb{K}$ are called **scalars**
- If $\mathbb{K} = \mathbb{R}$: real vector space
- If $\mathbb{K} = \mathbb{C}$: complex vector space

Example 1.1

$\mathbb{R}^2$ as a vector space

- **Addition**: $(x, y) + (x', y') = (x + x', y + y')$
- **Scalar multiplication**: $\lambda \cdot (x, y) = (\lambda x, \lambda y)$
- **Zero vector**: $(0, 0)$
- **Additive inverse**: $-(x, y) = (-x, -y)$

Example 1.2

$\mathbb{R}^n$ as a vector space

- **Addition**: $(x_1, \ldots, x_n) + (x'_1, \ldots, x'_n) = (x_1 + x'_1, \ldots, x_n + x'_n)$
- **Scalar multiplication**: $\lambda \cdot (x_1, \ldots, x_n) = (\lambda x_1, \ldots, \lambda x_n)$

Example 1.3

Space of functions $\mathscr{F}(\mathbb{R}, \mathbb{R})$

- **Addition**: $(f + g)(x) = f(x) + g(x)$

- **Scalar multiplication**: $(\lambda f)(x) = \lambda f(x)$
- **Zero vector**: $f(x) = 0$ for all x
- **Additive inverse**: $(-f)(x) = -f(x)$

1.1.1 Product of vector spaces

Definition 1.2

Let $\mathbb{K}$ be a commutative field and let $E_1, E_2, \ldots, E_n$ be vector spaces on the field $\mathbb{K}$. We define by $E = E_1 \times E_2 \times \cdots \times E_n$ the two internal operations $(+)$ and $(\cdot)$ as follows:

$$\begin{aligned}
&\forall \lambda \in \mathbb{K}, \forall (x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n) \in E : \\
&1)\ (x_1, x_2, \ldots, x_n) + (y_1, y_2, \ldots, y_n) = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n), \\
&2)\ \lambda \cdot (x_1, x_2, \ldots, x_n) = (\lambda \cdot x_1, \lambda \cdot x_2, \ldots, \lambda \cdot x_n).
\end{aligned}$$

Then $(E, +, \cdot)$ represents a vector space called the product space. The neutral element in this space is the ray of the neutral elements of each space, and we write:

$$0_E = (0_{E_1}, 0_{E_2}, \ldots, 0_{E_n}).$$

1.1.2 Calculus in vector spaces

Proposition 1.1.1. *Let E be a vector space on the field $\mathbb{K}$. Let $u \in E$ and $\lambda \in \mathbb{K}$. Then, we have:*

(1 $0 \cdot u = 0_E$

(2 $\lambda \cdot 0_E = 0_E$

(3 $(-1) \cdot u = -u$

(4 $\lambda \cdot u = 0_E \iff \lambda = 0$ *where* $u = 0_E$

(5 $\lambda \cdot u = 0_E \Longrightarrow (\lambda = 0_{\mathbb{K}}) \vee (u = 0_E)$

(6 *The operation that attaches to (u, v) the image $u + (-v)$ is called subtraction, and the vector $u + (-v)$ is denoted by $u - v$. Then, we have the following properties:*

$$\lambda(u - v) = \lambda u - \lambda v \quad \textit{and} \quad (\lambda - \mu)u = \lambda u - \mu u.$$

Definition 1.3

Let $n \geq 1$ be an integer, and let $v_1, v_2, \ldots, v_n$, n be a vector from E. Each ray of the form:

$$u = \lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_n v_n$$

(where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are ladders of the field $\mathbb{K}$) It is called linear mixing of rays $v_1, v_2, \ldots, v_n$. The scales $\lambda_1, \lambda_2, \ldots, \lambda_n$ are called linear mixing coefficients.

Remark 1.1.1. *If $n = 1$, then $u = \lambda_1 v_1$, and we say that u is in a linear relationship with v_1.*

Example 1.4

(1 In the space $\mathbb{R}^3$, the ray $(3,3,1)$ is a linear combination of the two rays $(1,1,0)$ and $(1,1,1)$ because:

$$(3,3,1) = 2(1,1,0) + (1,1,1).$$

(2 In the space $\mathbb{R}^2$, the vector $u = (2,1)$ is not linearly related to the vector $v_1 = (1,1)$ because there is no real λ until $u = \ lambdav_1$ which is equivalent to $(2,1) = (\lambda, \lambda)$.

(3 Let $E = \mathscr{F}(\mathbb{R}, \mathbb{R})$ be the space of real functions, and let f_0, f_1, f_2 and f_3 be functions defined by:

$$\forall x \in \mathbb{R} \quad f_0(x) = 1, \;\; f_1(x) = x, \;\; f_2(x) = x^2, \;\; f_3(x) = x^3.$$

then the function f defined by

$$\forall x \in \mathbb{R} \quad f(x) = x^3 - 2x^2 - 7x - 4$$

it is a linear combination of the functions f_0, f_1, f_2, f_3 because

$$f = f_3 - 2f_2 - 7f_1 - 4f_0.$$

(4 In the matrix space $\mathscr{M}_{2,3}(\mathbb{R})$ let the matrix

$$A = \begin{pmatrix} 1 & 1 & 3 \\ 0 & -1 & 4 \end{pmatrix}.$$

We can express matrix A as a linear combination of matrices that contain zeros in all their components except one, for example:

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + 3\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} + 4\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

1.2 Linear correlation and independence

Definition 1.4

Let $n \in \mathbb{N}^*$ be a natural number. We say that a family $\{v_1, v_2, \cdots, v_n\}$ of elements in the vector space E over the field $\mathbb{K}$ is linearly independent or a free family if, for every family of scalars $\{\lambda_i\}_{i \leq n} \in \mathbb{K}$, the following condition holds:

$$\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_n v_n = 0_E$$

where all its coefficients are zero, i.e.:

$$\lambda_1 = 0_{\mathbb{K}}, \quad \lambda_2 = 0_{\mathbb{K}}, \quad \cdots \quad \lambda_n = 0_{\mathbb{K}}.$$

0_E and $0_{\mathbb{K}}$ represent the zero of the vector space E and the zero of the commutative field $\mathbb{K}$, respectively.

Example 1.5

Let us consider in real vector space $\mathbb{R}^3$ the rays

$$a_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, a_2 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, a_3 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, b = \begin{pmatrix} 2 \\ -2 \\ -1 \end{pmatrix}$$

Hence, the ray b is a linear mixture of the rays $\{a_1, a_2, a_3\}$ and we have:

$$\begin{aligned} b &= \begin{pmatrix} 2 \\ -2 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \\ &= a_1 - a_2 + a_3. \end{aligned}$$

Remark 1.2.1. —

- *We say of any family of vector space elements, if they are not linearly independent, that they are linearly dependent.*
- *The empty set is linearly independent in any vector space.*

Example 1.6

The polynomials

$$P_1(X) = 1 - X, P_2(X) = 5 + 3X - 2X^2 \text{ and } P_3(X) = 1 + 3X - X^2.$$

form a linearly dependent set in the polynomial space $\mathscr{P}_n[X]$ because:
If we examine the coefficients of these polynomials, we can observe that the equation

$$aP_1(X) + bP_2(X) + cP_3(X) = 0.$$

has a non-trivial solution, which means there exist constants a, b, and c, not all equal to zero, that make this equation equal to zero.

$$3P_1(X) - P_2(X) + 2P_3(X) = 0.$$

Therefore, the polynomials are linearly dependent in the polynomial space.

Example 1.7

Let $\mathscr{F}(\mathbb{R}, \mathbb{R})$ be the space of real functions, and let the statement be $\{\cos, \sin\}$. To prove that this set is linearly independent: We assume that

$$\lambda \cos + \mu \sin = 0$$

That equivalent to

$$\forall x \in \mathbb{R} \quad \lambda \cos(x) + \mu \sin(x) = 0.$$

For $x = 0$ these equations give us: $\lambda = 0$.
For $x = \frac{\pi}{2}$ it gives us $\mu = 0$. That is, the set $\{\cos, \sin\}$ is linearly independent.
On the other hand, the set $\{\cos^2, \sin^2, 1\}$ is linearly related because we have the following trigonometric relationship:

$$\forall x \in \mathbb{R} : \quad \cos(x)^2 + \sin(x)^2 - 1 = 0.$$

Here, all the linear combination factors are non-zero because we have:

$$\lambda_1 = 1, \lambda_2 = 1, \lambda_3 = -1.$$

Theorem 1.1

Let $n \in \mathbb{N}^*$ we say about the family $\{v_1, v_2, \cdots, v_n\}$ of vector space elements E on The commutative field $\mathbb{K}$ is linearly dependent if there exists a family of scalers $\{\lambda_i\}_{i\leq n} \in \mathbb{K}$ that are not all null together, check:

$$\sum_{i=1}^{n} \lambda_i v_i = 0_E.$$

Example 1.8

From the previous example notice that the vectors

$$a_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, a_2 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, a_3 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, b = \begin{pmatrix} 2 \\ -2 \\ -1 \end{pmatrix}$$

linearly dependent

$$a_1 - a_2 + a_3 - b = 0_{\mathbb{R}^3}.$$

so

$$\exists \lambda_1 = 1, \lambda_2 = -1, \lambda_3 = 1, \lambda_4 = -1 : \lambda_1 a_1 + \lambda_2 a_2 + \lambda_3 a_3 + \lambda_4 b = 0_{\mathbb{R}^3}.$$

Not all are zero together.

1.2.1 The base or basis

A basis is a fundamental concept in linear algebra and holds significant applications and importance. When you have a basis for a vector space, you can represent and understand the elements in that space more comprehensively and easily. You can also perform various operations, such as linear transformations and coefficient calculations, with ease using this basis.

Definition 1.5

Let $v_1, \cdots, v_n$ be rays from the vector space E, we say of the set $\{v_1, \cdots, v_n\}$ that it is a generating set for the vector space E if every ray of E can be expressed as a linear combination of the rays $v_1, \cdots, v_n$. We write:

$$\forall v \in E, \quad \exists \lambda_1, \ldots, \lambda_p \in \mathbb{K} : \quad v = \lambda_1 v_1 + \cdots + \lambda_n v_n.$$

We also say that the set $\{v_1, \cdots, v_n\}$ generates the space E. This is also associated with the concept of the span generator if and only if:

$$E = Vect(v_1, \cdots, v_n).$$

Example 1.9

Take, for example, the following rays

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \text{ and } v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \text{ of } E = \mathbb{R}^3.$$

The set $\{v_1, v_2, v_3\}$ is generating $\mathbb{R}^3$ because each ray $v = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ from $\mathbb{R}^3$ writes

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = x \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Here the factors are

$$\lambda_1 = x, \lambda_2 = y, \lambda_3 = z.$$

Example 1.10

Let the following rays be

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \text{ of } E = \mathbb{R}^3.$$

The rays $\{v_1, v_2\}$ do not form a generative set for $\mathbb{R}^3$. For example, the vector $v = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ does not belong to the vector space $Vect(v_1, v_2)$.
If it is true, we will find $\lambda_1, \lambda_2 \in \mathbb{R}$ where $v = \lambda_1 v_1 + \lambda_2 v_2$. Who also writes:

$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \lambda_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

it gives us the following linear equations:

$$\begin{cases} \lambda_1 + \lambda_2 &= 0 \\ \lambda_1 + 2\lambda_2 &= 1 \\ \lambda_1 + 3\lambda_2 &= 0 \end{cases}$$

which has no solution.

Let $\mathscr{P}_n[X]$ be the real vector space of polynomials of degree $\leq n$ with real coefficients. Then, the polynomial set $\{1, X, X^2 \cdots, X^n\}$ forms a generating set for the space $\mathscr{P}_n[X]$.

Proposition 1.2.1. *Let $\mathscr{F} = \{v_1, v_2, \cdots, v_p\}$ be a generative set of E. Hence $\mathscr{F}' = \left\{v_1', v_2', \cdots, v_q'\right\}$ is also a generative set of E if and only if every vector of $\mathscr{F}'$ is written as a linear mixture in the set $\mathscr{F}$.*

Definition 1.6

Let E be a vector space on $\mathbb{K}$. We say that the set $\mathcal{B} = (v_1, v_2, \ldots, v_n)$ of E forms a basis for the space E if:

(1 $\mathcal{B}$ is a generating set for E.

(2 $\mathcal{B}$ is a linear independent set.

Theorem 1.2

Let $\mathcal{B} = (v_1, v_2, \ldots, v_n)$ be a basis of the vector space E.
Each vector $v \in E$ is written as a single linear combination in the elements of the set $\mathcal{B}$. That is, there are single scalers $\lambda_1, \ldots, \lambda_n \in \mathbb{K}$ where:

$$v = \lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_n v_n.$$

Remark 1.2.2. (1 $(\lambda_1, \ldots, \lambda_n)$ *are called the coordinates of v in the basis $\mathcal{B}$.*

(2 *The application of form*

$$\begin{array}{rrcl} \phi : & \mathbb{K}^n & \longrightarrow & E \\ & (\lambda_1, \lambda_2, \ldots, \lambda_n) & \longmapsto & \lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_n v_n \end{array}$$

It is a bijection from the vector space $\mathbb{K}^n$ to the vector space E.

1.2.2 Dimension in Vector Spaces

The dimension of a vector space is the number of vectors in a basis for that space. It represents the minimum number of independent directions needed to describe any vector in the space.

- **Mathematical perspective**: A basis is a set of linearly independent vectors that span the entire space. The dimension equals the number of basis vectors.

  - 2D space: requires 2 basis vectors
  - 3D space: requires 3 basis vectors
  - n-dimensional space: requires n basis vectors

- **Physical interpretation**: Dimension represents independent directions of movement:
  - A point on a line: 1 dimension
  - Movement on a plane: 2 dimensions
  - Motion in space: 3 dimensions

Dimension is a fundamental concept for analyzing mathematical structures and physical systems across science and engineering.

Definition 1.7

If the vector space E has a basis $\mathcal{B}$ with a finite number of elements n, then the vector space E has a finite dimension, and we write:

$$\dim(E) = Card(\mathcal{B}) = n.$$

Remark 1.2.3. *The zero space* $\{0\}$ *has a zero dimension, i.e.* $\dim(\{0\}) = 0$.

Example 1.11

(1 The canonical basis for the space $\mathbb{R}^2$ is:

$$\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

Hence the dimension of space $\mathbb{R}^2$ is 2.

(2 The vectors

$$\left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$$

It also forms the basis of the space $\mathbb{R}^2$, and any other basis of the space $\mathbb{R}^2$ contains the same number of elements.

(3 In general, the space $\mathbb{K}^n$ has n dimensions because each basis $(e_1, e_2, \cdots, e_n)$ contains n elements.

(4 $(\dim \mathscr{P}_n[X] = n+1)$ because the basis of the space $\mathscr{P}_n[X]$ is $(1, X, X^2, \cdots, X^n)$ which contains $n+1$ elements.

Theorem 1.3

In a vector space with finite dimension n, we have:

- Each linearly independent set has a maximum of n elements,
- Every linearly independent set consisting of n elements is a basis,
- Every generative set is composed of at least n elements,
- Every generated set consisting of n elements is a basis.

Definition 1.8

We call the range of a set of rays, the dimension of the vector space they generate.

Remark 1.2.4. *The following observations are easy consequences of the previous theorem:*

- *The range of a set consisting of* n *rays, at most is* n*.*
- *The range of a ray system consisting of* n *rays is* n *if and only if this system is linearly independent.*
- *The range of a ray system consisting of* n *rays is* n *if and only if this system forms the basis of the vector space it generates.*

1.2.3 Linear combination

Definition 1.9

Let $n \geq 1$ be an integer, and let $v_1, v_2, \ldots, v_n$, n be a vector from E. Each ray of the form:

$$u = \lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_n v_n$$

(where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are ladders of the field $\mathbb{K}$) It is called linear mixing of rays $v_1, v_2, \ldots, v_n$. The scales $\lambda_1, \lambda_2, \ldots, \lambda_n$ are called linear mixing coefficients.

Remark 1.2.5. *If $n = 1$, then $u = \lambda_1 v_1$, and we say that u is in a linear relationship with v_1.*

Example 1.12

(1 In the space $\mathbb{R}^3$, the ray $(3,3,1)$ is a linear combination of the two rays $(1,1,0)$ and $(1,1,1)$ because:

$$(3,3,1) = 2(1,1,0) + (1,1,1).$$

(2 In the space $\mathbb{R}^2$, the vector $u = (2,1)$ is not linearly related to the vector $v_1 = (1,1)$ because there is no real λ until $u = \ lambdav_1$ which is equivalent to $(2,1) = (\lambda, \lambda)$.

(3 Let $E = \mathscr{F}(\mathbb{R}, \mathbb{R})$ be the space of real functions, and let f_0, f_1, f_2 and f_3 be functions defined by:

$$\forall x \in \mathbb{R} \quad f_0(x) = 1, \ \ f_1(x) = x, \ \ f_2(x) = x^2, \ \ f_3(x) = x^3.$$

then the function f defined by

$$\forall x \in \mathbb{R} \quad f(x) = x^3 - 2x^2 - 7x - 4$$

it is a linear combination of the functions f_0, f_1, f_2, f_3 because

$$f = f_3 - 2f_2 - 7f_1 - 4f_0.$$

(4 In the matrix space $\mathscr{M}_{2,3}(\mathbb{R})$ let the matrix

$$A = \begin{pmatrix} 1 & 1 & 3 \\ 0 & -1 & 4 \end{pmatrix}.$$

We can express matrix A as a linear combination of matrices that contain zeros in all their components except one, for example:

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + 3\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} + 4\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

- $\vec{v} = (1,2,3)$: Magnitude $|\vec{v}| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$
- $\vec{u} = (0,-1,2)$: Magnitude $|\vec{u}| = \sqrt{0^2 + (-1)^2 + 2^2} = \sqrt{5}$
- Addition: $(1,2,3) + (0,-1,2) = (1,1,5)$

- Scalar multiplication: $-2(1,2,3) = (-2,-4,-6)$

1.3 Polar and Cylindrical Coordinates

1.3.1 Polar Coordinates

In 2D, polar coordinates (r, θ) represent a point using:

- r: distance from origin
- θ: angle from positive x-axis

Conversion formulas:

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$r = \sqrt{x^2 + y^2}$$
$$\theta = \arctan\left(\frac{y}{x}\right) \quad \text{(with quadrant adjustment)}$$

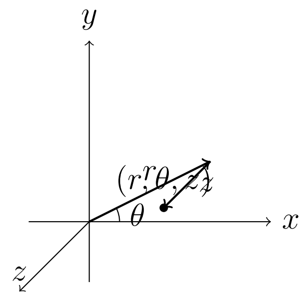
1.3.2 Cylindrical Coordinates

In 3D, cylindrical coordinates (r, θ, z) extend polar coordinates:

- r: distance from z-axis
- θ: angle from positive x-axis in xy-plane
- z: height above xy-plane

Conversion formulas:

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = z$$
$$r = \sqrt{x^2 + y^2}$$
$$\theta = \arctan\left(\frac{y}{x}\right)$$
$$z = z$$



1.4 Exercise Series N° 1: Vector Spaces and Operations

Exercise 1.1 (Vector Space Verification)

Verify if the following sets with given operations form vector spaces over $\mathbb{R}$:

1. $V = \{(x, y) \in \mathbb{R}^2 : x \geq 0\}$ with standard operations
2. $W = \{(x, y, z) \in \mathbb{R}^3 : x + y + z = 0\}$ with standard operations
3. $U = \{f : \mathbb{R} \to \mathbb{R} : f(0) = 1\}$ with pointwise addition and scalar multiplication

Solution: Example 1

$$V = \{(x, y) \in \mathbb{R}^2 : x \geq 0\}$$

1. **Closure under addition**: Take $(x_1, y_1), (x_2, y_2) \in V$ with $x_1 \geq 0$, $x_2 \geq 0$. Their sum is $(x_1 + x_2, y_1 + y_2)$. Since $x_1 + x_2 \geq 0$, closure holds.
2. **Closure under scalar multiplication**: Take $(x, y) \in V$ with $x \geq 0$ and $\lambda \in \mathbb{R}$. The product is $(\lambda x, \lambda y)$. But if $\lambda < 0$, then $\lambda x \leq 0$, which may violate $x \geq 0$ condition. For example, $\lambda = -1$ gives $(-x, -y) \notin V$ when $x > 0$.
3. **Conclusion**: V is **not** a vector space because it fails closure under scalar multiplication with negative scalars.

Solution: Example 2

$$W = \{(x, y, z) \in \mathbb{R}^3 : x + y + z = 0\}$$

1. **Contains zero vector**: $(0, 0, 0)$ satisfies $0 + 0 + 0 = 0$, so $0_W \in W$.