

Preface

These lessons are dedicated to second-year students in the Materials Science department, specializing in Chemistry, following the curriculum provided by the ministry. They can also help Physics students from the same department or Mathematics students and researchers in applied fields who need to understand the basic concepts of multivariable functions and multiple integrals.

They can serve as foundational lessons for the second year in the department and be a starting point for more advanced training in integral calculus. These lessons are written in a clear and simple manner to motivate students to learn the basic principles and concepts of mathematical analysis, attempting to simplify the definitions and explanations of differentiation and integration methods that require describing physical or chemical phenomena and knowing certain things about these phenomena, observations, or samples.

In these lessons, we have integrated well-solved examples and exercises. This work is divided into four chapters, the first of which is dedicated to simple and multiple integrals. The second chapter introduces integrals, while we discuss first and second-order differential equations in the third chapter, and finally we address numerical series, function series, and Fourier series in the last chapter.

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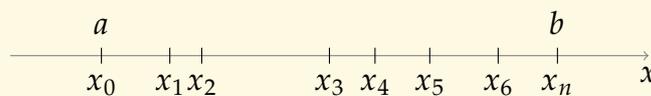
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Simple and Multiple Integrals **1**

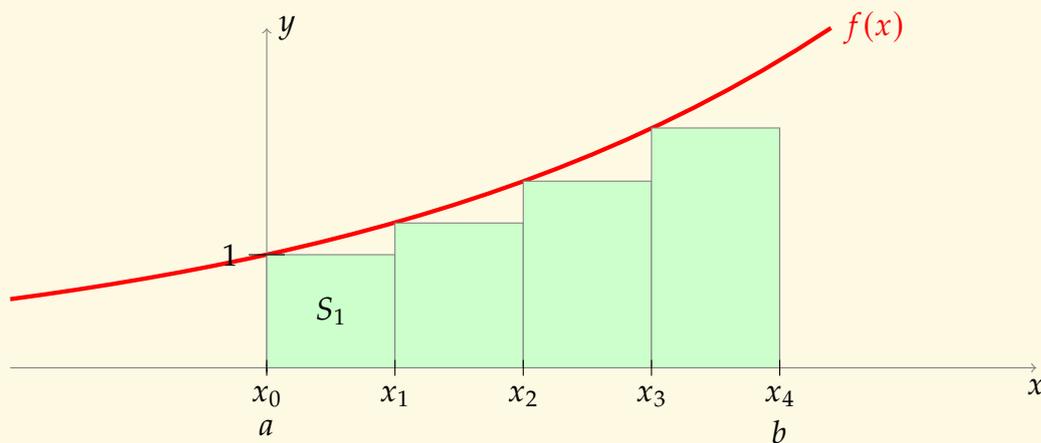
1.1 Riemann Integral, Definite Integral

Definition 1.1.1 Let $f : [a, b] \rightarrow \mathbb{R}$ be a function defined on the interval $[a, b]$. We divide the interval $[a, b]$ into n arbitrary parts, and let ξ_k be an arbitrary point where

$$\xi_k \in [x_{k-1}, x_k], 1 \leq k \leq n.$$



And let S_n be the sum of the areas of rectangles where each has height $f(\xi_k)$ and width $(x_k - x_{k-1})$.



$$S_n = \sum_{k=0}^n f(\xi_k)(x_k - x_{k-1})$$

If the sum S_n has a finite limit that does not depend on how the interval $[a, b]$ is divided when $n \rightarrow \infty$, then we denote this limit by

$$\int_a^b f(x) dx.$$

This limit is called the Riemann integral of the function f on the interval $[a, b]$. We say that f is Riemann integrable on the interval $[a, b]$ and write

$$\lim_{n \rightarrow \infty} S_n = \int_a^b f(x) dx.$$

1.1.1 Properties

The three main properties of the integral are the Chasles relation, positivity, and linearity.

Chasles Relation

Proposition 1.1.1 Let $a < c < b$. If f is integrable on the interval $[a, c]$ and the interval $[c, b]$, then f is integrable on the interval $[a, b]$ and we write

$$\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx.$$

We also have

$$\int_a^a f(x) dx = 0 \quad \text{and for } a < b \quad \int_b^a f(x) dx = - \int_a^b f(x) dx.$$

Positivity of the Integral

Proposition 1.1.2 Let $a \leq b$ be real numbers and f and g be functions integrable on the interval $[a, b]$. If $f \leq g$ then

$$\int_a^b f(x) dx \leq \int_a^b g(x) dx.$$

In particular, the integral of a positive function is positive: if $f \geq 0$ then

$$\int_a^b f(x) dx \geq 0$$

Linearity of the Integral

Proposition 1.1.3 Let f, g be functions integrable on the interval $[a, b]$.

(1) $f + g$ is integrable and we have

$$\int_a^b (f + g)(x) dx = \int_a^b f(x) dx + \int_a^b g(x) dx.$$

(2) For every λ , λf is integrable and we have

$$\int_a^b \lambda f(x) dx = \lambda \int_a^b f(x) dx.$$

Thus we have the linearity of the integral, i.e., for every λ, μ

$$\int_a^b (\lambda f(x) + \mu g(x)) dx = \lambda \int_a^b f(x) dx + \mu \int_a^b g(x) dx$$

(3) $f \cdot g$ is integrable on the interval $[a, b]$

But in general we have

$$\int_a^b (fg)(x) dx \neq \left(\int_a^b f(x) dx \right) \left(\int_a^b g(x) dx \right).$$

(4) $|f|$ is integrable on the interval $[a, b]$ and we have

$$\left| \int_a^b f(x) dx \right| \leq \int_a^b |f(x)| dx$$

Example 1.1.1

$$\int_0^1 (7x^2 - e^x) dx = 7 \int_0^1 x^2 dx - \int_0^1 e^x dx = 7 \frac{1}{3} - (e - 1) = \frac{10}{3} - e$$

We used the calculations we saw before: $\int_0^1 x^2 dx = \frac{1}{3}$ and $\int_0^1 e^x dx = e - 1$.

Example 1.1.2 Let $I_n = \int_1^n \frac{\sin(nx)}{1+x^n} dx$. Prove that $I_n \rightarrow 0$ when $n \rightarrow +\infty$.

$$|I_n| = \left| \int_1^n \frac{\sin(nx)}{1+x^n} dx \right| \leq \int_1^n \frac{|\sin(nx)|}{1+x^n} dx \leq \int_1^n \frac{1}{1+x^n} dx \leq \int_1^n \frac{1}{x^n} dx$$

It remains only to compute this last integral:

$$\int_1^n \frac{1}{x^n} dx = \int_1^n x^{-n} dx = \left[\frac{x^{-n+1}}{-n+1} \right]_1^n = \frac{n^{-n+1}}{-n+1} - \frac{1}{-n+1} \xrightarrow{n \rightarrow +\infty} 0$$

Because $(n^{-n+1} \rightarrow 0$ and $\frac{1}{-n+1} \rightarrow 0)$.

Remark 1.1.1 Note that although $f \cdot g$ is integrable, we generally have

$$\int_a^b (fg)(x) dx \neq \left(\int_a^b f(x) dx \right) \left(\int_a^b g(x) dx \right).$$

For example, let $f : [0, 1] \rightarrow \mathbb{R}$ be the function defined by $f(x) = 1$ if $x \in [0, \frac{1}{2}[$ and $f(x) = 0$ for $x \notin [0, \frac{1}{2}[$. Let $g : [0, 1] \rightarrow \mathbb{R}$ be the function defined by $g(x) = 1$ if $x \in [\frac{1}{2}, 1[$ and $g(x) = 0$ for $x \notin [\frac{1}{2}, 1[$.

Then $f(x) \times g(x) = 0$ for every $x \in [0, 1]$ and $\int_0^1 f(x)g(x) dx = 0$ where $\int_0^1 f(x) dx = \frac{1}{2}$ and $\int_0^1 g(x) dx = \frac{1}{2}$.

1.2 Calculation of Antiderivatives

Definition 1.2.1 Let $I = [a, b]$ be an interval in \mathbb{R} and let f be a function where

$$f : I \rightarrow \mathbb{R}$$

We say that F is an antiderivative of the function f where

$$F : I \rightarrow \mathbb{R}$$

if the following conditions are satisfied:

- (1) F is differentiable on the open interval I .
- (2)

$$\forall x \in I, F'(x) = f(x)$$

Theorem 1.2.1 Every continuous function $f : I \rightarrow \mathbb{R}$ has an antiderivative.

Theorem 1.2.2 Let the function $f : I \rightarrow \mathbb{R}$ where f has an antiderivative.

The set of antiderivatives of the function f is

$$\{F + c, c \in \mathbb{R}\}$$

where F is a particular antiderivative of the function f .

We denote by $\int f(t)dt$ the antiderivative of the function f and write:

$$F(x) = \int f(x)dx$$

1.2.1 Definite Integral

Let the function $f : [a, b] \rightarrow \mathbb{R}$ be continuous on the interval $[a, b]$ where $b \geq a$.

The integral can be defined in another way more commonly used in finding constant values for integrals through the following theorem:

Theorem 1.2.3 Let the function $F : [a, b] \rightarrow \mathbb{R}$ defined by

$$F(x) = \int_a^x f(t) dt$$

is an antiderivative of the function f , meaning that the function F is differentiable and satisfies:

$$F'(x) = f(x), \forall x \in [a, b]$$

Definition 1.2.2 We call the definite integral of the function f , which we denote by

$$\int_a^b f(x) dx$$

the real number $F(b) - F(a)$ where F is an antiderivative of the function f , and we write

$$\int_a^b f(x) dx = F(b) - F(a).$$

Example 1.2.1 Let's compute the following integrals:

(1) For $f(x) = e^x$, let $F(x) = e^x$ be its antiderivative, then

$$\int_0^1 e^x dx = [e^x]_0^1 = e^1 - e^0 = e - 1.$$

(2) For $g(x) = x^2$, let $G(x) = \frac{x^3}{3}$ be its antiderivative, then

$$\int_0^1 x^2 dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}.$$

(3)

$$\int_a^x \cos t dt = [\sin t]_{t=a}^{t=x} = \sin x - \sin a$$

is an antiderivative of the function $\cos x$.

(4) If a function is odd, its antiderivative is even (to be proven later) and we deduce that

$$\int_{-a}^a f(t) dt = 0.$$

1.2.2 Integration Methods

Integration by Parts

Theorem 1.2.4 Let u and v be functions of class \mathcal{C}^1 defined on the interval $[a, b]$. Then:

$$\int_a^b u(x)v'(x) dx = [uv]_a^b - \int_a^b u'(x)v(x) dx$$

The integration by parts formula for the antiderivative is the same but without limits:

$$\int u(x)v'(x) dx = [uv] - \int u'(x)v(x) dx.$$

Example 1.2.2 To compute the integral

$$\int_0^1 xe^x dx$$

we set $u(x) = x$ and $v'(x) = e^x$.

We know that the function $u'(x) = 1$ is the derivative of $u(x)$ and the function $v(x) = e^x$ is an antiderivative of v' . Using the integration by parts formula we find:

$$\begin{aligned} \int_0^1 xe^x dx &= \int_0^1 u(x)v'(x) dx \\ &= [u(x)v(x)]_0^1 - \int_0^1 u'(x)v(x) dx \\ &= [xe^x]_0^1 - \int_0^1 1 \cdot e^x dx \\ &= (1 \cdot e^1 - 0 \cdot e^0) - [e^x]_0^1 \\ &= e - (e^1 - e^0) \\ &= 1 \end{aligned}$$

Example 1.2.3 To compute the integral

$$\int_1^e x \ln x dx.$$

This time we set $u(x) = \ln x$ and $v'(x) = x$.

Then the function $u' = \frac{1}{x}$ is the derivative of $u(x)$ and the function $v = \frac{x^2}{2}$ is an antiderivative of v' . Using the integration by parts formula we find:

$$\begin{aligned}
\int_1^e \ln x \cdot x \, dx &= \int_1^e uv' = [uv]_1^e - \int_1^e u'v = \left[\ln x \cdot \frac{x^2}{2} \right]_1^e - \int_1^e \frac{1}{x} \frac{x^2}{2} \, dx \\
&= \left(\ln e \frac{e^2}{2} - \ln 1 \frac{1^2}{2} \right) - \frac{1}{2} \int_1^e x \, dx = \frac{e^2}{2} - \frac{1}{2} \left[\frac{x^2}{2} \right]_1^e \\
&= \frac{e^2}{2} - \frac{e^2}{4} + \frac{1}{4} = \frac{e^2 + 1}{4}
\end{aligned}$$

Example 1.2.4 To compute the integral

$$\int \arcsin x \, dx$$

To find an antiderivative of $\arcsin(x)$, we express it as a product by setting $u(x) = \arcsin(x)$ and $v'(x) = 1$, where we have $u'(x) = \frac{1}{\sqrt{1-x^2}}$ and $v(x) = x$. Then applying the integration by parts formula we find:

$$\begin{aligned}
\int 1 \cdot \arcsin(x) \, dx &= [x \arcsin(x)] - \int \frac{x}{\sqrt{1-x^2}} \, dx \\
&= [x \arcsin(x)] - \left[-\sqrt{1-x^2} \right] \\
&= x \arcsin(x) + \sqrt{1-x^2} + c
\end{aligned}$$

Example 1.2.5 Compute the integral

$$\int x^2 e^x \, dx.$$

We set $u(x) = x^2$ and $v'(x) = e^x$.

We know that the function $u'(x) = 2x$ is the derivative of $u(x)$ and the function $v(x) = e^x$ is an antiderivative of $v'(x)$. Using the integration by parts formula we find:

$$\int x^2 e^x \, dx = [x^2 e^x] - 2 \int x e^x \, dx$$

Applying integration by parts a second time to the second part of the equation:

$$\int x e^x \, dx = [x e^x] - \int e^x \, dx = (x-1)e^x + c$$

Finally we find

$$\int x^2 e^x \, dx = (x^2 - 2x + 2)e^x + c.$$

Integration by Change of Variable

Theorem 1.2.5 *If f is a function defined on the interval $I = [a, b]$ and let $\varphi : J \rightarrow I$ be a bijection of class \mathcal{C}^1 . For every $a, b \in J$ we have:*

$$\int_{\varphi(a)}^{\varphi(b)} f(x) dx = \int_a^b f(\varphi(t)) \cdot \varphi'(t) dt$$

If F is an antiderivative of f , then $F \circ \varphi$ is an antiderivative of $(f \circ \varphi) \cdot \varphi'$. In other words:

$$\left(\int f(x) dx \right) \circ \varphi = \int f(\varphi(t)) \varphi'(t) dt$$

The expression $\int f(x) dx = \int f(\varphi(t)) \varphi'(t) dt$ actually represents a change of variable, or more simply we set $x = \varphi(t)$ and then by differentiation we find $\frac{dx}{dt} = \varphi'(t)$ i.e., $dx = \varphi'(t) dt$, which gives us:

$$\int_{\varphi(a)}^{\varphi(b)} f(x) dx = \int_a^b f(\varphi(t)) \varphi'(t) dt.$$

Example 1.2.6 Compute the integral

$$\int_0^{\frac{\pi}{2}} \sin^2(x) \cos(x) dx$$

by setting

$$\sin(x) = t \implies \sin(x)' = \cos(x) = dt$$

Then the integration limits change from x to t as follows:

$$\begin{aligned} x = 0 &\implies t = \sin(0) = 0 \\ x = \frac{\pi}{2} &\implies t = \sin\left(\frac{\pi}{2}\right) = 1 \end{aligned}$$

Then we find:

$$\begin{aligned} \int_0^{\frac{\pi}{2}} \sin^2(x) \cos(x) dx &= \int_0^1 t^2 dt \\ &= \left. \frac{1}{3} t^3 \right|_0^1 \\ &= \frac{1}{3}. \end{aligned}$$

1.3 Review of Multivariable Functions

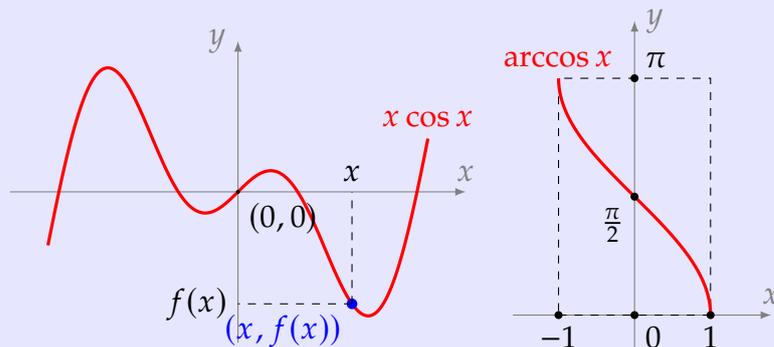
In this section we will study functions of multiple variables defined on \mathbb{R}^2 or \mathbb{R}^3 , which can also be studied in the general framework i.e., on \mathbb{R}^n , so these functions will be of the form:

$$f : E \subset \mathbb{R}^n \rightarrow \mathbb{R},$$

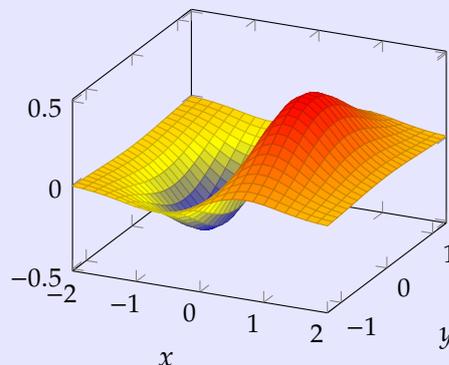
where $n \geq 1$ is a natural number.

In other words, the elements of the starting set E will be vectors of the form $x = (x_1, \dots, x_n)$, and the elements of the final set will be real numbers.

Example 1.3.1 (1) $n = 1, f : I \subset \mathbb{R} \rightarrow \mathbb{R}$: This is the simplest case, $x \mapsto f(x)$. Below are the graphs of the functions $x \mapsto x \cos x$ and $x \mapsto \arccos x$:



(2) $n = 2, f : E \subset \mathbb{R}^2 \rightarrow \mathbb{R}$. We denote the variables by (x, y) . Functions $(x, y) \mapsto f(x, y)$ are represented, for example, by surfaces:



A curve representing the function $(x, y) \mapsto -x \cdot e^{-x^2 - y^2}$.

Once $n > 2$, it becomes very difficult to obtain a graphical visualization of functions of several variables.

1.3.1 Limits

The concept of limits and continuity for functions of one variable can be generalized to functions of several variables without complication; it suffices to replace the absolute value with the Euclidean norm.

Let f be a function $f : E \subset \mathbb{R}^n \rightarrow \mathbb{R}$ defined in a neighborhood of the point $x_0 \in \mathbb{R}^n$ except possibly at x_0 .

Definition 1.3.1 The function has a limit equal to the real number ℓ when x approaches x_0 if:

$$\forall \epsilon > 0, \quad \exists \delta > 0, \quad \forall x \in E : \quad 0 < \|x - x_0\| < \delta \implies |f(x) - \ell| < \epsilon$$

And we write

$$\lim_{x \rightarrow x_0} f(x) = \ell \quad \text{or} \quad f(x) \xrightarrow{x \rightarrow x_0} \ell$$

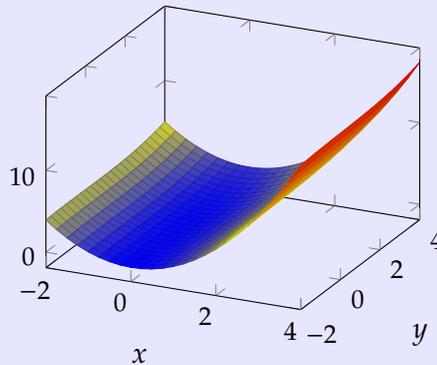
Similarly, we define the limit at infinity $\lim_{x \rightarrow x_0} f(x) = +\infty$ as follows:

$$\forall A > 0 \quad \exists \delta > 0 \quad \forall x \in E : \quad 0 < \|x - x_0\| < \delta \implies |f(x)| > A$$

Example 1.3.2 Let the function f be defined as follows

$$f(x, y) = x^2 + y \sin(x + y^2).$$

(1) Let's prove that f approaches 0 when $(x, y) \rightarrow (0, 0)$.



The function $f(x, y)$ is bounded using $|\sin(t)| \leq 1$: we find

$$|f(x, y)| = |x^2 + y \sin(x + y^2)| \leq x^2 + |y| |\sin(x + y^2)| \leq x^2 + |y|$$

Take $0 < \epsilon < 1$, $a = \sqrt{\frac{\epsilon}{2}}$ and $b = \frac{\epsilon}{2}$, then for $x \in]-a, a[$, we have $x^2 < \frac{\epsilon}{2}$, and for $y \in]-b, b[$ we have $|y| < \frac{\epsilon}{2}$.

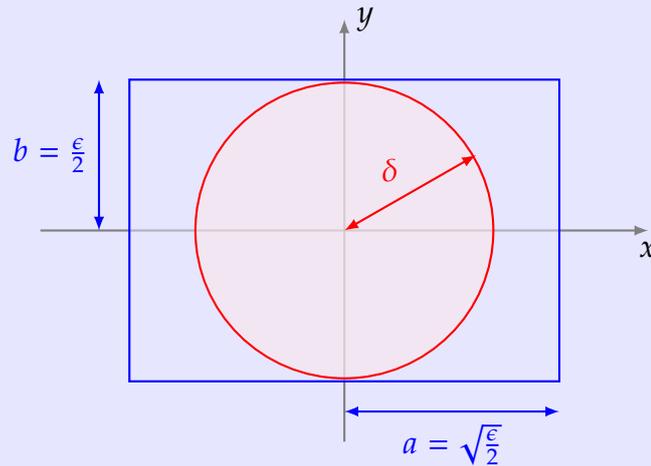
Then for $(x, y) \in]-a, a[\times]-b, b[$ we find:

$$|f(x, y)| \leq x^2 + |y| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

One of the values of δ that satisfies the limit is $\delta = \frac{\epsilon}{2}$. Indeed, if $\|(x, y)\| < \delta$ then $|x| < \delta = \frac{\epsilon}{2} \leq \sqrt{\frac{\epsilon}{2}}$ and $|y| < \delta = \frac{\epsilon}{2}$, so $|f(x, y)| < \epsilon$. We conclude: f has a limit of 0 when (x, y) approaches $(0, 0)$.

(2) Find an open set U containing 0 such that for every $(x, y) \in U$ we have $|f(x, y)| < \frac{1}{100}$.

For $\epsilon = \frac{1}{100}$ we have $a = \frac{1}{\sqrt{200}}$ and $b = \frac{1}{200}$. For every (x, y) in the set $] - a, a[\times] - b, b[$, we have $|f(x, y)| < \frac{1}{100}$.



Operations on Limits

The definition is rarely used in calculating limits; instead, we use general theorems: operations on limits for functions of several variables present no difficulty or novelty.

Proposition 1.3.1 Let $f, g : \mathbb{R}^n \rightarrow \mathbb{R}$ be defined in a neighborhood of $x_0 \in \mathbb{R}^n$ where f and g have limits at x_0 . We have the following properties:

$$\begin{aligned} \lim_{x_0} (f + g) &= \lim_{x_0} f + \lim_{x_0} g, & \lim_{x_0} (fg) &= \lim_{x_0} f \times \lim_{x_0} g \\ \lim_{x_0} \frac{1}{g} &= \frac{1}{\lim_{x_0} g}, & \lim_{x_0} \frac{f}{g} &= \frac{\lim_{x_0} f}{\lim_{x_0} g} \end{aligned}$$

1.4 Multiple Integrals

Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a function defined on the bounded set $D \subset \mathbb{R}^2$.

Definition 1.4.1 For every $\delta > 0$, we call a **partition of the set D** the set \mathcal{S}_δ of squares K_i of side δ that cover or include D in any step of the δ fence. We consider the two subpartitions:

- ▶ \mathcal{S}_δ^{ext} means **external covering**,
- ▶ \mathcal{S}_δ^{int} means **strict covering** (internal).

Since D is bounded, the subpartition contains a finite number of squares, and we have $\mathcal{S}_\delta^{int} \subset \mathcal{S}_\delta^{ext}$. In fact, the squares in the set $\mathcal{S}_\delta^{ext} \setminus \mathcal{S}_\delta^{int}$ cover exactly the boundary ∂D of D .

For any choice of points $(x_i, y_i) \in K_i \cap D$, we call the **Riemann sum** of f associated with the subpartition $\mathcal{S}_\delta^{ext/int}$ and the points $\{(x_i, y_i)\}$ the sums:

$$R_\delta^{ext/int}(f, \{(x_i, y_i)\}) = \sum_{K_i \in \mathcal{S}_\delta^{ext/int}} f(x_i, y_i) \delta^2,$$

where each term $f(x_i, y_i) \delta^2$ represents the **algebraic volume** of the parallelepiped with base K_i and height $f(x_i, y_i)$ with sign \pm that is the sign of f at (x_i, y_i) .

Definition 1.4.2 If the limits

$$\lim_{\delta \rightarrow 0} R_\delta^{ext/int}(f; \{(x_i, y_i)\})$$

exist, they are independent of the choice of points $(x_i, y_i) \in K_i \cap D$ and are identical. In this case we call the **double integral of f over D** this limit:

$$\iint_D f(x, y) dx dy = \lim_{\delta \rightarrow 0} R_\delta^{ext/int}(f; \{(x_i, y_i)\}).$$

We say that f is **Riemann integrable on D** if the integral $\iint_D f(x, y) dx dy$ is finite (a real number and not $\pm\infty$).

In the special case where the function f is continuous and D is bounded.

Definition 1.4.3 The double integral is a type of definite integral extended to functions of two variables. If the function $f(x, y)$ is defined in \mathbb{D} of the (xOy) plane, then

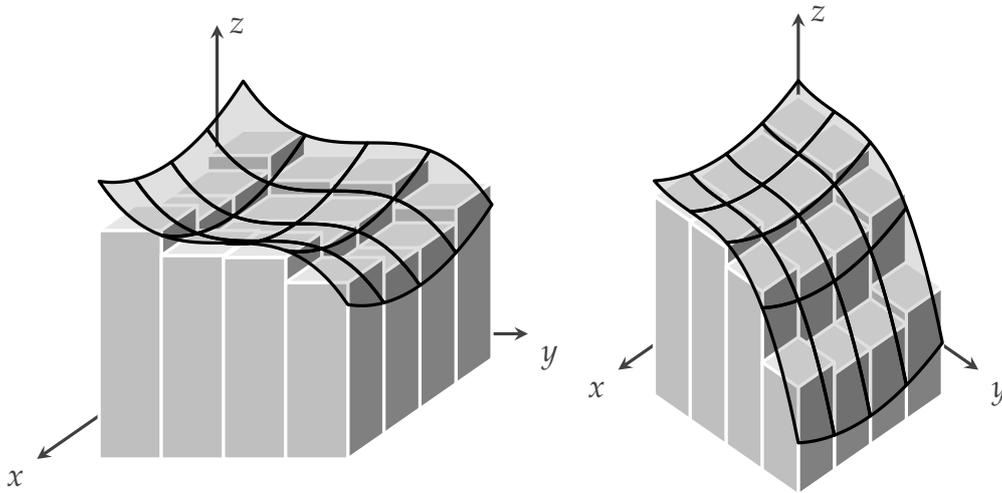
$$\iint_{\mathbb{D}} f(x, y) dx dy$$

is called the **double integral or multiple integral of the function $f(x, y)$ in \mathbb{D}** .

The importance of double integrals lies in finding surface areas, average centers, moments of inertia of plane surfaces, finding volumes under surfaces, and in electromagnetism, heat, sound waves, mechanics, and other topics.

To solve the double integral shown in the formula above, we first start with the inner integral which we integrate with respect to x considering the variable y constant, then we find the value of the outer integral which we integrate with respect to y .

Geometric Meaning of the Double Integral



Corollary 1.4.1 (1) The value $\iint_D f(x, y) \, dx \, dy$ is the algebraic volume of the portion between the graph of f and the xOy plane.
 (2) While $\iint_D |f(x, y)| \, dx \, dy$ is the volume of the portion of space between the graph of f and the xOy plane.

Example 1.4.1 The volume of the sphere

$$B = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 \leq 1\}$$

is twice the volume of the hemisphere

$$B^+ = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 \leq 1, z \geq 0\},$$

bounded between the graph of the function $z = \sqrt{1 - x^2 - y^2}$ and the xOy axis. We have therefore:

$$\text{Volume}(B) = 2 \iint_D \sqrt{1 - x^2 - y^2} \, dx \, dy, \quad \text{where } D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}.$$

To compute double integrals, we use the following properties and two specific methods.

Calculation of Double Integral - Fubini's Theorem

First case: Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a continuous function defined on the rectangle $D = [a, b] \times [c, d]$.

Theorem 1.4.2 $\iint_D f(x, y) dx dy = \int_a^b \left(\int_c^d f(x, y) dy \right) dx = \int_c^d \left(\int_a^b f(x, y) dx \right) dy.$

Corollary 1.4.3

$$\iint_{[a,b] \times [c,d]} f_1(x) f_2(y) dx dy = \int_a^b f_1(x) dx \int_c^d f_2(y) dy.$$

Remark 1.4.1 We can also write the expression:

$$\int_a^b dx \int_c^d dy f(x, y) = \int_a^b \left(\int_c^d f(x, y) dy \right) dx.$$

Example 1.4.2 .

(1) Let's compute the following double integral:

$$\begin{aligned} \iint_{[0,1] \times [0,\pi/2]} x \cos y dx dy &= \int_0^1 x dx \int_0^{\pi/2} \cos y dy \\ &= \left[\frac{1}{2} x^2 \right]_0^1 [\sin y]_0^{\pi/2} = \frac{1}{2}. \end{aligned}$$

(2) Let's compute the integral:

$$\begin{aligned} \iint_{[-1,1] \times [0,1]} (x^2 y - 1) dx dy &= \int_{-1}^1 dx \int_0^1 (x^2 y - 1) dy \\ &= \int_{-1}^1 dx \left[\frac{1}{2} x^2 y^2 - y \right]_{y=0}^{y=1} \\ &= \int_{-1}^1 \left(\frac{1}{2} x^2 - 1 \right) dx \\ &= \left[\frac{1}{6} x^3 - x \right]_{-1}^1 = -\frac{5}{3}. \end{aligned}$$

Second case: Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a continuous function defined on an arbitrary bounded set D , then:

- (1) For every $(x, y) \in D$ there exist values $a, b \in \mathbb{R}$ where $a \leq x \leq b$
- (2) For every $x \in [a, b]$ there exist values $c(x), d(x) \in \mathbb{R}$ where $c(x) \leq y \leq d(x)$

Such that:

$$D = \{ (x, y) \in \mathbb{R}^2 \mid x \in [a, b], y \in [c(x), d(x)] \}.$$

Note that the curves:

$$\partial D^- = \{ (x, y) \in \mathbb{R}^2 \mid x \in [a, b], y = c(x) \}$$

and:

$$\partial D^+ = \{ (x, y) \in \mathbb{R}^2 \mid x \in [a, b], y = d(x) \}$$

include the entire boundary of D . Conversely:

- (1) For every $(x, y) \in D$ there exist values $c, d \in \mathbb{R}$ where $c \leq y \leq d$
- (2) For every $y \in [c, d]$, there exist values $a(y), b(y) \in \mathbb{R}$ where $a(y) \leq x \leq b(y)$

Such that:

$$D = \{ (x, y) \in \mathbb{R}^2 \mid y \in [c, d], x \in [a(y), b(y)] \}.$$

In this case, the two curves are:

$$\partial D^- = \{ (x, y) \in \mathbb{R}^2 \mid y \in [c, d], x = a(y) \}$$

and:

$$\partial D^+ = \{ (x, y) \in \mathbb{R}^2 \mid y \in [c, d], x = b(y) \}$$

which include the entire boundary of D .

Based on the choice we make to describe D , we then have the following theorem:

Theorem 1.4.4

$$\iint_D f(x, y) \, dx \, dy = \int_a^b \left(\int_{c(x)}^{d(x)} f(x, y) \, dy \right) dx = \int_c^d \left(\int_{a(y)}^{b(y)} f(x, y) \, dx \right) dy$$

Example 1.4.3 Suppose D is the part of the xOy plane bounded below by the parabola arc $y = x^2$ and above by the line $y = 1$. We can then describe D as the set:

$$D = \{ (x, y) \in \mathbb{R}^2 \mid x \in [-1, 1], y \in [x^2, 1] \}.$$

Therefore we have:

$$\begin{aligned}
 \iint_D x^2 y \, dx \, dy &= \int_{-1}^1 x^2 dx \int_{x^2}^1 y \, dy \\
 &= \int_{-1}^1 x^2 \left[\frac{1}{2} y^2 \right]_{x^2}^1 dx \\
 &= \int_{-1}^1 \frac{1}{2} (x^2 - x^4) \, dx \\
 &= \frac{1}{2} \left[\frac{1}{3} x^3 - \frac{1}{5} x^5 \right]_{x=-1}^{x=1} = \frac{2}{15}.
 \end{aligned}$$

Calculation of Double Integral - Change of Variable

Consider the double integral:

$$\iint_D f(x, y) \, dx \, dy$$

and the change of variable:

$$(x, y) = h(u, v) = (x(u, v), y(u, v)).$$

To express the integral using the function $\tilde{f}(u, v) = f(x(u, v), y(u, v))$ we must express D and the product $dx \, dy$ in terms of (u, v) : thus

(1) We transform the region D into the region:

$$\tilde{D} = h^{-1}(D) = \{(u, v) \in \mathbb{R}^2 \mid (x, y) = h(u, v) \in D\}.$$

(2) The elements dx and dy transform to:

$$\begin{cases} dx = \frac{\partial x}{\partial u} du + \frac{\partial x}{\partial v} dv \\ dy = \frac{\partial y}{\partial u} du + \frac{\partial y}{\partial v} dv \end{cases} \quad \text{i.e.,} \quad \begin{pmatrix} dx \\ dy \end{pmatrix} = J_h(u, v) \begin{pmatrix} du \\ dv \end{pmatrix} \quad \text{where} \quad J_h(u, v) = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}$$

is the Jacobian matrix of the coordinate change.

It suffices to adopt the following formula, with the absolute value of the Jacobian determinant:

$$dx \, dy = \left| \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u} \right| du \, dv = \left| \det J_h(u, v) \right| du \, dv.$$

In particular, for the change to polar coordinates we have:

$$dx \, dy = \rho \, d\rho \, d\varphi.$$

We finally arrive at the following theorem:

Theorem 1.4.5

$$\iint_D f(x, y) dx dy = \iint_{h^{-1}(D)} f(x(u, v), y(u, v)) |\det J_h(u, v)| du dv.$$

Example 1.4.4 For:

$$B = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 \leq 1\}$$

we compute:

$$\text{Volume}(B) = 2 \iint_D \sqrt{1 - x^2 - y^2} dx dy, \quad \text{where } D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$$

with the change of variables to polar coordinates:

$$(x, y) = h(\rho, \varphi) = (\rho \cos \varphi, \rho \sin \varphi).$$

Because $x^2 + y^2 = \rho^2$ we have:

$$\begin{aligned} \sqrt{1 - x^2 - y^2} &= \sqrt{1 - \rho^2} \text{ and} \\ h^{-1}(B) &= \{(\rho, \varphi) \in [0, \infty[\times [0, 2\pi[\mid \rho \leq 1\} = [0, 1] \times [0, 2\pi[\end{aligned}$$

Therefore, knowing that $dx dy = \rho d\rho d\varphi$ and using Fubini's theorem to separate variables, we obtain:

$$\text{Volume}(B) = 2 \iint_{[0,1] \times [0,2\pi[} \sqrt{1 - \rho^2} \rho d\rho d\varphi = 2 \int_0^1 \sqrt{1 - \rho^2} \rho d\rho \int_0^{2\pi} d\varphi.$$

The integral in φ is simple: $\int_0^{2\pi} d\varphi = [\varphi]_0^{2\pi} = 2\pi$. For the other part, if we set $t = 1 - \rho^2$ we have:

$$\begin{aligned} \rho = 0 &\implies t = 1 \quad \text{and} \quad \rho = 1 \implies t = 0, \\ \sqrt{1 - \rho^2} &= \sqrt{t} = t^{1/2}, \\ dt = -2\rho d\rho &\implies \rho d\rho = -\frac{1}{2} dt, \end{aligned}$$

and finally we get:

$$\text{Volume}(B) = -\frac{2}{2} 2\pi \int_1^0 t^{1/2} dt = 2\pi \int_0^1 t^{1/2} dt = 2\pi \left[\frac{1}{\frac{1}{2} + 1} t^{\frac{1}{2} + 1} \right]_0^1 = 2\pi \frac{2}{3} \left[t^{\frac{3}{2}} \right]_0^1 = \frac{4\pi}{3}.$$

Example 1.4.5 Compute the value of the double integral:

$$\iint_{\mathbb{D}} (3y^2 - x) \, dx \, dy$$

given that:

$$\mathbb{D} = \{(x, y) : 0 \leq x \leq 2, 1 \leq y \leq 2\}$$

Then:

$$\begin{aligned} \iint_{\mathbb{D}} (3y^2 - x) \, dx \, dy &= \int_1^2 \left(\int_0^2 (3y^2 - x) \, dx \right) dy \\ &= \int_1^2 \left. 3xy^2 - \frac{1}{2}x^2 \right|_0^2 dy \\ &= \int_1^2 (6y^2 - 2) \, dy \\ &= 2y^3 - 2y \Big|_1^2 \\ &= 12 \end{aligned}$$

Example 1.4.6 Compute the value of the double integral:

$$\int_1^2 \int_y^{y^2} dx \, dy.$$

Then:

$$\begin{aligned} \int_1^2 \int_y^{y^2} dx \, dy &= \int_1^2 x \Big|_y^{y^2} dy \\ &= \int_1^2 (y^2 - y) \, dy \\ &= \left. \frac{1}{3}y^3 - \frac{1}{2}y^2 \right|_1^2 \\ &= \frac{5}{6}. \end{aligned}$$

Example 1.4.7 Compute the value of the double integral:

$$\int_0^\pi \int_0^x x \sin y \, dy \, dx$$

Then:

$$\begin{aligned}
 \int_0^\pi \int_0^x x \sin y \, dy \, dx &= \int_0^\pi \left(\int_0^x x \sin y \, dy \right) dx \\
 &= \int_0^\pi -x \cos y \Big|_0^x dx \\
 &= \int_0^\pi -x (\cos x - 1) dx \\
 &= \frac{1}{2} x^2 - x \sin x - \cos x \Big|_0^\pi \\
 &= \frac{\pi^2}{2} + 2.
 \end{aligned}$$

1.5 Triple Integrals

Let $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ be a function of three variables (x, y, z) and let $\Omega \subset \mathbb{R}^3$ be a bounded set on which the function f is defined.

Definition 1.5.1 We define the triple integral of the function f over Ω as the limit of the Riemann sum associated with the partition \mathcal{S}_δ of Ω into small cubes K_i of dimensions δ^3 where δ approaches zero:

$$\iiint_{\Omega} f(x, y, z) \, dx \, dy \, dz = \lim_{\delta \rightarrow 0} \sum_{K_i \in \mathcal{S}_\delta} f(x_i, y_i, z_i) \delta^3,$$

for any choice of points $(x_i, y_i, z_i) \in K_i$.

This definition is analogous to the definition of double integrals in dimension 3. Thus, triple integrals have exactly the same properties as double integrals, and the same existence theorems (f continuous on bounded Ω).

The geometric meaning of the triple integral is more abstract: by analogy, the (algebraic) volume of the portion of space between the graph of f and the xOy plane becomes the (algebraic) quadruple volume of the portion of quadruple space between the graph of f and the $Oxyz$ space.

Calculation of Triple Integral - Fubini's Theorem

Theorem 1.5.1 (1) If $\Omega = [a, b] \times [c, d] \times [e, g]$ is a parallelepiped, then:

$$\iiint_{\Omega} f(x, y, z) \, dx \, dy \, dz = \int_a^b dx \int_c^d dy \int_e^g dz f(x, y, z) \quad \text{in any desired order.}$$

(2) If:

$$\Omega = \left\{ (x, y, z) \in \mathbb{R}^3 \mid x \in [a, b], y \in [c(x), d(x)], z \in [e(x, y), g(x, y)] \right\}$$

is any bounded set, then:

$$\iiint_{\Omega} f(x, y, z) dx dy dz = \int_a^b dx \int_{c(x)}^{d(x)} dy \int_{e(x, y)}^{g(x, y)} dz f(x, y, z) \quad \text{forced order.}$$

Example 1.5.1 (1) Let's compute the integral:

$$\begin{aligned} \iiint_{[0,1] \times [1,2] \times [2,3]} (x^2 - 2yz) dx dy dz &= \int_2^3 dz \int_1^2 dy \int_0^1 dx (x^2 - 2yz) \\ &= \int_2^3 dz \int_1^2 dy \left[\frac{1}{3}x^3 - 2xyz \right]_{x=0}^{x=1} \\ &= \int_2^3 dz \int_1^2 dy \left(\frac{1}{3} - 2yz \right) \\ &= \int_2^3 \left[\frac{1}{3}y - y^2z \right]_{y=1}^{y=2} dz = \int_2^3 \left(\frac{2}{3} - 4z - \frac{1}{3} + z \right) dz \\ &= \int_2^3 \left(\frac{1}{3} - 3z \right) dz = \left[\frac{1}{3}z - \frac{3}{2}z^2 \right]_2^3 \\ &= \frac{3}{3} - \frac{27}{2} - \frac{2}{3} + \frac{12}{2} = \frac{1}{3} - \frac{15}{2} \\ &= -\frac{43}{6}. \end{aligned}$$

(2) If Ω is the complete cylinder, with base the disk $D = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 \leq 1, z = 0\}$ of height 3, we can write:

$$\begin{aligned} \Omega &= \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 \leq 1, 0 \leq z \leq 3\} \\ &= \{(x, y, z) \in \mathbb{R}^3 : x \in [-1, 1], y \in [-\sqrt{1-x^2}, \sqrt{1-x^2}], z \in [0, 3]\} \end{aligned}$$

Therefore:

$$\begin{aligned}
\iiint_{\Omega} (1 - 2yz) \, dx \, dy \, dz &= \int_0^3 dz \iint_D (1 - 2yz) \, dx \, dy \\
&= \int_0^3 dz \int_{-1}^1 dx \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (1 - 2yz) \, dy \\
&= \int_0^3 dz \int_{-1}^1 [y - y^2 z]_{y=-\sqrt{1-x^2}}^{y=\sqrt{1-x^2}} dx \\
&= \int_0^3 dz \int_{-1}^1 \left(\sqrt{1-x^2} - (1-x^2)z + \sqrt{1-x^2} + (1-x^2)z \right) dx \\
&= \int_0^3 dz \int_{-1}^1 2\sqrt{1-x^2} \, dx \\
&= 3 \int_{-\pi/2}^{\pi/2} 2 \cos^2 t \, dt = 3\pi.
\end{aligned}$$

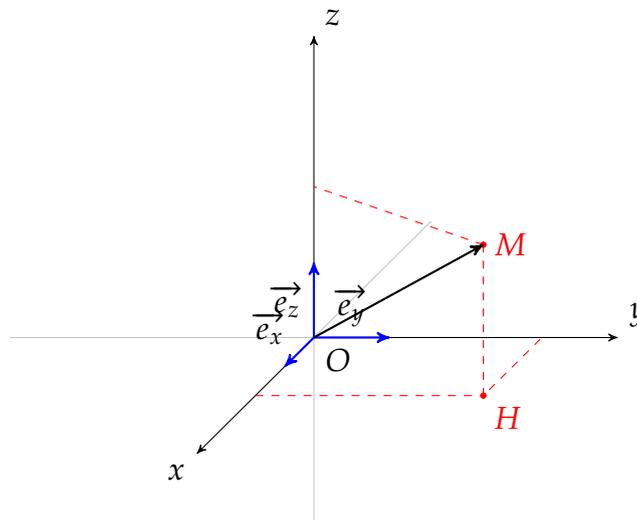
Calculation of Triple Integral - Change of Variable

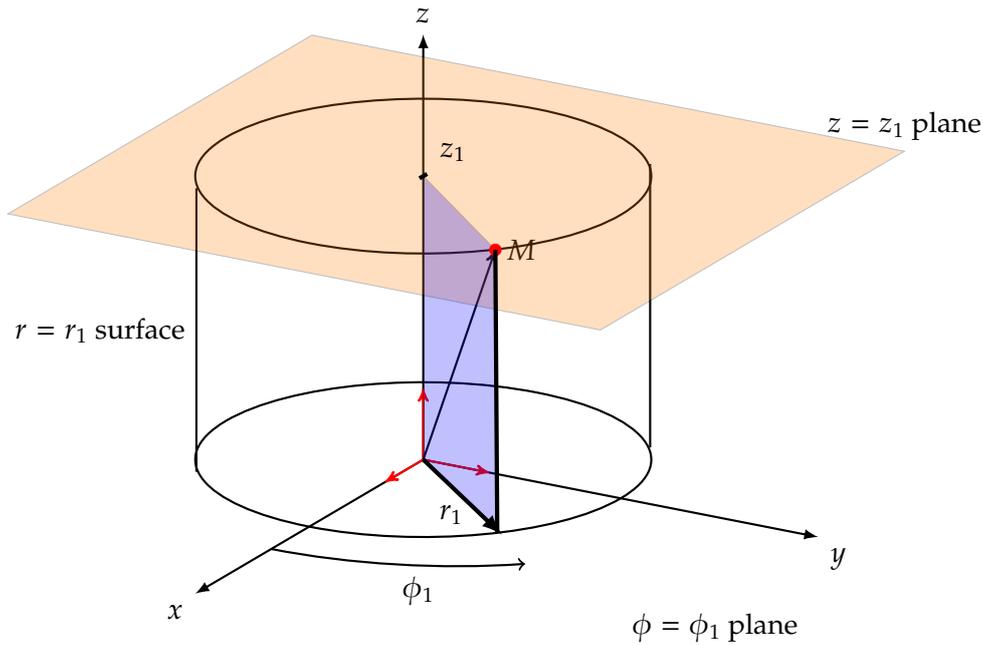
Theorem 1.5.2 If $(x, y, z) = h(u, v, w)$ is a change of variable then:

$$\begin{aligned}
&\iiint_{\Omega} f(x, y, z) \, dx \, dy \, dz \\
&= \iiint_{h^{-1}(\Omega)} f(x(u, v, w), y(u, v, w), z(u, v, w)) |\det J_h(u, v, w)| \, du \, dv \, dw
\end{aligned}$$

In particular, for changes to **cylindrical coordinates** or **spherical coordinates** we have:

$$dx \, dy \, dz = \rho \, d\rho \, d\varphi \, dz = r^2 \sin \theta \, dr \, d\varphi \, d\theta.$$





Example 1.5.2 Let's recall again the integral of the function $f(x, y, z) = 1 - 2yz$ over the complete cylinder Ω , with base the disk $D = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 \leq 1, z = 0\}$ of height 3. In cylindrical coordinates, we have:

$$\begin{aligned}\Omega &= \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 \leq 1, 0 \leq z \leq 3\} \\ &= \{(\rho, \varphi, z) \mid \rho \in [0, 1], \varphi \in [0, 2\pi[, z \in [0, 3]\}\end{aligned}$$

Therefore, since:

$$dx \, dy \, dz = \rho \, d\rho \, d\varphi \, dz$$

we obtain:

$$\begin{aligned}\iiint_{\Omega} (1 - 2yz) \, dx \, dy \, dz &= \int_0^3 dz \iint_D (1 - 2yz) \, dx \, dy \\ &= \int_0^3 dz \int_0^1 \rho \, d\rho \int_0^{2\pi} (1 - 2\rho \sin \varphi z) \, d\varphi \\ &= \int_0^3 dz \int_0^1 \rho \, d\rho [\varphi + 2\rho \cos \varphi z]_{\varphi=0}^{\varphi=2\pi} \\ &= \int_0^3 dz \int_0^1 (2\pi + 2\rho z - 2\rho z) \, \rho \, d\rho \\ &= \int_0^3 dz \int_0^1 2\pi \, \rho \, d\rho = 3\pi [\rho^2]_0^1 = 3\pi.\end{aligned}$$

1.6 Exercise Series No. 1

Exercise 1.6.1 Compute the following integrals:

$$\begin{array}{ll} 1) \int \sqrt{3x} \log x dx & 3) \int x^2 e^x dx \\ 2) \int_0^1 (7x^2 - e^x) dx & 4) \int x \sqrt{x^2 + 1} dx \end{array}$$

Exercise 1.6.2 Study the values of the following integral:

$$I_n = \int_0^1 \frac{\sin(\pi x)}{x+n} dx,$$

for every $n > 0$.

- (1) Prove that $0 \leq I_{n+1} \leq I_n$
- (2) Prove that $I_n \leq \ln \frac{n+1}{n}$ then deduce $\lim_{n \rightarrow +\infty} I_n$.
- (3) Compute the value of the integral:

$$\lim_{n \rightarrow +\infty} n I_n.$$

Exercise 1.6.3 Compute the following limits:

$$\begin{array}{ll} 1) \lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} \frac{\sqrt{x^2 y^2 + 1} - 1}{x^2 + y^2} & 3) \lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} \frac{x^2 + y^2}{\sqrt{x^2 + y^2 + 1} - 1} \\ 2) \lim_{\substack{x \rightarrow 0 \\ y \rightarrow 2}} \frac{\sin xy}{x^2 + y^2} & 4) \lim_{\substack{x \rightarrow \infty \\ y \rightarrow k}} \left(1 + \frac{y}{x}\right)^x \quad k \in \mathbb{R} \end{array}$$

Exercise 1.6.4 Study the continuity of the function f at the point $(x_0, y_0) = (0, 0)$:

$$f(x, y) = \begin{cases} \frac{x^3 + y^3}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

Then the function g at the point $(x_0, y_0) = (0, 1)$:

$$g(x, y) = \begin{cases} x + y, & (x, y) \neq (0, 1) \\ 0, & (x, y) = (0, 1) \end{cases}$$

Exercise 1.6.5 Study the values of the following integral:

$$I_n = \int_0^1 \frac{\sin(\pi x)}{x+n} dx,$$

for every $n > 0$.

- (1) Prove that $0 \leq I_{n+1} \leq I_n$
- (2) Prove that $I_n \leq \ln \frac{n+1}{n}$ then deduce $\lim_{n \rightarrow +\infty} I_n$.
- (3) Compute the value of the integral:

$$\lim_{n \rightarrow +\infty} nI_n.$$