Chapitre 2 : Etude des circuits en régime Transitoire

Circuit RC en régimes transitoires (charge et décharge), Circuits RL en régimes transitoires, Circuits RLC en régimes transitoires.

1) *Introduction*

The analysis of RC and RL circuits is carried out by applying Kirchhoff's laws, as we did for resistive circuits. The only difference is that applying Kirchhoff's laws to purely resistive circuits results in algebraic equations, while applying the laws to RC and RL circuits produces differential equations, which are more difficult to solve than algebraic equations. The differential equations resulting from analyzing RC and RL circuits are of the first order. Hence, the circuits are collectively known as ***first-order*** circuits.

A first-order circuit is characterized by a first-order differential equation.

In addition to there being two types of first-order circuits (RC and RL), there are two ways to excite the circuits.

1) The first way is by initial conditions of the storage elements in the circuits. In these so-called ***source-free circuits***, we assume that energy is initially stored in the capacitive or inductive element. The energy causes current to flow in the circuit and is gradually dissipated in the resistors. Although source free circuits are by definition free of independent sources, they may have dependent sources.
2) The second way of exciting first-order circuits is by independent sources (dc and ac sources).

2) *The Source-Free RC Circuit*

A source-free RC circuit occurs when its dc source is suddenly disconnected. The energy already stored in the capacitor is released to the resistors.

Consider a series combination of a resistor and an initially charged capacitor, as shown in Fig.2.1. (The resistor and capacitor may be the equivalent resistance and equivalent capacitance of combinations of resistors and capacitors.) Now to determine the circuit response. Since the capacitor is initially charged, we can assume that at time $t = 0$, the initial voltage is

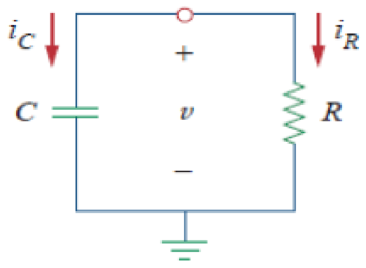

Fig.2.1.source-free RC circuit

$$v(0) = V_0 \qquad \ldots(2.1)$$

with the corresponding value of the energy stored as

$$w(0) = \frac{1}{2}CV_0^2 \qquad \ldots(2.2)$$

Applying KCL at the top node of the circuit in Fig. 2.1 yields

$$i_C + i_R = 0 \qquad \ldots(2.3)$$

By definition, $i_C = Cdv/dt$ and $i_R = v/R$. Thus,

$$C\frac{dv}{dt} + \frac{v}{R} = 0 \qquad \ldots(2.4a)$$

$$\therefore \frac{dv}{dt} + \frac{v}{RC} = 0 \qquad \ldots(2.4b)$$

This is a ***first-order differential equation***, since only the first derivative of u is involved. To solve it, we rearrange the terms as

$$\frac{dv}{v} = -\frac{1}{RC}dt \qquad \ldots(2.5)$$

Integrating both sides, we get

$$ln\ v = -\frac{t}{RC} + ln\ A$$

where $ln\ A$ is the integration constant. Thus,

$$ln\frac{v}{A} = -\frac{t}{RC} \qquad \ldots(2.6)$$

Taking powers of e produces

$$v(t) = Ae^{-t/RC}$$

But from the initial conditions, $v(O) = A = V_0$. Hence,

$$v(t) = V_0 e^{-t/RC} \qquad \ldots(2.7)$$

This shows that the voltage response of the RC circuit is an exponential decay of the initial voltage. Since the response is due to the initial energy stored and the physical characteristics of the circuit and not due to some external voltage or current source, it is called the ***natural response*** of the circuit.

The natural response of a circuit refers to the behavior (in terms of voltages and currents) of the circuit itself, with no external sources of excitation.

The natural response is illustrated graphically in Fig.2.2. Note that at $t = 0$, we have the correct initial condition as in Eq. (2.1). As t increases, the voltage decreases toward zero. The rapidity with which the voltage decreases is expressed in terms of the ***time constant***, denoted by $\boldsymbol{\tau}$, the lowercase Greek letter tau.

The time constant of a circuit is the time required for the response to decay to a factor of 1/e or 36.8 percent of its initial value.

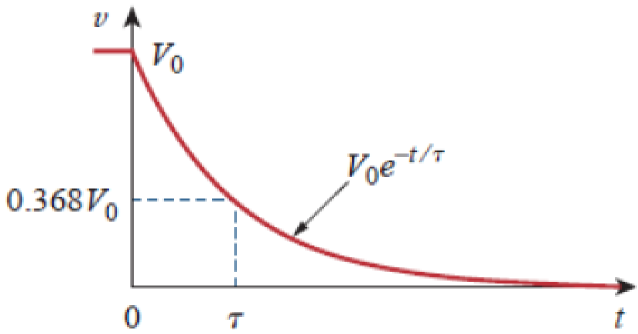

Fig.2.2. The voltage response of the RC circuit.

This implies that at $t = \tau$, Eq. (2.7) becomes

$$V_0 e^{-\tau/RC} = V_0 e^{-1} = 0.368V_0$$

or

$$\boxed{\tau = RC} \qquad \ldots(2.8)$$

In terms of the time constant, Eq. (2.7) can be written as

$$\boxed{v(t) = V_0 e^{-t/\tau}} \qquad \ldots(2.9)$$

It is evident from Table 7.1 that the voltage $v(t)$ is less than 1 percent of V_0 after 5τ (five time constants). *Thus, it is customary to assume that the capacitor is fully discharged (or charged) after five time constants.* In other words, it takes 5τ for the circuit to reach its final state or steady state when no changes take place with time. *Notice that for every time interval of* τ*, the voltage is reduced by 36.8 percent of its previous value,* $v(t+\tau) = v(t)/e = 0.368v(t)$ *, regardless of the value of* t.

TABLE 2.1
Values of $v(t)/V_0 = e^{-t/\tau}$.

t	$v(t)/V_0$
τ	0.36788
2τ	0.13534
3τ	0.04979
4τ	0.01832
5τ	0.00674

A circuit with a small time constant gives a fast response in that it reaches the steady state (or final state) quickly due to quick dissipation of energy stored, whereas a circuit with a large time constant gives a slow response because it takes longer to reach steady state (this is illustrated in Fig. 2.3). At any rate, whether the time constant is small or large, the circuit reaches steady state in five time constants.

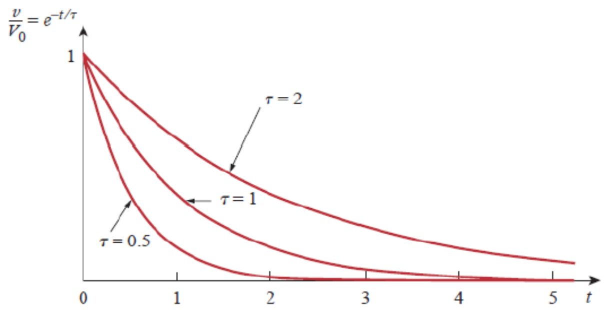

Figure 2.3. Plot of $v(t)/V_0 = e^{-t/\tau}$ for various values of the time constant.

With the voltage $v(t)$ in Eq. (2.9), we can find the current $i_R(t)$,

$$i_R(t) = \frac{v(t)}{R} = \frac{V_0}{R} e^{-t/\tau} \quad ...(2.10)$$

The power dissipated in the resistor is

$$p(t) = v i_R = \frac{V_0^2}{R} e^{-2t/\tau} \quad ...(2.11)$$

The energy absorbed by the resistor up to time t is

$$w_R(t) = \int_0^t p\,dt = \int_0^t \frac{V_0^2}{R} e^{-2t/\tau} dt = -\frac{\tau V_0^2}{2R} e^{-2t/\tau} \Big|_0^t = \frac{1}{2} C V_0^2 (1 - e^{-2t/\tau}),\ \tau = RC \quad .(2.12)$$

Notice that as $t \to \infty$, $w_R(\infty) \to \frac{1}{2} C V_0^2$, which is the same as $w_C(0)$, the energy initially stored in the capacitor. The energy that was initially stored in the capacitor is eventually dissipated in the resistor.

With these two items, we obtain the response as the capacitor voltage $v_C(t) = v(t) = v(0)e^{-t/\tau}$ other variables (capacitor current i_C, resistor voltage v_R, and resistor current i_R) can be determined. ***In finding the time constant $\tau = RC$, R is often the Thevenin equivalent resistance at the terminals of the capacitor; that is, we take out the capacitor C and find $R = R_{Th}$ at its terminals.***

3) *The Source-Free RL Circuit*

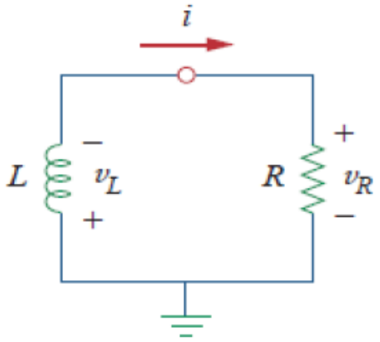

Fig.3.1 A source-free RL circuit

Our goal is to determine the circuit response (current $i(t)$ through the inductor). We select the inductor current as the response in order to take advantage of the idea $\underline{i}$ that the inductor current cannot change instantaneously. At $t = 0$, we assume that the inductor has an initial current I_0, or with the corresponding energy stored in the inductor as

$$i(0) = I_o \tag{3.1}$$

$$w(0) = \frac{1}{2}LI_0^2 \tag{3.2}$$

Applying KVL around the loop in Fig.3.1,

$$v_L + v_R = 0 \tag{3.3}$$

But $v_L = Ldi/dt$ and $v_R = iR$. Thus,

$$L\frac{di}{dt} + Ri = 0 \quad \Rightarrow \quad \frac{di}{dt} + \frac{R}{L}i = 0 \tag{3.4}$$

Rearranging terms and integrating gives

$$\int_{I_0}^{i(t)} \frac{di}{i} = -\int_0^t \frac{R}{L}dt \quad \Rightarrow \quad \ln i \,\Big|_{I_o}^{i(t)} = -\frac{Rt}{L}\Big|_0^t \Rightarrow \ln i(t) - \ln I_0 = -\frac{Rt}{L} + 0$$

$$\therefore \ \ln\frac{i(t)}{I_0} = -\frac{Rt}{L} \tag{3.5}$$

Taking the powers of e, we have

$$i(t) = I_0 e^{-Rt/L} \tag{3.6}$$

This shows that the natural response of the *RL* circuit is an exponential decay of the initial current. The current response is shown in Fig.3.2.

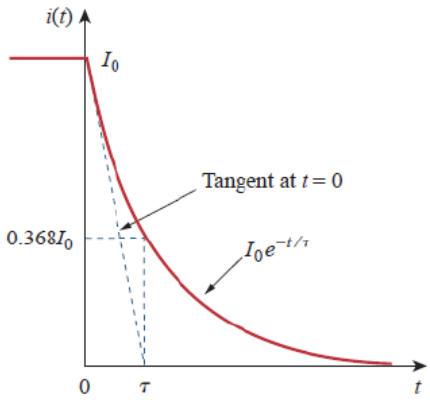

Fig.3.2. The current response of the RL circuit

It is evident from Eq. (3.6) that the time constant for the *RL* circuit is

$$\boxed{\tau = \frac{L}{R} \quad \text{in (s)}} \tag{3.7}$$

$$\therefore i(t) = I_0 e^{-t/\tau} \tag{3.8}$$

$$v_R(t) = iR = I_0 R e^{-t/\tau} \tag{3.9}$$

The power dissipated in the resistor is

$$p = v_R i = I_0^2 R e^{-2t/\tau} \tag{3.10}$$

$$w_R(t) = \int_0^t p\, dt = \int_0^t I_0^2\, R e^{-2t/\tau} dt = -\frac{1}{2}\tau I_0^2 R e^{-2t/\tau}\Big|_0^t, \qquad \tau = \frac{L}{R}$$

$$w_R(t) = \frac{1}{2} L I_0^2 (1 - e^{-2t/\tau}) \tag{3.11}$$

Note that as $t \to \infty$,$w_R(\infty) \to \frac{1}{2} L I_0^2$, which is the same as $w_L(0)$, the initial energy stored in the inductor as in Eq. (3.2). Again, energy initially stored in the inductor is eventually dissipated in the resistor.

With the two items, we obtain the response as the inductor current $i_L(t) = i(t) = i(0)e^{-t/\tau}$. Once we determine the inductor current i_L, other variables (inductor voltage v_L, resistor voltage v_R, and resistor current i_R) can be obtained. *Note that in general,* ***R*** *in Eq. (3.7) is the Thevenin resistance at the terminals of the inductor.*

5) Step Response of an RC Circuit

When the dc source of an RC circuit is suddenly applied, the voltage or current source can be modeled as a step function, and the response is known as a ***step response***.

The step response of a circuit is its behavior when the excitation is the step function, which may be a voltage or a current source.

The step response is the response of the circuit due to a sudden application of a dc voltage or current source.

Consider the RC circuit in Fig.5.1 (a) which can be replaced by the circuit in Fig.5.1 (b), where V_s is a constant dc voltage source. We assume an initial voltage V_0 on the capacitor, although this is not necessary for the step response.

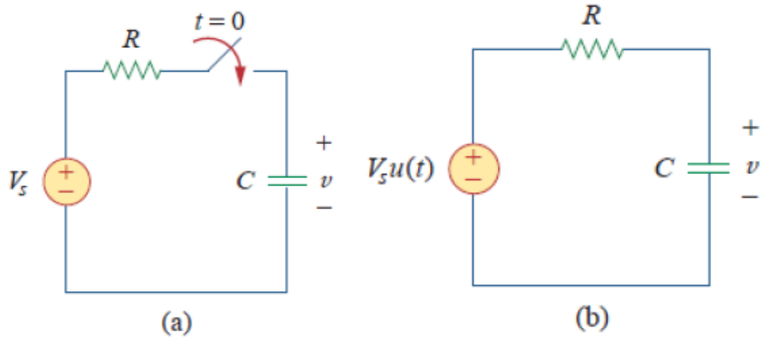

Fig.5.1 An RC circuit with voltage step input.

Since the voltage of a capacitor cannot change instantaneously,

$$v(0^-) = v(0^+) = V_0 \qquad (5.1)$$

where $v(0^-)$ is the voltage across the capacitor just before switching and $v(0^+)$ is its voltage immediately after switching. Applying KCL, we have

$$C\frac{dv}{dt} + \frac{v - V_s u(t)}{R} = 0$$

$$\therefore \frac{dv}{dt} + \frac{v}{RC} = \frac{V_s}{RC}u(t) \qquad (5.2)$$

where v is the voltage across the capacitor. For $t > 0$, Eq. (5.2) becomes

$$\frac{dv}{dt} + \frac{v}{RC} = \frac{V_s}{RC} \qquad (5.3)$$

Rearranging terms gives

$$\frac{dv}{dt} = -\frac{v - V_s}{RC}$$

$$\therefore \frac{dv}{v - V_s} = -\frac{dt}{RC} \qquad (5.4)$$

Integrating both sides and introducing the initial conditions,

$$ln\,(v - V_s)|_{V_0}^{v(t)} = -\frac{t}{RC}|_0^t$$

$$ln\,(v(t) - V_s) - ln(V_0 - V_s) = -\frac{t}{RC} + 0$$

$$ln\,\frac{v - V_s}{V_0 - V_s} = -\frac{t}{RC} \qquad (5.5)$$

Taking the exponential of both sides

$$\frac{v - V_s}{V_0 - V_s} = e^{-t/\tau}, \quad \tau = RC$$

$$v - V_s = (V_0 - V_s)e^{-t/\tau}$$

$$\therefore v(t) = V_s + (V_0 - V_s)e^{-\frac{t}{\tau}}, \quad t > 0 \qquad (5.6)$$

Thus,

$$\boxed{v(t) = \begin{cases} V_0, & t < 0 \\ V_s + (V_0 - V_s)e^{-t/\tau}, & t > 0 \end{cases}} \qquad (5.7)$$

This is known as the ***complete response*** (or ***total response***) of the RC circuit to a sudden application of a dc voltage source, assuming the capacitor is initially charged. The reason for the term "complete" will become evident a little later. Assuming that $V_s > V_0$, a plot of $v(t)$ is shown in Fig.5.2.

If we assume that the capacitor is uncharged initially, we set $V_0 = 0$ in Eq. (5.7) so that

$$v(t) = \begin{cases} 0, & t < 0 \\ V_s(1 - e^{-t/\tau}), & t > 0 \end{cases} \qquad (5.8)$$

which can be written alternatively as

$$v(t) = V_s(1 - e^{-t/\tau})u(t) \qquad (5.9)$$

This is the complete step response of the RC circuit when the capacitor is initially uncharged. The current through the capacitor is obtained from Eq. (5.8) using $i(t) = \frac{Cdv}{dt}$. We get

$i(t) = C\frac{dv}{dt} = \frac{C}{\tau}V_s e^{-\frac{t}{\tau}}, \qquad \tau = RC, t > 0$

$$\therefore\ i(t) = \frac{V_s}{R}e^{-t/\tau}u(t) \qquad (5.10)$$

Fig.5.3 shows the plots of capacitor voltage $v(t)$ and capacitor current $i(t)$.

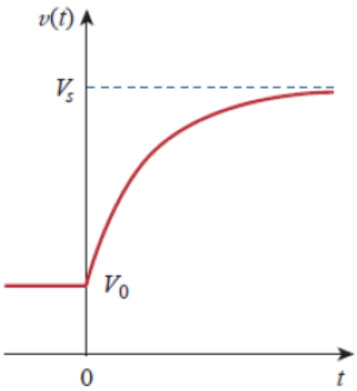

Fig.5.2 Response of an *RC* circuit with initially charged capacitor.

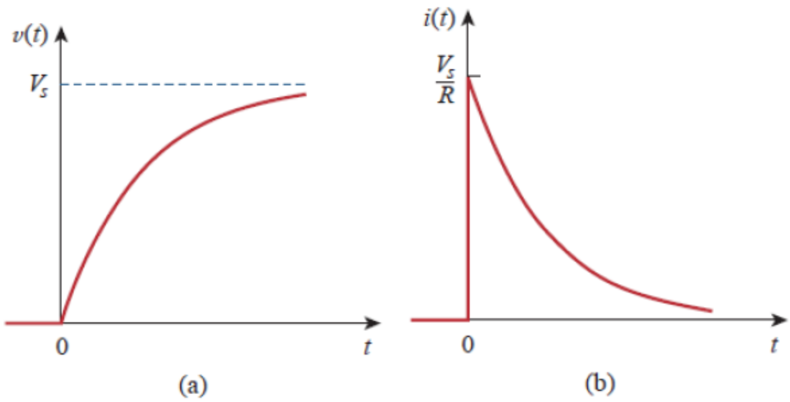

Fig.5.3 Step response of an *RC* circuit with initially uncharged capacitor: (a) voltage response, (b) current response.

Rather than going through the derivations above, there is a systematic approach—or rather, a short-cut method—for finding the step response of an *RC* or *RL* circuit. Let us reexamine Eq. (5.6), which is more general than Eq. (5.9). It is evident that $v(t)$ has two components.

Classically there are two ways of decomposing this into two components. The first is to break it into a "***natural response and a forced response***" and the second is to break it into a "***transient response and a steady-state response.***" Starting with the natural response and forced (a) response, we write the total or complete response as

Complete response = natural response (stored energy) + forced response (independent source)

or

$$v = v_n - v_f \qquad (5.11)$$

where $v_n = V_o e^{-t/\tau}$ and $v_f = V_s(1 - e^{-t/\tau})$

We are familiar with the natural response v_n of the circuit, as discussed response, (b) current response. in Section 2. v_f is known as the ***forced response*** because it is produced by the circuit when an external "***force***" (a voltage source in this case) is applied. It represents what the circuit is forced to do by the input excitation. The natural response eventually dies out along with the transient component of the forced response, leaving only the steady- state component of the forced response.

Another way of looking at the complete response is to break into two components—one temporary and the other permanent, i.e.,

$$\text{Complete response} = \underbrace{\text{transient response}}_{\text{temporary part}} + \underbrace{\text{steady-state response}}_{\text{permanent part}}$$

or

$$v = v_t - v_{ss} \tag{5.12}$$

where $v_t = (V_o - V_s)e^{-t/\tau}$ and $v_{ss} = V_s$

The ***transient response*** v_t is temporary; it is the portion of the complete response that decays to zero as time approaches infinity. Thus,

> The transient response is the circuit's temporary response that will die out with time.

The ***steady-state response*** v_{ss} is the portion of the complete response that remains after the transient reponse has died out. Thus,

> The steady-state response is the behavior of the circuit a long time after an external excitation is applied.

The first decomposition of the complete response is in terms of the source of the responses, while the second decomposition is in terms of the permanency of the responses. Under certain conditions, the natural response and transient response are the same. The same can be said about the forced response and steady-state response.

Whichever way we look at it, the complete response in Eq. (5.6) may be written as

$$v(t) = v(\infty) + [v(O) - v(\infty)]e^{-t/\tau} \tag{5.13}$$

where $v(O)$ is the initial voltage at $t = 0^+$ and $v(\infty)$ is the final or steady- state value. Thus, to find the step response of an RC circuit requires three things:

1. The initial capacitor voltage $v(O)$.

2. The final capacitor voltage $v(\infty)$.

3. The time constant τ.

We obtain item 1 from the given circuit for $t < 0$ and items 2 and 3 from the circuit for $t > 0$. Once these items are determined, we obtain the response using Eq. (5.13). This technique equally applies to RL circuits, as we shall see in the next section.

Note that if the switch changes position at time $t = t_0$ instead of at $t = 0$, there is a time delay in the response so that Eq. (5.13) becomes

$$v(t) = v(\infty) + [v(t_0) - v(\infty)]e^{-(t-t_0)/\tau} \tag{5.14}$$

where $v(t_0)$ is the initial value at $t = t_0^+$. Keep in mind that Eq. (5.13) or (5.14) applies only to step responses, that is, when the input excitation is constant.

6) Step Response of an RL Circuit

Consider the RL circuit in Fig.6.1 (a), which may be replaced by the circuit in Fig.6.1 (b). Rather than apply Kirchhoff's laws, we will use the simple technique in Eqs. (5.11) through (5.14). Let the response be the sum of the transient response and the steady-state response,

$$i = i_t + i_{ss} \tag{6.1}$$

We know that the transient response is always a decaying exponential, that is,

$$i_t = Ae^{-t/\tau}, \qquad \tau = \frac{L}{R} \tag{6.2}$$

where A is a constant to be determined.

The steady-state response is the value of the current a long time after the switch in Fig.6.1 (a) is closed. We know that the transient response essentially dies out after five time constants. At that time, the inductor becomes a short circuit, and the voltage across it is zero. The entire source voltage V_s appears across R. Thus, the steady-state response is

$i_{ss} = \frac{V_s}{R}$ (6.3)

Substituting Eqs. (6.2) and (6.3) into Eq. (6.1) gives

$$i = Ae^{-t/\tau} + \frac{V_s}{R} \quad (6.4)$$

We now determine the constant A ffom the initial value of i. Let I_0 be the initial current through the inductor, which may come from a source other than V_s. Since the current through the inductor cannot change instantaneously,

$$i(0^+) = i(0^-) = I_0 \quad (6.5)$$

Thus, at $t = 0$, Eq. (6.4) becomes

$$I_0 = A + \frac{V_s}{R}$$

From this, we obtain A as

$$A = I_0 - \frac{V_s}{R}$$

Substituting for A in Eq. (6.4), we get

$$i(t) = \frac{V_s}{R} + (I_0 - \frac{V_s}{R})e^{-t/\tau} \quad (6.6)$$

This is the complete response of the RL circuit. It is illustrated in Fig.6.2. The response in Eq. (6.6) may be written as

$$i(t) = i(\infty) + [i(0) - i(\infty)]e^{-t/\tau} \quad (6.7)$$

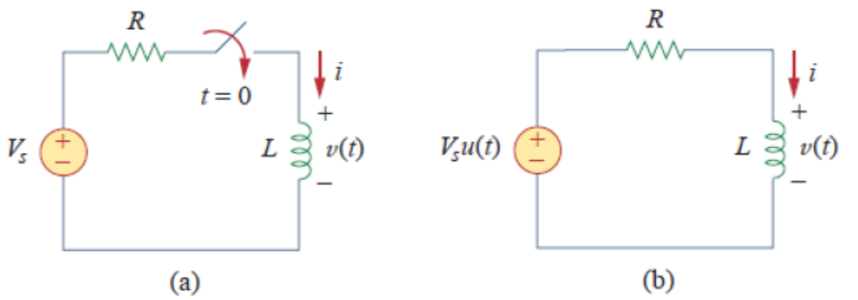

Fig.6.1 An RL circuit with a step input voltage.

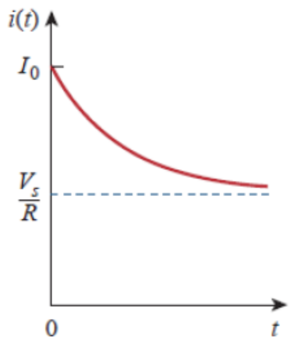

Fig.6.2 Total response of the RL circuit with initial inductor current I_0.

where $i(0)$ and $i(\infty)$ are the initial and final values of i, respectively. Thus, to find the step response of an RL circuit requires three things:

1. The initial inductor current $i(0)$ at $t = 0$.

2. The final inductor current $i(\infty)$.

3. The time constant τ.

We obtain item 1 from the given circuit for $t < 0$ and items 2 and 3 from the circuit for $t > 0$. Once these items are determined, we obtain the response using Eq. (6.7). Keep in mind that this technique applies only for step responses.

Again, if the switching takes place at time $t = t_0$ instead of $t = 0$, Eq. (6.7) becomes

$$i(t) = i(\infty) + [i(t_0) - i(\infty)]e^{-(t-t_0)/\tau} \quad (6.8)$$

If $I_0 = 0$, then

$$i(t) = \begin{cases} 0, & t < 0 \\ \frac{V_s}{R}(1 - e^{-t/\tau}), & t > 0 \end{cases} \quad (6.9a)$$

or

$$i(t) = \frac{V_s}{R}(1 - e^{-t/\tau})u(t) \quad (6.9b)$$

This is the step response of the RL circuit with no initial inductor current. The voltage across the inductor is obtained from Eq. (6.9) using $v = Ldi/dt$. We get

$$v(t) = L\frac{di}{dt} = V_s\frac{L}{\tau R}e^{-t/\tau}, \tau = \frac{L}{R}, t > 0$$

or

$$v(t) = V_s e^{-t/\tau}u(t) \quad (6.10)$$

Fig.6.3 shows the step responses in Eqs. (6.9) and (6.10).

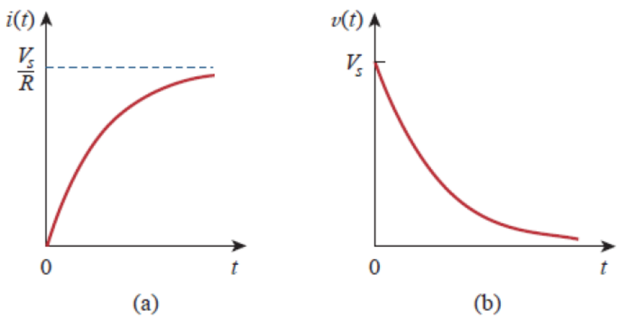

Fig.6.3 Step responses of an RL circuit with no initial inductor current: (a) current response, (b) voltage response.