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Module: Applied Mathematics

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Exercise Series N° 1: Vector Spaces and Operations

Exercise 0.1 (Vector Space Verification)

Verify if the following sets with given operations form vector spaces over $\mathbb{R}$:

1. $V = \{(x, y) \in \mathbb{R}^2 : x \geq 0\}$ with standard operations
2. $W = \{(x, y, z) \in \mathbb{R}^3 : x + y + z = 0\}$ with standard operations
3. $U = \{f : \mathbb{R} \to \mathbb{R} : f(0) = 1\}$ with pointwise addition and scalar multiplication

Exercise 0.2 (Linear Combinations and Span)

For the vectors in $\mathbb{R}^3$: $\vec{v}_1 = (1, 2, 3)$, $\vec{v}_2 = (2, 5, 7)$, $\vec{v}_3 = (1, 3, 5)$

1. Determine if $\vec{w} = (4, 11, 18)$ is in the span of $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$
2. Find the general form of vectors in $span\{\vec{v}_1, \vec{v}_2\}$
3. Express $\vec{v}_3$ as a linear combination of $\vec{v}_1$ and $\vec{v}_2$ if possible

Exercise 0.3 (Linear Independence Testing)

Determine if the following sets are linearly independent:

1. $\{(1, 2, 3), (2, 5, 7), (1, 3, 5)\}$ in $\mathbb{R}^3$
2. $\{\sin x, \cos x, e^x\}$ in $\mathscr{F}(\mathbb{R}, \mathbb{R})$
3. $\{1, x, x^2, x^3\}$ in $\mathscr{P}_3(\mathbb{R})$

Exercise 0.4 (Basis and Dimension)

For each case, determine if the set forms a basis and find the dimension:

1. $\{(1, 1, 0), (1, 0, 1), (0, 1, 1)\}$ for $\mathbb{R}^3$
2. $\{1, 1 + x, 1 + x + x^2, 1 + x + x^2 + x^3\}$ for $\mathscr{P}_3(\mathbb{R})$
3. $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ for $M_{2\times 2}(\mathbb{R})$

Exercise 0.5 (Coordinate Systems and Transformations)

Perform the following coordinate transformations:

1. Convert $P(2, 2\sqrt{3})$ from Cartesian to polar coordinates
2. Convert $Q(4, \frac{\pi}{3}, 5)$ from cylindrical to Cartesian coordinates
3. Convert $R(2, \frac{\pi}{4}, \frac{\pi}{6})$ from spherical to Cartesian coordinates

Exercise 0.6 (Vector Operations in Different Coordinates)

Given vectors in different coordinate systems:

1. $\vec{u} = (3, \frac{\pi}{4})$ in polar, $\vec{v} = (2, \frac{\pi}{6})$ in polar - find $\vec{u} + \vec{v}$ in Cartesian
2. $\vec{a} = (2, \frac{\pi}{3}, 4)$ in cylindrical, $\vec{b} = (1, \frac{\pi}{6}, -2)$ in cylindrical - find dot product
3. $\vec{p} = (3, \frac{\pi}{4}, \frac{\pi}{3})$ in spherical, convert to Cartesian and find magnitude

Exercise 0.7 (Subspaces and Direct Sums)

For the following subspaces of $\mathbb{R}^3$:

1. $U = \{(x, y, z) : x + y + z = 0\}$, $V = \{(x, y, z) : x = y = z\}$ - find $U \cap V$ and $U + V$
2. $W = span\{(1, 2, 3), (2, 5, 7)\}$, $Z = span\{(1, 0, 1), (0, 1, 2)\}$ - find basis for $W \cap Z$
3. Show that $\mathbb{R}^3 = U \oplus W$ where $U = \{(x, y, 0)\}$, $W = \{(0, 0, z)\}$

Exercise 0.8 (Linear Transformations and Coordinates)

Consider the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T(x, y, z) = (x + y, y + z)$:

1. Find matrix representation relative to standard bases
2. Find kernel and image of T
3. If $B = \{(1, 1, 0), (1, 0, 1), (0, 1, 1)\}$ is basis for $\mathbb{R}^3$, find $[T]_B$

Exercise 0.9 (Applications to Geometry)

Solve the following geometric problems using vector methods:

1. Find distance between point $P(1, 2, 3)$ and plane $x + 2y + 3z = 6$
2. Find angle between lines $L_1 : (x, y, z) = (1, 2, 3) + t(1, 1, 1)$ and $L_2 : (x, y, z) = (0, 1, 2) + s(2, -1, 1)$
3. Find volume of parallelepiped determined by vectors $(1, 2, 3)$, $(2, 5, 7)$, $(1, 3, 5)$

Exercise 0.10 (Advanced Vector Space Problems)

Solve these advanced problems:

1. Show that $W = \{f \in \mathscr{F}(\mathbb{R}, \mathbb{R}) : f'' + f = 0\}$ is a subspace and find its dimension
2. Prove that if $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is basis for $\mathbb{R}^3$, then so is $\{\vec{v}_1 + \vec{v}_2, \vec{v}_2 + \vec{v}_3, \vec{v}_3 + \vec{v}_1\}$
3. Find basis for the solution space of the system:

$$\begin{cases} x + 2y + 3z = 0 \\ 2x + 5y + 7z = 0 \end{cases}$$