Integral calculus

The indefinite integral

- grating irrational functions

Integration of irrational functions, examples

Integration of irrational functions of the form $\int \frac{P_n(x)}{\sqrt{ax^2+bx+c}}\,dx$,

Integrating irrational functions

Integrals of the form $\int R\left[x, \left(\frac{ax+b}{cx+d}\right)^{\frac{p_1}{q_1}}, \left(\frac{ax+b}{cx+d}\right)^{\frac{p_2}{q_2}}, \ldots\right]dx$

where, R is a rational function and, $p_1, q_1, p_2, q_2, \ldots$ are integers,

we can solve using substitution $\frac{ax+b}{cx+d} = z^n$,

where the power n is the least common multiple of $q_1, q_2, \ldots$.

Integration of irrational functions of the form $\int \frac{P_n(x)}{\sqrt{ax^2+bx+c}}\,dx$,

where $P_n(x)$ is an n-th degree polynomial.

Set $$\int \frac{P_n(x)}{\sqrt{ax^2+bx+c}}\,dx = Q_{n-1}(x)\cdot\sqrt{ax^2+bx+c} + \lambda\int\frac{dx}{\sqrt{ax^2+bx+c}},$$

where $Q_{n-1}(x)$ is an $(n-1)$ -th degree polynomial of undetermined coefficients and λ is a constant.

Coefficients of the polynomial Q and the constant λ we obtain by deriving the above identity.

Integration of irrational functions of the form $\int \frac{dx}{(x-\alpha)^n\sqrt{ax^2+bx+c}}$.

Given integral can be solved using the substitution $x-\alpha = 1/t$.

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Integrals of the form $\int R(x, \sqrt{ax^2 + bx + c})\, dx.$

Euler's substitutions

Integrating irrational functions using Euler's substitutions examples

Integrals of the form $\int R(x, \sqrt{ax^2 + bx + c})\, dx.$

Euler's substitutions

1. $\sqrt{ax^2 + bx + c} = t \pm \sqrt{a}\, x, \;\; a > 0,$

2. $\sqrt{ax^2 + bx + c} = \sqrt{a(x - x_1)(x - x_2)} = t(x - x_1) = t(x - x_2),$

3. $\sqrt{ax^2 + bx + c} = tx \pm \sqrt{c}, \;\; c > 0.$

Binomial integral

Integral of the form

$$\int x^m (a + bx^n)^p\, dx$$

is called the binomial integral where, a and b are real numbers while m, n and p are rational numbers.

If m, n and p all are integers then the integrand is a rational function integration of which is shown above.

There are only three cases the binomial integral can be solved by elementary functions:

1. if m and n are fractions and p is an integer then, the integral can be solved using substitution $x =$ ^{s}, where s is the least common denominator of m and n.

2. if p is a fraction and $(m + 1) / n$ is an integer, then the integral can be solved using substitution $a + bx^n = t^s$, where s is denominator of p.

3. if p is a fraction and $(m + 1) / n + p$ is an integer then, the integral can be solved using substitution $ax^{-n} + b = t^s$, where s is denominator of p.

Integral calculus

The indefinite integral

- Trigonometric integrals

Trigonometric integrals of the form $\int \sin^m x, \cos^n x\, dx$

Integrals of the rational functions containing sine and cosine, $\int R(\sin x, \cos x)\, dx$

Trigonometric integrals of the form

$$\int \sin^m x, \cos^n x\, dx$$

where m and n are integers, we use the following substitutions;

1) if m is a positive odd integer then, $\cos x = t$
2) if n is a positive odd integer then, $\sin x = t$
3) if $m + n$ is a negative even integer then,

$$\tan x = t \quad \text{or} \quad x = \tan^{-1} t, \quad dx = \frac{dt}{1+t^2}, \quad \sin x = \frac{t}{\sqrt{1+t^2}} \quad \text{and} \quad \cos x = \frac{1}{\sqrt{1+t^2}}.$$

If m and n are positive even integers then the integrand expression can be transformed using the following trigonometric identities,

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x), \quad \cos^2 x = \frac{1}{2}(1 + \cos 2x) \quad \text{and} \quad \sin x \cos x = \frac{1}{2}\sin 2x.$$

Trigonometric integrals of the form

$$\int \sin(m\,x)\sin(n\,x)\,dx, \quad \int \sin(m\,x)\cos(n\,x)\,dx, \quad \int \cos(m\,x)\cos(n\,x)\,dx,$$

in these cases we use the following product to sum formulas,

$$1)\quad \sin mx \sin nx = \frac{1}{2}[\cos(m-n)x - \cos(m+n)x],$$

$$2)\quad \sin mx \cos nx = \frac{1}{2}[\sin(m-n)x + \sin(m+n)x],$$

$$3)\quad \cos mx \cos nx = \frac{1}{2}[\cos(m-n)x + \cos(m+n)x].$$