University of Biskra
Mathematics Department
Module: Analysis 2

First year license
2023/2024

Final Exam

Exercise 1 (./04pts)

Let f a real function defined by

$$f(x) = \frac{5(x-2)}{x(x-5)}.$$

1. Calculates the antiderivative of f ***(02pts)***.
2. Calculates the value of the surface delimited by the curve of f, the axe $y = 0$, the axe $x = 1$ and $x = 3$ ***(02pts)***.

Exercise 2 (./05pts)

Let consider the following integrals

$$I(x) = \int e^x \cos^2(x/2)dx \text{ and } J(x) = \int e^x \sin^2(x/2)dx.$$

1. Compute $F(x) = I(x) + J(x)$.
2. Compute $G(x) = I(x) - J(x)$.
3. Deduce the expressions of I and J.

Note: $\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$.

Exercise 3 (./07pts)

Discuss the solutions of the following differential equation according to the real parameter n.

$$y' - y = xy^n, \text{ with } n \in \mathbb{R}, \ x > 0 \text{ and } y > 0.$$

Exercise 4 (./04pts)

1. Solve the following second order differential equation.

$$y'' + 2y' + 5y = 4e^{-x}.$$

2. Determines the solution that passed from the origin $(0,0)$ and the point $(\pi/4, 0)$.

Good luck

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Correction of the Final Exam

Solution of the Exercise 1

1. $\int f(x)dx =?$ ***(02pts)***.From the expression of f, one notes that the problem concerns the calculation of an integral of a rational (fractional) function whose degree of the polynomial of the dominant is less than that of the nominator. Then f can be simplified as follows:

$$f(x) = \frac{5(x-2)}{x(x-5)} = \frac{a}{x} + \frac{b}{x-5}, \text{with } a,\ b \text{ and } c \text{ are a real contants that will be determined.}$$

Let's determine the value of the above constants

$$\begin{cases} xf(x) &= \frac{5(x-2)}{(x-5)} = a + \frac{bx}{x-5} & \text{and if we put } x=0 \text{ then we get } a=2 \\ (x-5)f(x) &= \frac{5(x-2)}{x} \quad = \frac{a(x-5)}{x} + b & \text{and if we put} x=5 \text{ then we get } b=3 \end{cases}$$

so,

$$\int f(x)dx = \int \frac{2}{x} + \frac{3}{x-5} dx = 2\ln(|x|) + 3\ln(|x-5|) + c \ \text{ with } c \in \mathbb{R}.$$

2. Calculates the surface delimited by the curve of f, the axe $y = 0$, the axe $x = 1$ and $x = 3$ ***(02pts)***.Before calculating the Surface we must study the sign of f on the interval $[1;3]$. The following table summarizes the subintervals

values	1	2	3	5	
x	+	+	+	+	+
$x-2$	-	-	+	+	+
$x-5$	-	-	-	-	+
$f(x)$	+	+	-	-	+

$$\begin{aligned} \int_1^3 |f(x)|dx &= \int_1^2 f(x)dx + \int_2^3 (-f(x))dx \\ &= \int_1^2 \frac{5\,(x-2)}{x\,(x-5)}dx - \int_2^3 \frac{5\,(x-2)}{x\,(x-5)}dx \\ &= \left[2\ln(x) + 3\ln(5-x)|_1^2\right] - \left[2\ln(x) + 3\ln(5-x)|_2^3\right] \\ &= 4\ln(3) - 5\ln(2). \end{aligned}$$

Solution of the Exercise 2

1. $F(x) =?$

$$\begin{aligned} F(x) &= I(x) + J(x) \\ &= \int \sin^2(x/2)e^x dx + \int \cos^2(x/2)e^x dx \\ &= \int \left(\sin^2(x/2) + \cos^2(x/2)\right) e^x dx = \int e^x dx \\ &= e^x + c_1, \ \text{with } c_1 \in R. \end{aligned} \tag{1}$$

2. $G(x) =?$

$$\begin{aligned} G(x) &= I(x) - J(x) \\ &= \int \sin^2(x/2)e^x dx - \int \cos^2(x/2)e^x dx \\ &= \int \left(\sin^2(x/2) - \cos^2(x/2)\right) e^x dx \\ &= \int \cos(x)e^x dx \end{aligned} \tag{2}$$

To calculate the latter, we use integration by parts. So posing:

$$\left\{ \begin{array}{lcl} u &=& \cos(x) \\ v' &=& e^x \end{array} \right. \Longrightarrow \left\{ \begin{array}{lcl} u' &=& -\sin(x) \\ v &=& e^x \end{array} \right.$$

hence,

$$\begin{aligned} G(x) &= \int \cos(x)e^x dx \\ &= \cos(x)e^x + \int \sin(x)e^x dx \end{aligned} \tag{3}$$

To calculate the latter, we use also integration by parts. So posing:

$$\left\{ \begin{array}{lcl} u &=& \sin(x) \\ v' &=& e^x \end{array} \right. \Longrightarrow \left\{ \begin{array}{lcl} u' &=& \cos(x) \\ v &=& e^x \end{array} \right.$$

hence,

$$\begin{aligned} G(x) &= \cos(x)e^x + \int \sin(x)e^x dx \\ &= \cos(x)e^x + \sin(x)e^x - \int \sin(x)e^x dx \\ &= \cos(x)e^x + \sin(x)e^x - G(x). \end{aligned} \tag{4}$$

From the formula (4) we deduce that

$$G(x) = \frac{(\cos(x) + \sin(x))\, e^x}{2} + c_2, \text{ with } c_2 \in R.$$

3. I(x)=? and J(x)=? To find the expressions of I and J we must solve the following system

$$\left\{ \begin{array}{lcll} I + J &=& F(x), & ....(E_1) \\ I - J &=& G(x), & ....(E_2) \end{array} \right.$$

hence

- From $(E_1) + (E_2)$ we get $2I = F(x) + G(x) \Longrightarrow I = \frac{F(x)+G(x)}{2}$ i.e.

$$I = \frac{1}{4}(2 + \sin(x) + \cos(x))e^x + k_1, \text{ with } k_1 \in R.$$

- From $(E_1) - (E_2)$ we get $2J = F(x) - G(x) \Longrightarrow J = \frac{F(x)-G(x)}{2}$ i.e.

$$J = \frac{1}{4}(2 - \sin(x) - \cos(x))e^x + k_2, \text{ with } k_2 \in R.$$

Solution of the Exercise 3

$$y' - y = xy^n. \tag{5}$$

For the resolution of this equation three situations are possible, namely:

case $n = 0$ resolution of a linear differential equation with second member

$$y' - y = x.$$

- homogeneous solution :

$$y' - y = 0 \Rightarrow \frac{y'}{y} = 1 \Rightarrow \int \frac{dy}{y} = \int 1dx \Rightarrow \ln(y) = x + c \Rightarrow y = ke^x.$$

- the general solution (using the variation of the constant method): Let $k \equiv k(x)$, then

$$y = k(x)e^x \text{ and } y' = k'(x)e^x + k(x)e^x.$$

$$\begin{aligned}
&\Rightarrow \quad (k'(x)e^x + k(x)e^x) - (k(x)e^x) = x \\
&\Rightarrow \quad k'(x) = xe^{-x} \\
&\Rightarrow \quad k(x) = \int xe^{-x}dx \text{ (to be computed by integration by parts)} \\
&\Rightarrow \quad k(x) = -(x+1)e^{-x}.
\end{aligned}$$

The integral is calculated using the method of integration by parts and this by considering:

$$\begin{cases} u &= x \\ v' &= e^{-x} \end{cases} \Longrightarrow \begin{cases} u' &= 1 \\ v &= -e^{-x} + c \end{cases}$$

Finally, we conclude that the general solution of the considered equation is given by:

$$y = \left(-(x+1)e^{-x} + c\right) e^x = ce^x - (x+1).$$

case $n = 1$ is a linear equation without second member (with separate variables).

$$y' - (x+1)y = 0 \Rightarrow \frac{y'}{y} = x + 1 \Rightarrow \int \frac{dy}{y} = \int x + 1dx \Rightarrow \ln(y) = \frac{1}{2}x^2 + x + c.$$

Hence the general solution of the equation is given by:

$$y = e^{\frac{1}{2}x^2 + x + c}$$

case $n \neq 0$ **and** $n \neq 1$ resolution of a Bernoulli's differential equation.

$$y' - y = xy^n \Rightarrow y^{-n}y' - y^{1-n} = x$$

The first step in solving a Bernoulli differential equation is linearizing the given equation using the substitution $z = y^{1-n}$.if we put $z = y^{1-n}$ then $z' = (1-n)y^{-n}y'$, and by replacing these two expressions in the original equation we will have

$$z' - (1-n)z = (1-n)x$$

- The homogeneous solution of the new equation.

$$\begin{aligned}
z' - (1-n)z = 0 \quad &\Rightarrow \quad \frac{z'}{z} = (1-n) \\
&\Rightarrow \quad \int \frac{dz}{z} = \int (1-n)dx \\
&\Rightarrow \quad \ln(z) = (1-n)x + c \\
&\Rightarrow \quad z = ke^{(1-n)x}
\end{aligned}$$

- the general solution (using the variation of the constant method): Let $k \equiv k(x)$, then

$$y = ke^{(1-n)x} \text{ and } y' = k'(x)e^{(1-n)x} + (1-n)k(x)e^{(1-n)x}.$$

Hence,

$$\Rightarrow \left(k'(x)e^{-(n-1)x} + (1-n)k(x)e^{-(n-1)x}\right) - \left((1-n)k(x)e^{-(n-1))x}\right) = (1-n)x$$
$$\Rightarrow k'(x) = (1-n)xe^{(n-1)x}$$
$$\Rightarrow k(x) = \int (1-n)xe^{(n-1)x}dx \text{ (to be computed by integration by parts)}$$
$$\Rightarrow k(x) = -\left(x + \frac{1}{n-1}\right)e^{-(n-1)x} + c.$$

The integral is calculated using the method of integration by parts and this by considering:

$$\begin{cases} u &= x \\ v' &= (1-n)e^{-(1-n)x} \end{cases} \Longrightarrow \begin{cases} u' &= 1 \\ v &= -e^{-(1-n)x} \end{cases}$$

Finally, we conclude that the general solution of the considered equation is given by:

$$z = \left(-\left(x + \frac{1}{n-1}\right)e^{-(n-1)x} + c\right)e^{-(n-1)x} = ce^{-(n-1)x} - \left(x + \frac{1}{n-1}\right).$$

Finally, as $z = y^{1-n}$ we conclude that the general solution of the original equation given in (5) is

$$y = z^{\frac{1}{1-n}} = \left(ce^{-(n-1)x} - \left(x + \frac{1}{n-1}\right)\right)^{\frac{1}{1-n}}.$$

Solution of the Exercise 4 (./04pts)

1. Solve the following second order differential equation.

$$y'' + 2y' + 5y = 4e^{-x}.$$

   (a) The homogenous solution

$$y_h'' + 2y_h' + 5y_h = 0. \tag{6}$$

   Let put $R^2 = y^{''}$, $R = y'$ and $1 = y$ then (6) becomes $R^2 + 2R + 5 = 0$.

$$\Delta = 2^2 - 4 * 5 = -16 < 0 \Rightarrow \sqrt{\Delta} = 4i.$$

   This implied that the solution of the second order equation are complex, where

$$R_1 = \frac{-2 + \sqrt{\Delta}}{2} = -1 + 2i \quad and \quad R_2 = \frac{-2 - \sqrt{\Delta}}{2} = -1 - 2i.$$

   Consequently the solution y_h is

$$y_h = (c_1\cos(2x) + c_2\sin(2x))e^{-x}$$

   (b) The particular solution: The particular solution written in the general form of $4e^{-x} \Rightarrow y_p = ae^{-x}$.

$$y_p = ae^{-x} \Rightarrow y_p^{'} = -ae^{-x} \text{ and } y_p^{''} = ae^{-x}$$

   so,

$$ae^{-x} + 2\left(-ae^{-x}\right) + 5\left(ae^{-x}\right) = 4e^{-x} \Rightarrow a = 1 \Rightarrow y_p = e^{-x}$$

   (c) The general solution is given as follow:

$$y_g = y_h + y_p = (c_1\cos(2x) + c_2\sin(2x) + 1)e^{-x}.$$

2. Determines the solution that passed from the origin $(0,0)$ and the point $(\pi/4, 0)$.

$$\begin{cases} (c_1\cos(0) + c_2\sin(0) + 1)e^0 &= 0 \\ \\ (c_1\cos(\pi/2) + c_2\sin(\pi/2) + 1)\,e^{-\pi/4} &= 0 \end{cases} \Rightarrow \begin{cases} c_1 + 1 &= 0, \\ \\ c_2 + 1 &= 0, \end{cases} \Rightarrow \begin{cases} c_1 &= -1 \\ \\ c_2 &= -1 \end{cases}$$

$$y_0 = (1 - \cos(2x) - \sin(2x))e^{-x}.$$