University of Biskra
Mathematics Department
Module: Analysis 2

First year license
2023/2024

Catch-up Exam

Exercise 1 (./06pts)

Calculate the following integrals:

1. $\int \frac{x3+2x^2+3x-1}{x^2+2x+1}$.
2. $\int \ln(x^2+2x-3)dx$.
3. $\int \cos^3(x)\sin^2(x)dx$.

Exercise 2 (./04pts)

Let I_n, $n \in \mathbb{R}_+^*$, defined as follows

$$I_n = \int_0^{\infty} x^{n-1} n e^{-x} dx$$

1. Show that $I_n = (n-1)I_{n-1}$, for all $n \in \mathbb{R}_+^*$.
2. Check that if $n \in \mathbb{N}$ then $I_n = (n-1)!$.
3. If we know that the value of $I_{1/2} = \sqrt{\pi}$, then what is the value of $I_{7/2}$.

Exercise 3 (./04pts)

Solve the following second order differential equation.

$$y'' + 4y' - 5y = sin(x) + e^x.$$

Exercise 4 (./06pts)

Let consider the following differential equation

$$x^2\left(y' + y^2\right) = xy - 1, \tag{1}$$

1. Check that $y = \frac{1}{x}$ is a solution of the equation (1).
2. Finds the general solution of the equation (1).

Solution :

Exo1 :

$$F(x) = \int \ln(x^2+2x-3)\,dx = \int 1 \cdot \ln(x^2+2x-3)\,dx$$

$$\begin{cases} u' = 1 \\ v = \ln(x^2+2x-3) \end{cases} \Rightarrow \begin{cases} u = x \\ v' = \dfrac{2x+2}{x^2+2x-3} \end{cases}$$ 0,5

$$\Rightarrow F(x) = x\ln(x^2+2x-3) - \underbrace{\int \frac{2x^2+2x}{x^2+2x-3}\,dx}_{I}$$ 0,5

$$x^2+2x-3=0 \Rightarrow \Delta = 16 \Rightarrow \begin{cases} x_1 = -3 \\ x_2 = 1 \end{cases}$$ 0,5 0,5

$$\Rightarrow I = \int 2 + \frac{a}{x+3} + \frac{b}{x-1}\,dx$$ 0,5

$$\frac{2x^2+2x}{x^2+2x-3} = 2 + \frac{a}{x+3} + \frac{b}{x-1} \Rightarrow \begin{cases} a = -3 \\ b = 1. \end{cases}$$ 0,5

$$\Rightarrow I = \int 2 + \frac{-3}{x+3} + \frac{1}{x-1}\,dx$$

$$= 2x - 3\ln(|x+3|) + \ln(|x-1|) + C \;/\; C \in \mathbb{R}$$ 0,5

$$\Rightarrow F(x) = x\ln(x^2+2x-3) + 2x - 3\ln(|x+3|) + \ln(|x-1|) + C \;/ C\in\mathbb{R}$$ 0,5

$$G(x) = \int \cos^3(x)\sin^2(x)\,dx = \int \cos^2(x)\sin^2(x)\cos(x)\,dx$$

we put $t = \sin(x) \Rightarrow dt = \cos(x)\,dx.$ 0,5

$$\Rightarrow G(t) = \int \underline{(1-t^2)}\,t^2\,dt = \int t^2 - t^4\,dt$$ 0,5

0,5 $= \frac{1}{3}t^3 - \frac{1}{5}t^5 + C \;/\; C\in\mathbb{R}$ (1 than

$$\boxed{G(x) = \frac{1}{3}\sin^3(x) - \frac{1}{5}\sin^5(x) + C}$$ 0.5

EXO2 $I_n = \int_0^{+\infty} x^n e^{-x}, \quad n \in \mathbb{R}_+^*$ 0.5

① $I_n = \int_0^{+\infty} x^n e^{-x}\,dx = ?$ $\begin{cases} u = x^n \\ v' = e^{-x} \end{cases} \Rightarrow \begin{cases} u' = n x^{n-1} \\ v = -e^{-x}. \end{cases}$

$$\Rightarrow I_n = \cancel{-x^n e^{-x}\Big|_0^{+\infty}} + \int_0^{+\infty} n x^{n-1} e^{-x}\,dx$$ 0.5

$$= n \int_0^{+\infty} x^{n-1} e^{-x}\,dx.$$

$$\boxed{I_n = n I_{n-1}}$$ 0.5

② let $n \in \mathbb{N}$.

- we have for $n = 0 \Rightarrow I_0 = \int_0^{+\infty} e^{-x}\,dx = 1 = 0!$ ——— $(*)$ 0.5
- from the first question we have

$$I_n = n I_{n-1}$$

$$= n * (n-1) I_{n-2}$$

$$= n * (n-1) * \cdots * 1 * I_0 \quad / \text{As } I_0 = 1 \text{ then}$$

$$\boxed{I_n = n!} \text{ ——— } (**)$$ 0.5

0.5

from $(*)$ and $(**)$ we deduce that $I_n = n!\ \forall n \in \mathbb{N}$.

3) we have $I_{1/2} = \sqrt{\pi}/2$

we have from the first question $I_n = n\, I_{(n-1)}$ so

$$I_{7/2} = \frac{7}{2} I_{5/2}$$
$$= \frac{7}{2} \times \frac{5}{2} I_{3/2}$$
$$= \frac{7}{2} \times \frac{5}{2} \times \frac{3}{2} I_{1/2}$$
$$= \frac{7}{2} \times \frac{5}{2} \times \frac{3}{2} \frac{\sqrt{\pi}}{2}$$
$$\boxed{I_{7/2} = \frac{105}{16}\sqrt{\pi}}$$

1

Exo 4:

$$x^2 (y' + y^2) = xy - 1 \quad (1)$$

① $y = 1/x$ is a solution of (1) ?

$y = 1/x \Rightarrow y' = -1/x^2 \xrightarrow{\text{in } (1)} x^2\left(-\frac{1}{x^2} + \left(\frac{1}{x}\right)^2\right) = x\frac{1}{x} - 1.$

$\Rightarrow 0 = 0$ so $y = 1/x$ is a solution of (1).

0.5

② To find the general solution of (1) we must transform it to a Bernoulli equation ($n=2$) using the substitution $z = y - y_0 = y - 1/x$.

0.5

$\Rightarrow y = z + 1/x \Rightarrow y' = z' - \frac{1}{x^2}.$

0.5

$\cancel{x^2\left(z' - \frac{1}{x^2} + \frac{1}{x^2}\right) = x\left(z + \frac{1}{x}\right) - 1}$

3

$(1) \Rightarrow x^2\left(\left(z' - \frac{1}{x^2}\right) + \left(z + \frac{1}{x}\right)^2\right) = x\left(z + \frac{1}{x}\right) - 1.$

$\Rightarrow z' + z/x = -z^2$

$\Rightarrow -z^{-2}z' - z^{-1}/x = 1$ —— Bernoulli $n=2$. —— $(**)$

$u = z^{-1} \Rightarrow u' = -z^{-2}z'$ —— ~~$(***)$~~ 0.5

$(**) \Rightarrow u' - u/x = 1$ —— $(***)$ 0.5

$\Rightarrow u' - u/x = 0$ —— $(****)$

$\Rightarrow u_H = Kx \; / K \in \mathbb{R}_+.$ 0.5

$u_G = K(x)x \Rightarrow u_G' = K'(x)x + K(x)$, substitute u_G and u_G' in $(***)$ we get:

$xK'(x) + \cancel{K(x)} - \cancel{\frac{K(x)x}{x}} = 1.$ 0.5

$\Rightarrow K'(x) = \frac{1}{x} \Rightarrow \boxed{K(x) = \ln(|x|) + c} \; / \in \in \mathbb{R}$

So the solution of $(***)$ is given by

$\boxed{u_G = (\ln(|x|) + c)x}$ $c \in \mathbb{R}$ 0.5

hence $z = \frac{1}{u_G} = \left[(\ln(|x|) + c)\cdot x\right]^{-1}.$ 0.5

Consequently:

$y = \frac{1}{(\ln(|x|) + c)x} + \frac{1}{x} \; / \; c \in \mathbb{R}.$ 0.5

End

EXO3: /04

$$y''+4y'-5y=e^x$$

① Homogenous solution

$$y''+4y'-5y=0 \Rightarrow R^2+4y-5=0$$

$$\Delta = 36 \Rightarrow \sqrt{\Delta}=6$$ 0,5

$$\Rightarrow \begin{cases} R_1 = \dfrac{-4-6}{2} = -5 & 0{,}5 \\ R_2 = \dfrac{-4+6}{2} = 1. & 0{,}5 \end{cases}$$

$$\Rightarrow \boxed{y_H = C_1 e^x + C_2 e^{-5x}}$$ 0,5

② Particular solution: $y_p = axe^x$ / as e^x is a Homogenous solution

0,5

$$y_p = axe^x$$

$$\Rightarrow y_p' = axe^x + ae^x$$

$$y_p'' = axe^x + 2ae^x$$ 0,5

$$\Rightarrow (axe^x+2ae^x)+4(axe^x+ae^x)-5axe^x=e^x$$

$$\Rightarrow 6a=1 \Rightarrow \boxed{a=1/6} \Rightarrow \cancel{y}$$

$$\Rightarrow \boxed{y_p = \tfrac{1}{6} x e^x}$$ 0,5

③ general solution: $y_G = y_H + y_p$

$$\Rightarrow \boxed{y_G = \left(C_1 + \tfrac{1}{6}x\right)e^x + C_2 e^{-5x}}.$$ 0,5