Chapter III

Kinematic and static analysis of mechanism

III Kinematic and static analysis of mechanism

In this part, we will be interested in systems of solids connected to each other by frictionless joints (perfect joints), solids are non-deformable and we will quite often neglect the actions of gravity in front of other mechanical actions. The Fundamental Principle of Statics (FPS) therefore applies to each solid of the mechanism studied. The objective is both to study the kinematics of a mechanism (input output relationship) and the mechanical actions between the solids of the system studied. Each solid being in contact with one or more others, each connection between two solids will be described by one of the elementary joints presented previously.

III.1 Equivalent joint

Let us assume that there are several connections between two parts S_1 and S_2, made with or without intermediate parts. The connection equivalent to all the connections located between parts S_1 and S_2 is the theoretical reference joints L_{12}, which has the same behavior as this association of joints, i.e. it transmits the same mechanical action and allows the same movement. An illustration in terms of a joint graph is given in Figure III.1 where joints L_1, L_2, L_3 and L_4 as well as the material system S_3 are kinematically and mechanically equivalent to joint L_{12}. The connections that can exist between the connected solids are either in parallel or in series. Let us now see what this implies as a condition on the kinematic torsors and mechanical actions.

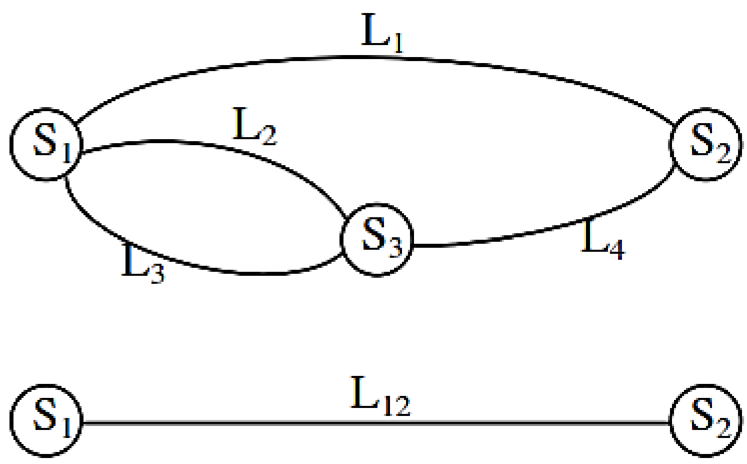


Figure III. 1 : Example of equivalent joint.

III.1.1 Parallel joints

We say that ***n*** joints ***L_1, L_2, ..., L_i, ..., L_n*** are arranged in parallel between two solids ***S_1*** and ***S_2*** if each joint directly connects these two solids. An illustration in terms of a joint graph is given in Figure II.2.

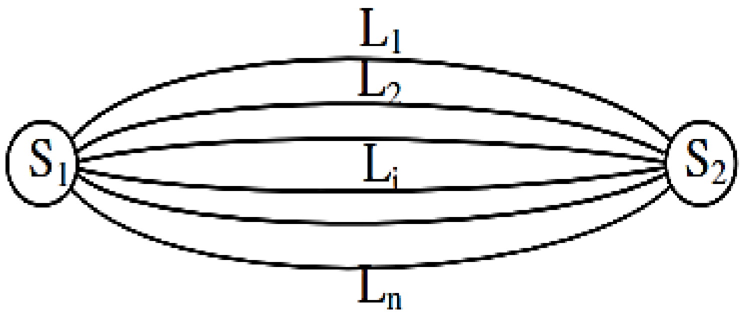


Figure III. 2 : *n* parallel joints.

a) ***Static torsor***

The components of mechanical actions transmissible between ***S_1*** and ***S_2*** are the set of actions transmissible by the joints ***L_i***. Indeed, by applying the fundamental principle of statics to the solid ***S_2*** for example, the following relation immediately follows:

$$\left\{\left[T_{Eq}\right]_{S1/S2}\right\}_M = \sum_{i=1}^{n}\left\{[T_i]_{S1/S2}\right\}_M$$

With : ***(i)*** represent the joint ***L_i***

Therefore, for a component of the static torsor of the equivalent joint not to be zero, it suffices that a single corresponding component of a joint ***L_i*** is not zero.

b) ***Kinematic torsor***

To obtain the kinematic torsor of the equivalent joints, it suffices to write that the kinematic torsor of the equivalent joints and must be compatible with all the kinematic torsors of the joints ***L_i*** for $i \in [1; n]$, that's to say :

$$\left\{\left[T_{Eq}\right]_{S1/S2}\right\}_M = \left\{[T_1]_{S1/S2}\right\}_M = \left\{[T_2]_{S1/S2}\right\}_M = \cdots\cdots\cdots = \left\{[T_n]_{S1/S2}\right\}_M$$

Example (Kinematic study)

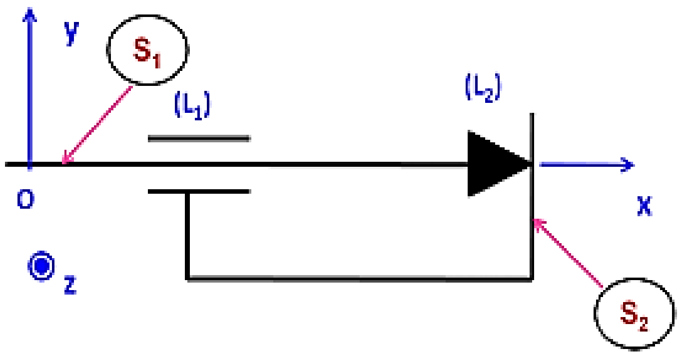
We consider two solids S_1 and S_2 assembled by two parallel joints:

L_1: Sliding pivot connection with axis $(O, \vec{x})$

L_2: Point connection with normal $(O, \vec{x})$

The kinematic torsors of these two connections are written at point ***0***, in the same basis ***R***.

$$[T_{L1}]_R = \begin{Bmatrix} \alpha_1 & u_1 \\ 0 & 0 \\ 0 & 0 \end{Bmatrix}_R \qquad [T_{L2}]_R = \begin{Bmatrix} \alpha_2 & 0 \\ \beta_2 & v_2 \\ \gamma_2 & w_2 \end{Bmatrix}_R$$



Solution

We suppose that the ***L₁₂*** is the equivalent joints of two joins ***L₁*** and ***L₂*** . The ***L₁₂*** torsor is written in the following form in the same basis ***R***:

$$[T_{L12}]_R = \begin{Bmatrix} \alpha & u \\ \beta & v \\ \gamma & w \end{Bmatrix}_R$$

The two joints ***L₁*** and ***L₂*** are parallel, so we have :

$$[T_{L12}]_R = [T_{L1}]_R = [T_{L2}]_R$$

$$\begin{Bmatrix} \alpha & u \\ \beta & v \\ \gamma & w \end{Bmatrix}_R = \begin{Bmatrix} \alpha_1 & u_1 \\ 0 & 0 \\ 0 & 0 \end{Bmatrix}_R = \begin{Bmatrix} \alpha_2 & 0 \\ \beta_2 & v_2 \\ \gamma_2 & w_2 \end{Bmatrix}_R$$

The equality of the torsors gives :

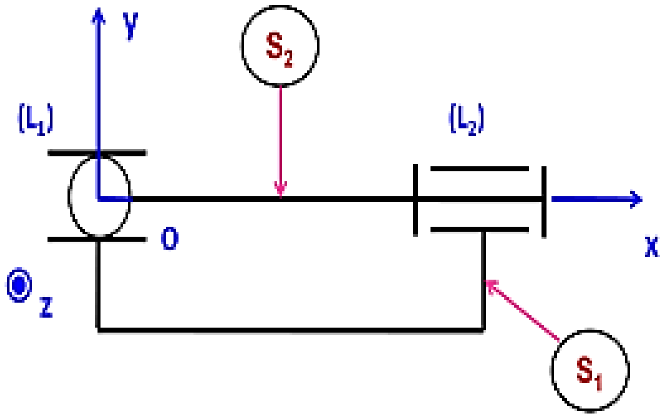
$$\begin{array}{ll} \alpha = \alpha_1 = \alpha_2 & u = u_1 = 0 \\ \beta = 0 = \beta_2 & v = 0 = v_2 \\ \gamma = 0 = \gamma_2 & w = 0 = w_2 \end{array}$$

So : the torsor of equivalent joints $[T_{L12}]_R = \begin{Bmatrix} \alpha = \alpha_1 = \alpha_2 & 0 \\ 0 & 0 \\ 0 & 0 \end{Bmatrix}_R$

The equivalent joint is the pivot along the $(O, \vec{x})$ axis with a single degrees of freedom.

Example 1

We consider a shaft ***S₂***, with axis $(O,\vec{x})$ mounted in a frame ***S₁*** via two joints ***L₁*** and ***L₂***. Link ***L₁*** is an annular linear joint with axis $(O,\vec{x})$, link ***L₂*** is an pivot along the axis $(O,\vec{x})$.



The statically torsors of two joints are written in the same basis ***O*** by the following formula:

$$[T_{S1}]_0 = \begin{Bmatrix} 0 & 0 \\ Y_1 & 0 \\ Z_1 & 0 \end{Bmatrix}_0 \qquad [T_{S2}]_0 = \begin{Bmatrix} X_2 & 0 \\ Y_2 & M_2 \\ Z_2 & N_2 \end{Bmatrix}_0$$

1- Determine the static torsor of the equivalent joint to the two parallel joints ***L₁*** and ***L₂***.
2- Determine the degree of hyperstaticity of the equivalent joint as well as the hyperstatic unknowns.
3- By a kinematic study, determine the kinematic torsor of the equivalent joint to the two joints ***L₁*** and ***L₂***.

Solution

1- The two statically torsors ***L₁*** and ***L₂*** are in the same basis ***0***, so maintain their shape :

$$[T_{S1}]_0 = \begin{Bmatrix} 0 & 0 \\ Y_1 & 0 \\ Z_1 & 0 \end{Bmatrix}_0 \qquad [T_{S2}]_0 = \begin{Bmatrix} X_2 & 0 \\ Y_2 & M_2 \\ Z_2 & N_2 \end{Bmatrix}_0$$

The equivalent torseur is written in the form :

$$[T_{S12}]_0 = \begin{Bmatrix} X & L \\ Y & M \\ Z & N \end{Bmatrix}_0$$

The two joints ***L₁*** and ***L₂*** are parallel so :

$$[T_{S12}]_0 = [T_{S1}]_0 + [T_{S2}]_0$$

$$\begin{Bmatrix} X & L \\ Y & M \\ Z & N \end{Bmatrix}_0 = \begin{Bmatrix} 0 & 0 \\ Y_1 & 0 \\ Z_1 & 0 \end{Bmatrix}_0 + \begin{Bmatrix} X_2 & 0 \\ Y_2 & M_2 \\ Z_2 & N_2 \end{Bmatrix}_0 = \begin{Bmatrix} X_2 & 0 \\ Y_1 + Y_2 & M_2 \\ Z_1 + Z_2 & N_2 \end{Bmatrix}_0$$

$$\begin{cases} X = X_2 \\ Y = Y_1 + Y_2 \\ Z = Z_1 + Z_2 \\ L = 0 \\ M = M_2 \\ N = N_2 \end{cases} \quad \text{with} \quad r_s = 5$$

So : $$[T_{S12}]_0 = \begin{Bmatrix} X_2 & 0 \\ Y_1 + Y_2 & M_2 \\ Z_1 + Z_2 & N_2 \end{Bmatrix}_0$$

The torsor of equivalent joint is the pivot along the $(O, \vec{x})$

2- The degree of hyperstaticity ***h***

$$h = I_s - r_s$$

With : $I_s = I_{s1} + I_{s2}$

The torsor $[T_{S1}]_0 = \begin{Bmatrix} 0 & 0 \\ Y_1 & 0 \\ Z_1 & 0 \end{Bmatrix}_0$ have $I_{s1} = 2$ and the torsor $[T_{S2}]_0 = \begin{Bmatrix} X_2 & 0 \\ Y_2 & M_2 \\ Z_2 & N_2 \end{Bmatrix}_0$ have $I_{s2} = 5$

So : $I_s = 2 + 5 = 7$

From the equations of equivalent joint we have $r_s = 5$

So : $h = 7 - 5,$ $h = 2$

The mobility ***m*** is calculated by :

$$m = E_s - r_s \quad \text{with} \quad E_s = 6.(N_L - 1) = 6.(2 - 1) = 6$$

$$m = 6 - 5 = 1$$

3- Kinematic torsor of equivalent joint ***L₁₂***

The kinematic torsors of two joints are :

$$[T_{K1}]_0 = \begin{Bmatrix} \alpha_1 & u_1 \\ \beta_1 & 0 \\ \gamma_1 & 0 \end{Bmatrix}_0 \qquad [T_{K2}]_0 = \begin{Bmatrix} \alpha_2 & 0 \\ 0 & 0 \\ 0 & 0 \end{Bmatrix}_0$$

The equivalent torseur is written in the form :

$$[T_{K12}]_0 = \begin{Bmatrix} \alpha & u \\ \beta & v \\ \gamma & w \end{Bmatrix}_0$$

The two joints ***L_1*** and ***L_2*** are parallel so the compatibility of joints requires that:

$$[T_{K12}]_0 = [T_{K1}]_0 = [T_{K2}]_0$$

$$\begin{Bmatrix} \alpha & u \\ \beta & v \\ \gamma & w \end{Bmatrix}_0 = \begin{Bmatrix} \alpha_1 & u_1 \\ \beta_1 & 0 \\ \gamma_1 & 0 \end{Bmatrix}_0 = \begin{Bmatrix} \alpha_2 & 0 \\ 0 & 0 \\ 0 & 0 \end{Bmatrix}_0$$

So we have :

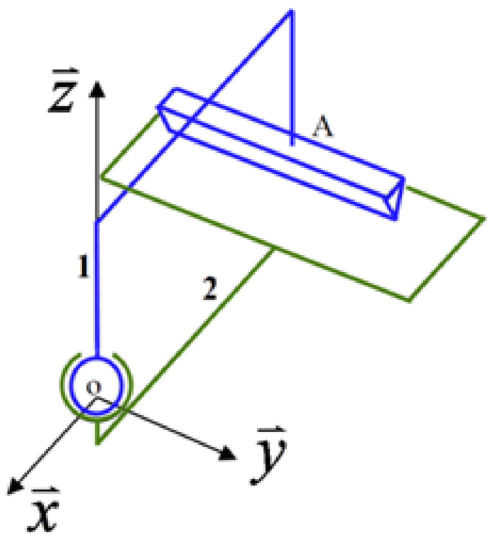
$$\begin{cases} \alpha = \alpha_1 = \alpha_2 \\ \beta = \beta_1 = 0 \\ \gamma = \gamma_1 = 0 \\ u = u_1 = 0 \\ v = 0 \\ w = 0 \end{cases} \quad \text{with} \quad r_c = 5$$

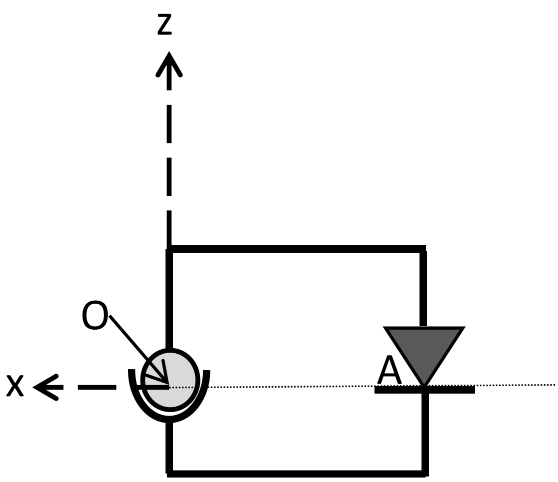
The equivalent torsor of joints is written as follow : $[T_{K12}]_0 = \begin{Bmatrix} \alpha = \alpha_1 = \alpha_2 & 0 \\ 0 & 0 \\ 0 & 0 \end{Bmatrix}_0$

The torsor of equivalent of joints is the pivot along the $(O, \vec{x})$

Example 2

The following Figure shows a diagram of a mechanism consisting of two links. A ball joint with center ***O*** and a rectilinear linear link (plane cylinder) with normal $\vec{z}$ and contact along (A, $\vec{y}$). with $\overrightarrow{OA} = -a.\vec{x}$.





- The static torsors of the ball joint ***S₁*** and the rectilinear linear joint ***S₂*** are defined respectively at point ***O*** and at point ***A*** by:

$$[S_1]_O = \begin{bmatrix} X_1 & 0 \\ Y_1 & 0 \\ Z_1 & 0 \end{bmatrix}_O \qquad [S_2]_A = \begin{bmatrix} 0 & L_2 \\ 0 & 0 \\ Z_2 & 0 \end{bmatrix}_A$$

- The kinematic torsors of the ball joint C_1 and of the rectilinear linear joint C_2 are defined respectively at point O and at point A by:

$$\left[K_1\right]_O = \begin{bmatrix} w_{x1} & 0 \\ w_{y1} & 0 \\ w_{z1} & 0 \end{bmatrix}_O \qquad \left[K_2\right]_A = \begin{bmatrix} 0 & V_{x2} \\ w_{y2} & V_{y2} \\ w_{z2} & 0 \end{bmatrix}_A$$

1- Draw the joint graph of the mechanisms.

2- Are the two joints in series or in parallel?

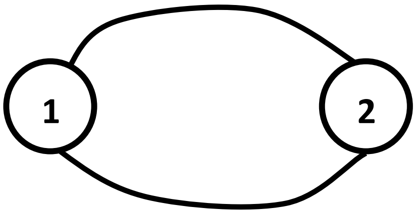
3- Calculate the cyclomatic number N_c.

4- Using a static approach, determine the equivalent joints at point **O** of the two joints.

5- Using a kinematic approach, determine the equivalent joint at point **O** of the two joints.

Solution

1- Joint graph (structure graph)



- **LJ :** Linear joint in A, along axis $\vec{y}$ with Normal axis $\vec{z}$

- **CBJ :** Center Ball Joint (O).

2- Les deux liaisons sont en parallèle

3- Nombre cyclomatique N_c

$\boldsymbol{N_c} = N_j - N_L + 1 = 1$

4- Equivalent joint (Static Approach)

The equivalent joint will be calculated at point **O**, so the torsor $\left[S_2\right]_A$ must be written at point **O**.

$$\left[S_2\right]_O = \left[S_2\right]_A + \overrightarrow{OA} \wedge \overrightarrow{\Omega_{2SA}}$$

$$\left[S_2\right]_O = \begin{bmatrix} 0 & L_2 \\ 0 & 0 \\ Z_2 & 0 \end{bmatrix}_A + \begin{pmatrix} -a \\ 0 \\ 0 \end{pmatrix} \wedge \begin{pmatrix} 0 \\ 0 \\ Z_2 \end{pmatrix}$$

$$\left[S_2\right]_O = \begin{bmatrix} 0 & L_2 \\ 0 & 0 \\ Z_2 & 0 \end{bmatrix}_A + \begin{pmatrix} 0 \\ a.Z_2 \\ 0 \end{pmatrix} \quad \text{so} \quad \left[S_2\right]_O = \begin{bmatrix} 0 & L_2 \\ 0 & a.Z_2 \\ Z_2 & 0 \end{bmatrix}_O$$

The two joints are in parallel so:

$$\left[S_{Eq}\right]_O = \left[S_1\right]_O + \left[S_2\right]_O = \begin{bmatrix} X_1 & 0 \\ Y_1 & 0 \\ Z_1 & 0 \end{bmatrix}_O + \begin{bmatrix} 0 & L_2 \\ 0 & a.Z_2 \\ Z_2 & 0 \end{bmatrix}_O$$

$$\left[S_{Eq}\right]_O = \begin{bmatrix} X_1 & L_2 \\ Y_1 & a.Z_2 \\ Z_1 + Z_2 & 0 \end{bmatrix}_O$$ the joint represent a pivot of axis $\vec{z}$

5- Equivalent joint (Kinematic Approach)

The same for the torsor $\left[K_2\right]_A$, the torsor is written in the point ***A*** and must be written at point ***O***.

$$\left[K_2\right]_O = \left[C_2\right]_A + \overrightarrow{OA} \wedge \overrightarrow{\Omega_{2CA}}$$

$$\left[K_2\right]_O = \begin{bmatrix} 0 & V_{x2} \\ w_{y2} & V_{y2} \\ w_{z2} & 0 \end{bmatrix}_A + \begin{pmatrix} -a \\ 0 \\ 0 \end{pmatrix} \wedge \begin{pmatrix} 0 \\ w_{y2} \\ w_{z2} \end{pmatrix}$$

$$\left[K_2\right]_O = \begin{bmatrix} 0 & V_{x2} \\ w_{y2} & V_{y2} \\ w_{z2} & 0 \end{bmatrix}_A + \begin{pmatrix} 0 \\ a.w_{z2} \\ -a.w_{y2} \end{pmatrix} \quad \text{so} \quad \left[C_2\right]_O = \begin{bmatrix} 0 & V_{x2} \\ w_{y2} & V_{y2} + a.w_{z2} \\ w_{z2} & -a.w_{y2} \end{bmatrix}_0$$

The two joints are in parallel so:

$$\left[C_{Eq}\right]_O = \left[C_1\right]_O = \left[C_2\right]_O \quad \text{with} \quad \begin{bmatrix} X & L \\ Y & M \\ Z & N \end{bmatrix}_O = \begin{bmatrix} w_{x1} & 0 \\ w_{y1} & 0 \\ w_{z1} & 0 \end{bmatrix}_O = \begin{bmatrix} 0 & V_{x2} \\ w_{y2} & V_{y2} + a.w_{z2} \\ w_{z2} & -a.w_{y2} \end{bmatrix}_0$$

For the equality we obtain:

$$X = w_{x1} = 0 \qquad L = 0 = V_{x2}$$

$$Y = w_{y1} = w_{y2} \qquad M = 0 = V_{y2} + a.w_{z2}$$

$$Z = w_{z1} = w_{z2} \qquad N = 0 = -a.w_{y2} \quad , \qquad w_{y2} = 0$$

$$\left[C_{Eq}\right]_O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ w_{z1} & 0 \end{bmatrix}_O$$ Joint pivot along axis $\vec{z}$

Example 3

1- Draw the graph of the structure of the slide on a ball joint axis mechanisms, shown in the following figure.

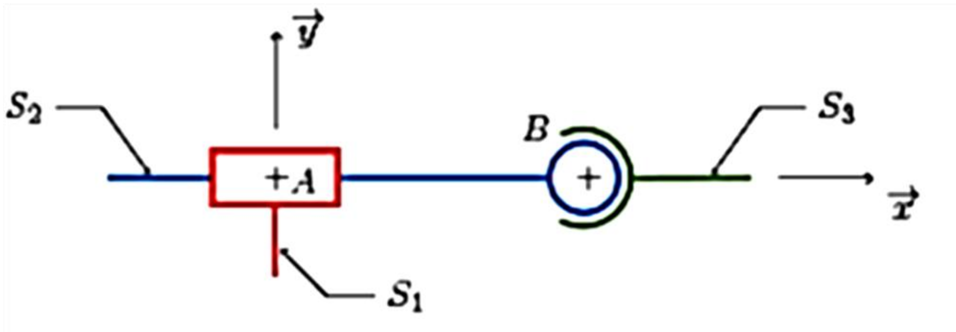
2- Give their kinematic chain type and their cyclomatic number.

3- Name each of the joints (Center, axis,…).

4- Determine their equivalent joints using the kinematic approach and then the static approach.

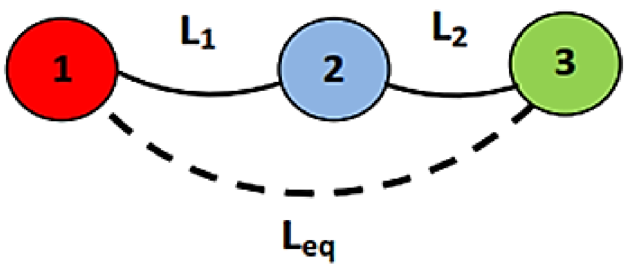
With the Kinematic and static joints are expressed by:

$$[K_{L1}]_A = \begin{bmatrix} 0 & V_{x1} \\ 0 & 0 \\ 0 & 0 \end{bmatrix}_A, [K_{L2}]_B = \begin{bmatrix} w_{x2} & 0 \\ w_{y2} & 0 \\ w_{z2} & 0 \end{bmatrix}_B, \quad [S_{L1}]_A = \begin{bmatrix} 0 & L_1 \\ Y_1 & M_1 \\ Z_1 & N_1 \end{bmatrix}_A, [S_{L2}]_B = \begin{bmatrix} X_2 & 0 \\ Y_2 & 0 \\ Z_2 & 0 \end{bmatrix}_B$$



Solution :

1- Graph of the structure (joint graph)



2- kinematic chain type and the cyclomatic number $\boldsymbol{N_c}$

$$\boldsymbol{N_c} = N_j - N_L + 1 = 2 - 3 + 1 = 0$$

No cycle is obtained because the chain is open (The joints are in series)

3- Name each of the joints (Center, axis,…).

L_1: Slide along axis $\vec{x}$

L_2: Ball joint with Center ***B***

4- Equivalent joints at point ***B***

a- kinematic approach

The torsor $[K_{L1}]_A$ is written at point ***A*** and must be written at point ***B***

$$[K_{L1}]_B = [K_{L1}]_A + \overrightarrow{BA} \wedge \overrightarrow{\Omega_A} = \begin{bmatrix} 0 & V_{x1} \\ 0 & 0 \\ 0 & 0 \end{bmatrix}_A + \begin{bmatrix} -AB \\ 0 \\ 0 \end{bmatrix} \wedge \begin{bmatrix} V_{x1} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & V_{x1} \\ 0 & 0 \\ 0 & 0 \end{bmatrix}_B$$

$$[K_{L1}]_B = \begin{bmatrix} 0 & V_{x1} \\ 0 & 0 \\ 0 & 0 \end{bmatrix}_B \quad \text{and} \quad [K_{L2}]_B = \begin{bmatrix} w_{x2} & 0 \\ w_{y2} & 0 \\ w_{z2} & 0 \end{bmatrix}_B$$

$$[K_{L/Eq}]_B = [K_{L1}]_B + [K_{L2}]_B = \begin{bmatrix} 0 & V_{x1} \\ 0 & 0 \\ 0 & 0 \end{bmatrix}_B + \begin{bmatrix} w_{x2} & 0 \\ w_{y2} & 0 \\ w_{z2} & 0 \end{bmatrix}_B$$

$$[K_{L/Eq}]_B = \begin{bmatrix} w_{x2} & V_{x1} \\ w_{y2} & 0 \\ w_{z2} & 0 \end{bmatrix}_B$$

The Equivalent joints represent a linear annular joint along the axis $\vec{x}$.

b- Static Approach

The torsor $[S_{L1}]_A$ is written at point ***A*** and must be written at point ***B***

$$[S_{L1}]_B = [S_{L1}]_A + \overrightarrow{BA} \wedge \overrightarrow{\Omega_A} = \begin{bmatrix} 0 & L_1 \\ Y_1 & M_1 \\ Z_1 & N_1 \end{bmatrix}_A + \begin{bmatrix} -AB \\ 0 \\ 0 \end{bmatrix} \wedge \begin{bmatrix} 0 \\ Y_1 \\ Z_1 \end{bmatrix} = \begin{bmatrix} 0 & L_1 \\ Y_1 & M_1 \\ Z_1 & N_1 \end{bmatrix}_A + \begin{bmatrix} 0 & 0 \\ 0 & AB.Z_1 \\ 0 & -AB.Y_1 \end{bmatrix}_B$$

$$[S_{L1}]_B = \begin{bmatrix} 0 & L_1 \\ Y_1 & M_1 + AB.Z_1 \\ Z_1 & N_1 - AB.Y_1 \end{bmatrix}_B, \quad [S_{L2}]_B = \begin{bmatrix} X_2 & 0 \\ Y_2 & 0 \\ Z_2 & 0 \end{bmatrix}_B$$

The two joints are in series so:

$$[S_{L/Eq}]_B = [S_{L1}]_B = [S_{L2}]_B$$

$$[S_{L/Eq}]_B = \begin{bmatrix} X & L \\ Y & M \\ Z & N \end{bmatrix}_B = \begin{bmatrix} 0 & L_1 \\ Y_1 & M_1 + AB.Z_1 \\ Z_1 & N_1 - AB.Y_1 \end{bmatrix}_B = \begin{bmatrix} X_2 & 0 \\ Y_2 & 0 \\ Z_2 & 0 \end{bmatrix}_B$$

we obtain : $X = 0 = X_2$, $Y = Y_1 = Y_2$, $Z = Z_1 = Z_2$

$L = L_1 = 0$, $M = M_1 + AB.Z_1 = 0$, $N = N_1 - AB.Y_1 = 0$

So : $\left[S_{L/Eq}\right]_B = \begin{bmatrix} X & L \\ Y & M \\ Z & N \end{bmatrix}_B = \begin{bmatrix} 0 & 0 \\ Y & 0 \\ Z & 0 \end{bmatrix}_B$, with : $\boldsymbol{Y = Y_1 = Y_2}$ and $\boldsymbol{Z = Z_1 = Z_2}$

III.2 Kinematic and static analysis of closed chains

III.2.1 Geometric study of a closed chain mechanism

To achieve the geometric study of a closed-loop system (figure III.3), it is sufficient to write the vector relation connecting the characteristic points of each solid.

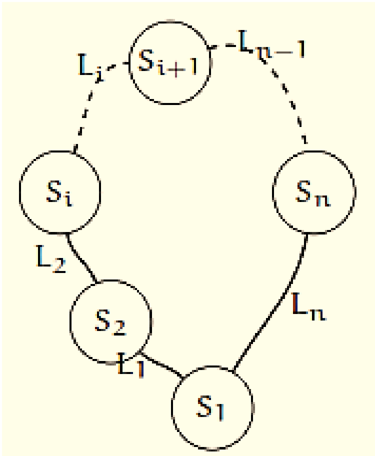


Figure III. 3: Closed chains.

Let $\boldsymbol{O_i}$ be the characteristic point of the solid $\boldsymbol{S_i}$, the closing relation of the geometric chain is written as:

$$\overrightarrow{0_1 0_2} + \overrightarrow{0_2 0_3} + \cdots + \overrightarrow{0_{\iota-1} 0_\iota} + \cdots + \overrightarrow{0_{n-1} 0_n} + \overrightarrow{0_n 0_1} = \vec{0}$$

By projecting this vector equation into an orthonormal basis, we obtain 3 scalar equations linking the different geometric parameters. In the case of a plane mechanism, we obtain 2 scalar equations, deduced from the projection of this relation onto the axes of the plane.

example

Example: Walking robot

Consider the walking robot in Figure III.4. The six legs of the robot are identical and are synchronized three by three using the alternating “tripod” technique. Within each flank, the center leg is angularly offset from the other two by the angle π (rad).

- The housing (0) is assumed to be fixed;

- The crankpin (1) pivots relative to the housing around the axis (O, z_0) and in axis pivot connection (B, z_0) with the lug (2);
- The lug (2) pivots around the axis (A, z_0) with the guide (3), it is simultaneously driven by the crankpin (2) by a connection at B;
- The guide (3) slides along (O, y_0) relative to the housing (0).

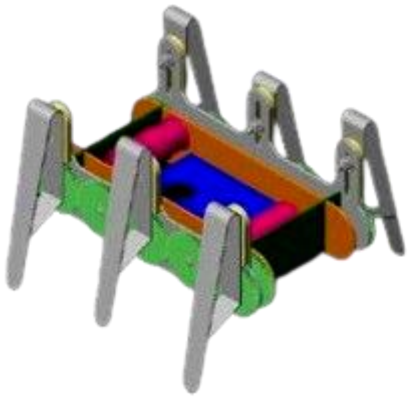
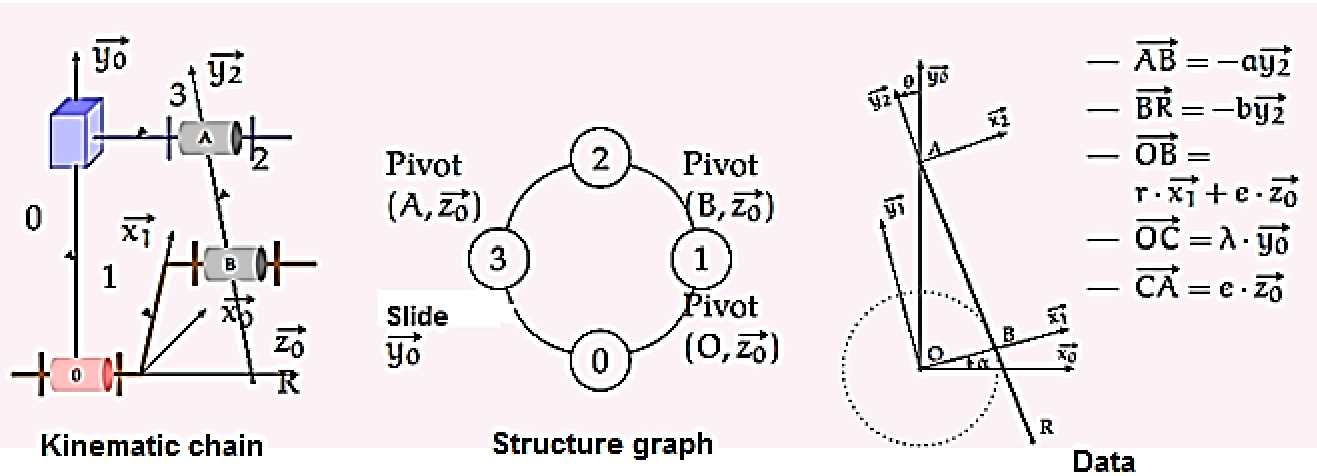


Figure III. 4 : Walking robot.



We recognize on the structure graph a simple closed chain. From the two base change figures we determine:

- $\overrightarrow{\Omega_{1/0}} = \dot{\alpha}.\vec{z_0}$
- $\overrightarrow{\Omega_{2/0}} = \overrightarrow{\Omega_{2/3}} + \overrightarrow{\Omega_{3/0}} = \dot{\theta}\vec{z_0} + \vec{0}$

The geometric closure is written:

$$\overrightarrow{OC} + \overrightarrow{CA} + \overrightarrow{AB} + \overrightarrow{BO} = \vec{0}$$

Either in projection in $(\vec{x_0}, \vec{y_0}, \vec{z_0})$:

$$\begin{cases} a.\sin\theta - r.\cos\alpha = 0 \\ \lambda - a.\cos\theta - r.\sin\alpha = 0 \\ e - e = 0 \end{cases}$$

From these relations, it is quite easy to obtain the relations between θ and α and then between λ and α.

- For θ: $\sin\theta = \frac{r}{a}.\cos\alpha$ with $\theta = \arcsin\left(\frac{r}{a}.\cos\alpha\right)$
- For λ :

  $\lambda = a.\cos\theta + r.\sin\alpha = a.\sqrt{1 - sin^2\theta} + r.\sin\alpha$ so :

  $\lambda = a.\sqrt{1 - (\frac{r}{a}.\cos\alpha)^2} + r.\sin\alpha$

III.2.2 Kinematic analysis of closed chains

Consider a closed chain mechanism composed of ***n*** solids and ***n*** links (figure III.5). For each link L_i, we can write the kinematic torsor between the two solids S_i and S_{i+1} of the link at the point O_i characteristic of the link.

$$[K_{(i+1)/i}] = \begin{Bmatrix} \vec{\Omega}_{(i+1)/i} \\ \vec{V}_{O_i \epsilon (i+1)/i} \end{Bmatrix}_{O_i}$$

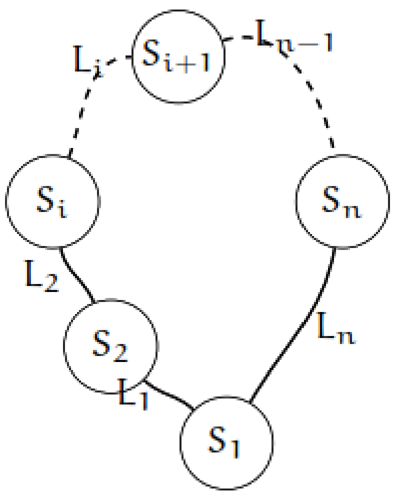


Figure III. 5: Closed chain mechanism.

The kinematic closure is obtained by writing the sum of the torsors at the same point O_i :

$$[K_{1/2}]_{O_i} + [K_{2/3}]_{O_i} + \cdots + [K_{(i-1)/i}]_{O_i} + [K_{i/(i+1)}]_{O_i} + [K_{n/1}]_{O_i} = [0]$$

This relationship allows us to obtain 2 vector equations, and after projection, 6 scalar equations.

From this system of equations, we deduce the degree of mobility of the mechanism.

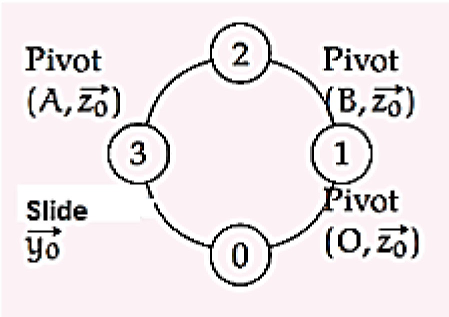
Note: This sum of torsors can only be calculated if the torsors are written at the same point.

Example: Walking robot (The continuation)

The kinematic closure is written by :

$$[K_{3/2}] = [K_{i/3}]_{O_i} + \cdots + [K_{(i-1)/i}]_{O_i} + [K_{i/(i+1)}]_{O_i} + [K_{n/1}]_{O_i} = [0]$$

$$[K_{3/2}] + [K_{2/1}] + [K_{1/0}] + [K_{0/3}] = [0]$$



Structure graph

With :

$$[K_{3/2}]_A = \begin{Bmatrix} \vec{\Omega}_{3/2} = \omega_{32}.\vec{z_0} \\ 0 \end{Bmatrix}_A \qquad [K_{1/0}]_A = \begin{Bmatrix} \vec{\Omega}_{1/0} = \dot{\alpha}.\vec{z_0} \\ 0 \end{Bmatrix}_0$$

$$[K_{2/1}]_A = \begin{Bmatrix} \vec{\Omega}_{2/1} = \omega_{21}.\vec{z_0} \\ 0 \end{Bmatrix}_B \qquad [K_{0/3}]_C = \begin{Bmatrix} 0 \\ \vec{V}_{0/3} = -\dot{\lambda}.\vec{y_0} \end{Bmatrix}_C$$

The kinematic closure on point ***O*** is written by :

$$[K_{3/2}]_O = [K_{3/2}]_A + \vec{\Omega}_{3/2} \wedge \overrightarrow{AO} = \omega_{32}.\vec{z_0} \wedge (-e.\vec{z_0} - \lambda.\vec{y_0}) = \lambda.\omega_{32}.\vec{x_0}$$

$$[K_{2/1}]_O = [K_{2/1}]_B + \vec{\Omega}_{2/1} \wedge \overrightarrow{BO} = \omega_{21}.\vec{z_0} \wedge (-r.\vec{x_1} - e.\vec{z_0}) = -r.\omega_{21}.\vec{y_1}$$

$$[K_{2/1}]_O = -r.\omega_{21}.\cos\alpha\,.\vec{y_0} + r.\omega_{21}.\sin\alpha\,.\vec{x_0}$$

$$[K_{3/2}]_O = \begin{Bmatrix} \omega_{32}.\overrightarrow{z_0} \\ \lambda.\omega_{32}.\overrightarrow{x_0} \end{Bmatrix}_0 = \begin{Bmatrix} 0 & \lambda.\omega_{32} \\ 0 & 0 \\ \omega_{32} & 0 \end{Bmatrix}_{0(\overrightarrow{x_0},\overrightarrow{y_0},\overrightarrow{z_0})}$$

$$[K_{2/1}]_O = \begin{Bmatrix} \omega_{21}.\overrightarrow{z_0} \\ -r.\omega_{21}.\cos\alpha.\overrightarrow{y_0} + r.\omega_{21}.\sin\alpha.\overrightarrow{x_0} \end{Bmatrix}_O = \begin{Bmatrix} 0 & r.\omega_{21}.\sin\alpha \\ 0 & -r.\omega_{21}.\cos\alpha \\ \omega_{21} & 0 \end{Bmatrix}_{0(\overrightarrow{x_0},\overrightarrow{y_0},\overrightarrow{z_0})}$$

$$[K_{1/0}]_O = \begin{Bmatrix} \dot{\alpha}.\overrightarrow{z_0} \\ 0 \end{Bmatrix}_0 = \begin{Bmatrix} 0 & 0 \\ 0 & 0 \\ \omega_{10} & 0 \end{Bmatrix}_{0(\overrightarrow{x_0},\overrightarrow{y_0},\overrightarrow{z_0})} , [K_{0/3}]_O = [K_{0/3}]_C = \begin{Bmatrix} 0 \\ -\dot{\lambda}.\overrightarrow{y_0} \end{Bmatrix}_O = \begin{Bmatrix} 0 & 0 \\ 0 & -\dot{\lambda} \\ 0 & 0 \end{Bmatrix}_{0(\overrightarrow{x_0},\overrightarrow{y_0},\overrightarrow{z_0})}$$

Where the the six equations system of kinematic closure is written by :

$$[K_{3/2}]_O + [K_{2/1}]_O + [K_{1/0}]_O + [K_{0/3}]_O = 0$$

$$\begin{Bmatrix} 0 & \lambda.\omega_{32} \\ 0 & 0 \\ \omega_{32} & 0 \end{Bmatrix}_O + \begin{Bmatrix} 0 & r.\omega_{21}.\sin\alpha \\ 0 & -r.\omega_{21}.\cos\alpha \\ \omega_{21} & 0 \end{Bmatrix}_O + \begin{Bmatrix} 0 & 0 \\ 0 & 0 \\ \omega_{10} & 0 \end{Bmatrix}_O + \begin{Bmatrix} 0 & 0 \\ 0 & -\dot{\lambda} \\ 0 & 0 \end{Bmatrix}_O = 0$$

$$\begin{cases} 0 = 0 \\ 0 = 0 \\ \omega_{32} + \omega_{21} + \omega_{10} = 0 \\ \lambda.\omega_{32} + r.\omega_{21}.\sin\alpha + 0 + 0 = 0 \\ 0 - r.\omega_{21}.\cos\alpha - \dot{\lambda} = 0 \\ 0 = 0 \end{cases} \Rightarrow \begin{cases} 0 = 0 \\ 0 = 0 \\ -\dot{\theta} + \omega_{21} + \dot{\alpha} = 0 \\ -\lambda.\dot{\theta} + r.\omega_{21}.\sin\alpha = 0 \\ -r.\omega_{21}.\cos\alpha - \dot{\lambda} = 0 \\ 0 = 0 \end{cases}$$

With : $\vec{\Omega}_{3/2} = \vec{\Omega}_{3/0} - \vec{\Omega}_{2/0}$, we have : $\omega_{32} = -\dot{\theta}$

We recognize a system of 6 equations with 4 unknowns. The rank of this system is $r_K = 3$. To solve this system, we must choose a parameter. It is often judicious for the kinematic study, to choose either the input parameter (here is $\dot{\alpha}$) or the output parameter, we call this parameter the pilot parameter. Here, it is sufficient to choose a single parameter α. The number of parameters that it is necessary to impose to solve the system is the degree of mobility of the mechanism. The mobility of the mechanism is therefore m = 1.

$$\begin{cases} -\dot{\theta} + \omega_{21} = -\dot{\alpha} \\ -\lambda.\dot{\theta} + r.\omega_{21}.\sin\alpha = 0 \\ -r.\omega_{21}.\cos\alpha - \dot{\lambda} = 0 \end{cases}$$

After solving the sytem equations we obtain :

$$\dot{\lambda} = -\frac{r.\lambda.\cos\alpha}{r.\sin\alpha - \lambda}.\dot{\alpha} \quad , \qquad \dot{\theta} = \frac{r.\sin\alpha}{r.\sin\alpha - \lambda}.\dot{\alpha} \quad , \qquad \omega_{21} = -\frac{\lambda}{r.\sin\theta - \lambda}.\dot{\alpha}$$

With : $\lambda = a.\cos\theta + r.\sin\alpha$

So : $\dot{\theta} = \frac{r.\sin\alpha}{r.\sin\alpha.\cos\theta - r.\sin\alpha}.\dot{\alpha} = -\frac{r.\sin\alpha}{a.\cos\theta}.\dot{\alpha}$

This relation represent the derivate of the geometric relation between θ and α.

III.2.3 Static analysis of closed chains

Consider a mechanism (figure II.6) formed of ***N*** solids connected by ***L*** links. We apply the F.P.S to each solid except the frame, thus for the solid **S_i**:

$$\{A_{2\to i}\} + \{A_{k\to i}\} + \{A_{(i+1)\to i}\} + \{F_{ext\to i}\} = \{0\}$$

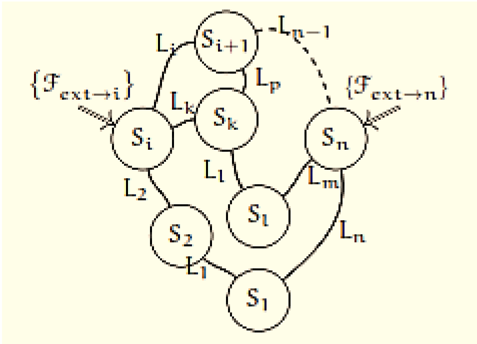


Figure III. 6: Closed chain mechanism

For each equilibrium, we can therefore write a system of 6 equations. It is possible to study the equilibrium of N−1 solids (the other equilibria are deduced from these), from which, for the whole mechanism a total number of equations of:

$$E_s = 6 \cdot (N-1)$$

Note: Writing the F.P.S assumes that the system is in equilibrium, we will assume here that the masses are negligible and/or the speeds are constant in order to apply it.

We will see later that it is possible to perform the calculations with zero forces, in fact the objective is not the equilibrium of the parts or the study of the movement but the determination of the mobilities and the hyperstaticity of the mechanism and the result of this calculation do not depend on external forces.

Each action transmitted torsors by a ***L_i*** bond of the solid Si−1 in $S_i(\{A_{(i-1)\to i}\})$ comporte ***n_{si}*** unknowns. Table III.1 shows some links and their associated unknowns.

Table III. 1 : Links and their associated unknowns.

Links	Static Torsor	n_{si}

pivot	$\begin{Bmatrix} X_i & 0 \\ Y_i & M_i \\ Z_i & N_i \end{Bmatrix}_{P_{i(\overrightarrow{x_i},\overrightarrow{y_i},\overrightarrow{z_i})}}$	$n_{si} = 5$
Slide Pivot	$\begin{Bmatrix} 0 & 0 \\ Y_i & M_i \\ Z_i & N_i \end{Bmatrix}_{P_{i(\overrightarrow{x_i},\overrightarrow{y_i},\overrightarrow{z_i})}}$	$n_{si} = 4$
Spherical link (kneecap, ball joint)	$\begin{Bmatrix} X_i & 0 \\ Y_i & 0 \\ Z_i & 0 \end{Bmatrix}_{P_{i(\overrightarrow{x_i},\overrightarrow{y_i},\overrightarrow{z_i})}}$	$n_{si} = 3$

The total number of static unknowns for the ***L*** connections is therefore:

$$I_s = \sum_{i=1}^{L} n_{si}$$

The global study of the mechanism is the study of a linear system of ***E_s*** equations with ***I_s*** unknowns. This system is a linear system whose rank is noted ***r_s***. The degree of hyperstaticity ***h*** corresponds to the number of unknowns of connection that cannot be determined by the resolution of the system:

$$h = I_s - r_s$$

- If $h = 0$, so it is possible to determine all the unknowns of connections or links, the system is isostatic.
- If $h > 0$, there are more unknowns than independent equations, the system is hyperstatic. The number of undetermined link unknowns represents the degree of hyperstaticity ***h***.

Example: Walking robot (Static study)

We continue the previous example of the walking robot. We complete this example by adding mechanical actions.

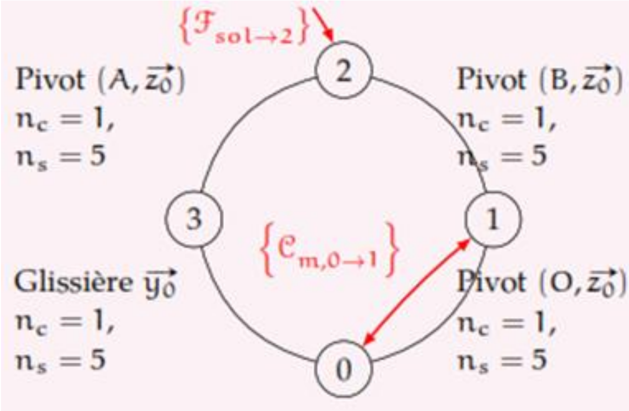
- A motor torque of 0 ---1 modeled by the following torsor :

$$\{C_{m,0\to1}\} = \begin{Bmatrix} \vec{0} \\ C_m.\vec{z}_0 \end{Bmatrix}_{\forall P}$$

- A mechanical action, modeling the action of the ground on the foot (2) modeled by the sliding torsor at point R:

$$\{F_{sol\to2}\} = \begin{Bmatrix} F.\vec{y}_0 \\ \vec{0} \end{Bmatrix}_R$$

The structure graph consists of a single loop:



We start by writing the different action torsors transmissible by the links and we identify the link unknowns:

$$[A_{1\to 2}] = \begin{Bmatrix} X_{12} & L_{12} \\ Y_{12} & M_{12} \\ Z_{12} & 0 \end{Bmatrix}_{B_{(?,?,\overrightarrow{z_0})}} \qquad [A_{2\to 3}] = \begin{Bmatrix} X_{23} & L_{23} \\ Y_{23} & M_{23} \\ Z_{23} & 0 \end{Bmatrix}_{A_{(?,?,\overrightarrow{z_0})}}$$

$$[A_{3\to 0}] = \begin{Bmatrix} X_{30} & L_{30} \\ 0 & M_{30} \\ Z_{30} & N_{30} \end{Bmatrix}_{\forall P_{(?,,\overrightarrow{y_0},\vec{0})}} \qquad [A_{0\to 1}] = \begin{Bmatrix} X_{01} & L_{01} \\ Y_{01} & M_{01} \\ Z_{01} & 0 \end{Bmatrix}_{O_{(?,?,\overrightarrow{z_0})}}$$

Each pivot link has n_s = 5 "static" unknowns. The sliding links also. Overall assessment: The system has 4 solids.

- It is possible to isolate at most ***N−1 = 3*** independent sets, i.e. ***E_C = 6*** .(N − 1) = 18 equations of statics with ***I_c*** = 20 unknowns.

From this simple assessment and before any calculation, we can say:

- That the mechanism is at least hyperstatic of order h = ***I_c*** - ***E_s*** = 2, in fact the rank of the static equation system is at most 18, with :

$$r_s \leq \min\,(I_c, E_c)$$

Degree of hyperstaticity : To determine the degree of hyperstaticity, we must isolate N − 1 = 3 solids (or set of solids) independent of the mechanism in order to determine all the unknown bonds.

1) ***Isolate the solid 3:*** It is subjected to two mechanical actions of connections, the slide between (0) and (3) and the pivot at point A between (2) and (3). We choose to write the FPS at A in the base $(\overrightarrow{x_0},,\overrightarrow{y_0},\overrightarrow{z_0})$.

$$[A_{0\to3}] + [A_{23}] = [0]$$

$$-\begin{Bmatrix} X_{30} & L_{30} \\ 0 & M_{30} \\ Z_{30} & N_{30} \end{Bmatrix}_{A(\overrightarrow{x_0},\overrightarrow{y_0},\overrightarrow{z_0})} + \begin{Bmatrix} X_{23} & L_{23} \\ Y_{23} & M_{23} \\ Z_{23} & 0 \end{Bmatrix}_{A(\overrightarrow{x_0},\overrightarrow{y_0},\overrightarrow{z_0})} = \begin{Bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{Bmatrix}_{A(\overrightarrow{x_0},\overrightarrow{y_0},\overrightarrow{z_0})}$$

Which allows us to write the first system of 6 equations (3 for the resultant, 3 for the moment):

$$\begin{cases} X_{23} - X_{30} = 0 \\ Y_{23} = 0 \\ Z_{23} - Z_{30} = 0 \end{cases} \quad and \quad \begin{cases} L_{23} - L_{30} = 0 \\ M_{23} - M_{30} = 0 \\ N_{30} = 0 \end{cases}$$

2) ***Isolate solid 2:*** It is subject to the two mechanical actions of connections (pivot at A, and pivot at B) and to the external mechanical action at R.

$$[A_{1\to2}] + [A_{3\to2}] + [F_{sol\to2}] = [0]$$

We choose to write the FPS in A in the base $(\overrightarrow{x_0}, \overrightarrow{y_0}, \overrightarrow{z_0})$:

$$\overrightarrow{M_{A,1\to2}} = \overrightarrow{M_{B,1\to2}} + \overrightarrow{AB} \wedge (X_{12}.\overrightarrow{x_0} + Y_{12}.\overrightarrow{y_0} + Z_{12}.\overrightarrow{z_0})$$

$$\overrightarrow{M_{A,1\to2}} = L_{12}.\overrightarrow{x_0} + M_{12}.\overrightarrow{y_0} - a.\overrightarrow{y_2} \wedge (X_{12}.\overrightarrow{x_0} + Y_{12}.\overrightarrow{y_0} + Z_{12}.\overrightarrow{z_0})$$

$$\overrightarrow{M_{A,1\to2}} = (L_{12} - a.\cos\theta\,.Z_{12}).\overrightarrow{x_0} + (M_{12} - a.\sin\theta\,.Z_{12}).\overrightarrow{y_0} + a.(\cos\theta\,.X_{12} + \sin\theta\,.Y_{12}).\overrightarrow{z_0}$$

$$\overrightarrow{M_{A,Sol\to2}} = \vec{0} + \overrightarrow{AR} \wedge F.\overrightarrow{y_0}$$

$$\overrightarrow{M_{A,Sol\to2}} = -(a+b).\overrightarrow{y_2} \wedge F.\overrightarrow{y_0} = F.(a+b).\sin\theta\,.\overrightarrow{z_0}$$

The fundamental principle of static (FPS) is therefore written in the base $(\overrightarrow{x_0}, \overrightarrow{y_0}, \overrightarrow{z_0})$ and at point A as:

$$\begin{Bmatrix} X_{12} & L_{12} - a.\cos\theta\,.Z_{12} \\ Y_{12} & M_{12} - a.\sin\theta\,.Z_{12} \\ Z_{12} & a.(\cos\theta\,.X_{12} + \sin\theta\,.Y_{12}) \end{Bmatrix} - \begin{Bmatrix} X_{23} & L_{23} \\ Y_{23} & M_{23} \\ Z_{23} & 0 \end{Bmatrix} + \begin{Bmatrix} 0 & 0 \\ F & 0 \\ 0 & F.(a+b).\sin\theta \end{Bmatrix} = \begin{Bmatrix} O & 0 \\ 0 & 0 \\ 0 & 0 \end{Bmatrix}_{A,(\overrightarrow{x_0},\overrightarrow{y_0},\overrightarrow{z_0})}$$

Finaly, the six equations the second equilibrium are:

$$\begin{cases} X_{12} - X_{23} + 0 = 0 \\ Y_{12} - Y_{23} + F = 0 \\ Z_{12} - Z_{23} + 0 = 0 \end{cases} \text{ and } \begin{cases} L_{12} - a.\cos\theta\,.Z_{12} - L_{23} + 0 = 0 \\ M_{12} - a.\sin\theta\,.Z_{12} - M_{23} + 0 = 0 \\ a.\cos\theta\,.X_{12} + a.\sin\theta\,.Y_{12} - 0 + F.(a+b).\sin\theta = 0 \end{cases}$$

3) ***Isolate the solid 1:*** It is subjected to 2 mechanical links actions (pivot at B and O) and to the engine torque.

$$[A_{0\to1}] - [A_{1\to2}] + \left[C_{m,0\to1}\right] = [0]$$

We choose to write the FPS in point O in the base $(\overrightarrow{x_0}, \overrightarrow{y_0}, \overrightarrow{z_0})$:

$$\overrightarrow{M_{O,1\to2}} = \overrightarrow{M_{A,1\to2}} + r.\overrightarrow{x_1} \wedge (X_{12}.\overrightarrow{x_0} + Y_{12}.\overrightarrow{y_0} + Z_{12}.\overrightarrow{z_0})$$

$$\overrightarrow{M_{O,1\to2}} = (L_{12} + r.Z_{12}.sin\ \alpha).\overrightarrow{x_0} + (M_{12} - r.Z_{12}.\cos\alpha).\overrightarrow{y_0} + (-r.X_{12}.\sin\alpha + r.Y_{12}.\cos\alpha).\overrightarrow{z_0}$$

The static equation become :

$$\begin{Bmatrix} X_{01} & L_{01} \\ Y_{01} & M_{01} \\ Z_{01} & 0 \end{Bmatrix}_{O_{(\overrightarrow{x_0},\overrightarrow{y_0},\overrightarrow{z_0})}} - \begin{Bmatrix} X_{12} & L_{12} + r.Z_{12}.sin\ \alpha \\ Y_{12} & M_{12} - r.Z_{12}.\cos\alpha \\ Z_{12} & -r.X_{12}.\sin\alpha + r.Y_{12}.\cos\alpha \end{Bmatrix}_{O_{(\overrightarrow{x_0},\overrightarrow{y_0},\overrightarrow{z_0})}} + \begin{Bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & C_m \end{Bmatrix}_{O_{(\overrightarrow{x_0},\overrightarrow{y_0},\overrightarrow{z_0})}} = \begin{Bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{Bmatrix}_O$$

The six equations of static are therefore written in the base $(\overrightarrow{x_0},\overrightarrow{y_0},\overrightarrow{z_0})$ and at point O as:

$$\begin{cases} X_{01} - X_{12} + 0 = 0 \\ Y_{01} - Y_{12} + 0 = 0 \\ Z_{01} - Z_{12} + 0 = 0 \end{cases} \text{ and } \begin{cases} L_{01} - L_{12} + r.Z_{12}.sin\ \alpha + 0 = 0 \\ M_{01} - M_{12} - r.Z_{12}.\cos\alpha + 0 = 0 \\ 0 - r.X_{12}.\sin\alpha + r.Y_{12}.\cos\alpha + C_m = 0 \end{cases}$$

The complete equilibrium is translated by the following 18 equations with 20 unknowns.

$$\begin{cases} X_{23} - X_{30} = 0 \\ Y_{23} = 0 \\ Z_{23} - Z_{30} = 0 \\ X_{12} - X_{23} = 0 \\ Y_{12} - Y_{23} + F = 0 \\ Z_{12} - Z_{23} = 0 \\ X_{01} - X_{12} = 0 \\ Y_{01} - Y_{12} = 0 \\ Z_{01} - Z_{12} = 0 \end{cases} \text{ and } \begin{cases} L_{23} - L_{30} = 0 \\ M_{23} - M_{30} = 0 \\ N_{30} = 0 \\ L_{12} - a.\cos\theta\,.Z_{12} - L_{23} = 0 \\ M_{12} - a.\sin\theta\,.Z_{12} - M_{23} = 0 \\ a.\cos\theta\,.X_{12} + a.\sin\theta\,.Y_{12} + F.(a+b).\sin\theta = 0 \\ L_{01} - L_{12} + r.Z_{12}.sin\ \alpha = 0 \\ M_{01} - M_{12} - r.Z_{12}.\cos\alpha = 0 \\ -r.X_{12}.\sin\alpha + r.Y_{12}.\cos\alpha + C_m = 0 \end{cases}$$

It remains to determine the rank ***r_s*** of this system.

Rather than looking for the rank of the entire system, it is often better to reorganize the system, and solve it piece by piece. We can see that the system can be devised into two parts:

$$\begin{cases} X_{01} - X_{12} = 0 \\ X_{12} - X_{23} = 0 \\ X_{23} - X_{30} = 0 \\ Y_{23} = 0 \\ Y_{12} - Y_{23} + F = 0 \\ Y_{01} - Y_{12} = 0 \\ a.\cos\theta\,.X_{12} + a.\sin\theta\,.Y_{12} + F.(a+b).\sin\theta = 0 \\ -r.X_{12}.\sin\alpha + r.Y_{12}.\cos\alpha + C_m = 0 \end{cases} \text{ and } \begin{cases} Z_{23} - Z_{30} = 0 \\ Z_{12} - Z_{23} = 0 \\ Z_{01} - Z_{12} = 0 \\ L_{23} - L_{30} = 0 \\ M_{23} - M_{30} = 0 \\ N_{30} = 0 \\ L_{12} - a.\cos\theta\,.Z_{12} - L_{23} = 0 \\ M_{12} - a.\sin\theta\,.Z_{12} - M_{23} = 0 \\ L_{01} - L_{12} + r.Z_{12}.sin\ \alpha = 0 \\ M_{01} - M_{12} - r.Z_{12}.\cos\alpha = 0 \end{cases}$$

We can written the first subsystem by the following equations :

$$\begin{cases} \boldsymbol{X_{01}} - \boldsymbol{X_{12}} = 0 \\ X_{12} - \boldsymbol{X_{23}} = 0 \\ X_{23} - \boldsymbol{X_{30}} = 0 \\ \boldsymbol{Y_{23}} = 0 \\ \boldsymbol{Y_{12}} - Y_{23} = -F \\ \boldsymbol{Y_{01}} - Y_{12} = 0 \\ a.\cos\theta\,.X_{12} + a.\sin\theta\,.Y_{12} = -F.(a+b).\sin\theta \\ -r.X_{12}.\sin\alpha + r.Y_{12}.\cos\alpha = -C_m \end{cases}$$

The first subsystem has 8 equations with 7 unknowns. The subsystem **rank** is therefore **7**. There is therefore an additional equation which will link $\boldsymbol{C_m}$ to $\boldsymbol{F}$ which can be quickly determined by :

$$C_m = -r.F.\frac{-a.\cos\alpha\,.\cos\theta + b.\sin\alpha\,.\sin\theta}{a.\cos\theta}$$

Now, it is possible to determine all the unknowns of the first subsystem :

$$\begin{cases} \boldsymbol{X_{01}} = \boldsymbol{X_{12}} = \dfrac{C_m - r.F.\cos\alpha}{r.\sin\alpha} \\ \boldsymbol{X_{23}} = X_{12} = \dfrac{C_m - r.F.\cos\alpha}{r.\sin\alpha} \\ \boldsymbol{X_{30}} = X_{23} = \dfrac{C_m - r.F.\cos\alpha}{r.\sin\alpha} \\ \boldsymbol{Y_{23}} = 0 \\ \boldsymbol{Y_{12}} = -F \\ \boldsymbol{Y_{01}} = -F \\ a.\cos\theta\,.X_{12} = a.\sin\theta\,.F - F.(a+b).\sin\theta \\ X_{12} = \dfrac{C_m - r.F.\cos\alpha}{r.\sin\alpha} \end{cases}$$

The second subsystem has 10 equations with 13 unknowns, the rank is 10, it is not possible to determine all the unknowns, it is necessary to fix at least 3 to solve: here it is wise to choose Z_{23}, L_{23} and M_{23}.

$$\begin{cases} \boldsymbol{Z_{30}} = Z_{23} \\ \boldsymbol{Z_{12}} = Z_{23} \\ \boldsymbol{Z_{01}} - Z_{12} = 0 \\ \boldsymbol{L_{30}} = L_{23} \\ \boldsymbol{M_{30}} = M_{23} \\ \boldsymbol{N_{30}} = 0 \\ \boldsymbol{L_{12}} - a.\cos\theta\,.Z_{12} = L_{23} \\ \boldsymbol{M_{12}} - a.\sin\theta\,.Z_{12} = M_{23} \\ \boldsymbol{L_{01}} - L_{12} + r.Z_{12}.sin\,\alpha = 0 \\ \boldsymbol{M_{01}} - M_{12} - r.Z_{12}.\cos\alpha = 0 \end{cases}$$

In conclusion,

— The rank r_s of the complete system is therefore $r_s = 7 + 10 = 17$

— 3 unknowns of links are not determinable.

The mechanism is therefore ***hyperstatic*** of degree: $h = I_s - r_s = 3$

The system of $E_s = 18$ equations and $I_s = 20$ unknowns of links, of rank $r_s = 17$ therefore includes 17 equations useful for determining the unknowns and 1 additional equation: $E_s - r_s = 1$.

The number of additional equations of the static study gives the degree of mobility of the mechanism: $m = E_s - r_s$

III.3 Isostatic solutions of hyperstatic problems

A hyperstatic mechanism is a mechanism in which the links are superabundant, so we could obtain the same operation with a simpler structure. Is it wise to try to transform it to make it isostatic?

The main quality of a hyperstatic mechanism is its rigidity. The counterpart of this quality is its main defect, hyperstatic mechanisms are more difficult to produce and therefore more expensive. We therefore reserve hyperstatic solutions whenever rigidity must prevail over cost, in other cases we prefer isostatic solutions.

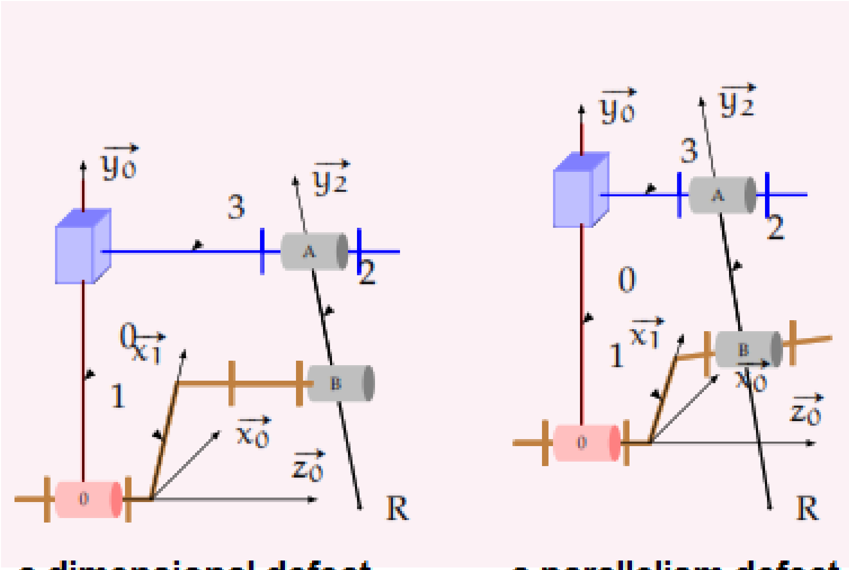
III.3.1 Influence of the degree of hyperstaticity on the realization of the mechanism

The hyperstatic unknowns correspond to the rating conditions between the links that the mechanism must respect in order to function correctly despite the hyperstaticity.

Thus, for the 4 legs of a chair (hyperstaticity of degree h = 1) to touch the ground, it is necessary that the 4 legs are coplanar. This condition will imply stricter realization conditions than for a 3-legged stool.

Guided example (continuation) : Walking robot - Influence of geometric and dimensional defects

We see in the two figures below the influence of two defects on the assembly of the mechanism, a defect in the length of one of the arms, or a defect in parallelism, will prohibit the assembly of the mechanism.



a dimensional defect **a parallelism defect**

- When the degree of hyperstaticity is linked to an unknown resultant, this implies that a dimensional constraint must be respected when producing the mechanism.
- When the degree of hyperstaticity is linked to an unknown moment, this implies that an angular constraint must be respected when producing the mechanism (parallelism, perpendicularity, flatness, etc.).

III.3.2 Systematic search for isostatic solutions.

From the static study, we identify ***the unknowns of superabundant links***. For each unknown of non-determinable link, we must ***add a degree of freedom in the kinematic chain***. It may also be necessary to ***add parts*** in the mechanism.

It is therefore a question of ***replacing certain links*** by links that allow cancelling the superabundant unknowns.

Guided example (continuation) : Walking robot - Make the mechanism isostatic

How to make the walking robot mechanism isostatic?

The static study is translated by the previous system of equations:

$$\begin{cases} Z_{23} - Z_{30} = 0 \\ Z_{12} - Z_{23} = 0 \\ Z_{01} - Z_{12} = 0 \\ L_{23} - L_{30} = 0 \\ M_{23} - M_{30} = 0 \\ N_{30} = 0 \\ L_{12} - a.\cos\theta\,.Z_{12} - L_{23} = 0 \\ M_{12} - a.\sin\theta\,.Z_{12} - M_{23} = 0 \\ L_{01} - L_{12} + r.Z_{12}.sin\,\alpha = 0 \\ M_{01} - M_{12} - r.Z_{12}.\cos\alpha = 0 \end{cases} \longrightarrow \begin{cases} Z_{30} = Z_{23} \\ Z_{12} = Z_{23} \\ Z_{01} - Z_{12} = 0 \\ L_{30} = L_{23} \\ M_{30} = M_{23} \\ N_{30} = 0 \\ L_{12} - a.\cos\theta\,.Z_{12} = L_{23} \\ M_{12} - a.\sin\theta\,.Z_{12} = M_{23} \\ L_{01} - L_{12} + r.Z_{12}.sin\,\alpha = 0 \\ M_{01} - M_{12} - r.Z_{12}.\cos\alpha = 0 \end{cases}$$

For the mechanism to become isostatic, it is necessary to cancel the superabundant static unknowns. Here this amounts to saying that it is necessary that:

Case 1 : $Z_{23} = 0\,, L_{23} = 0\ and\ \ M_{23} = 0$

But this is not the only possibility, we could choose the triplet :

Case 2 : $Z_{12} = 0\,, L_{12} = 0\ and\ \ M_{12} = 0$ or

Case 3 : $Z_{12} = 0\,, L_{23} = 0\ and\ \ M_{23} = 0$

Case 1 : $Z_{23} = 0\,, L_{23} = 0\ and\ \ M_{23} = 0$

For the first case, the torsor of the transmissible actions of the link $[A_{2\to3}]$ becomes :

$$[A_{2\to3}] = \begin{Bmatrix} X_{23} & L_{23} \\ Y_{23} & M_{23} \\ Z_{23} & 0 \end{Bmatrix}_{A(\vec{?},\vec{?},\overrightarrow{z_0})} \text{ becomes } [A_{2\to3}] = \begin{Bmatrix} X_{23} & 0 \\ Y_{23} & 0 \\ 0 & 0 \end{Bmatrix}_{A(\vec{?},\vec{?},\overrightarrow{z_0})}$$

We recognize a sphere-cylinder link with center A and axis $(A, \overrightarrow{z_0})$ which gives the equivalent isostatic diagram of the first case, presented in the figure III.7.

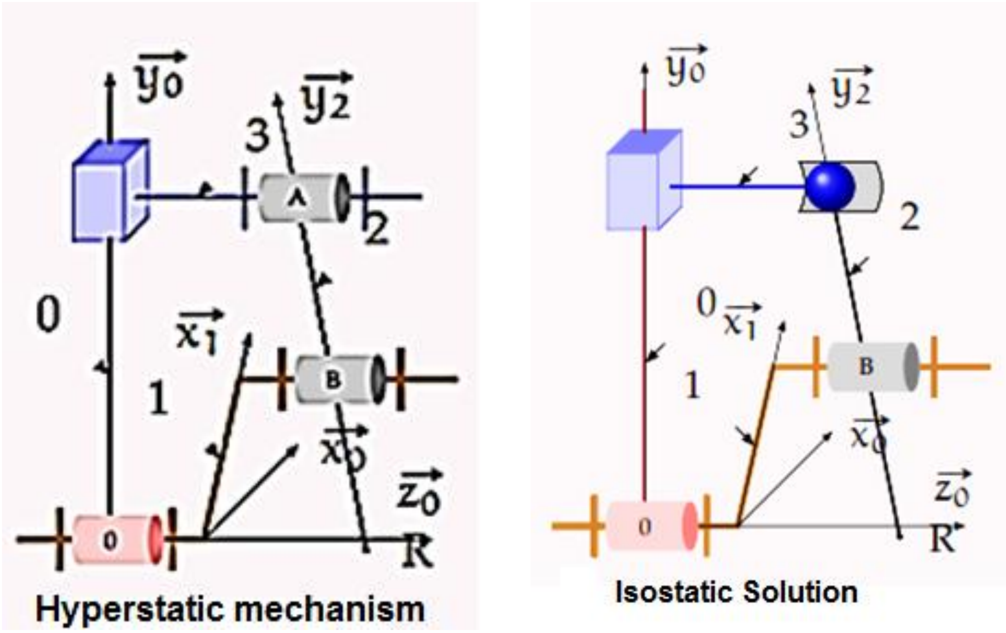


Figure III. 7 : Isostatic solution (Case 1).

<u>Case 2 :</u> $Z_{12} = 0\,, L_{12} = 0 \; and \;\; M_{12} = 0$

The second possibility corresponds to the modification of the pivot link in ***B*** into a sphere-cylinder link. Which gives the equivalent isostatic diagram of the case 2, presented in the figure III.8.

$[A_{1\to 2}] = \begin{Bmatrix} X_{12} & L_{12} \\ Y_{12} & M_{12} \\ Z_{12} & 0 \end{Bmatrix}_{B_{(\vec{?},\vec{?},\vec{z_0})}}$ the torsor becomes $[A_{1\to 2}] = \begin{Bmatrix} X_{12} & 0 \\ Y_{12} & 0 \\ 0 & 0 \end{Bmatrix}_{B_{(\vec{?},\vec{?},\vec{z_0})}}$

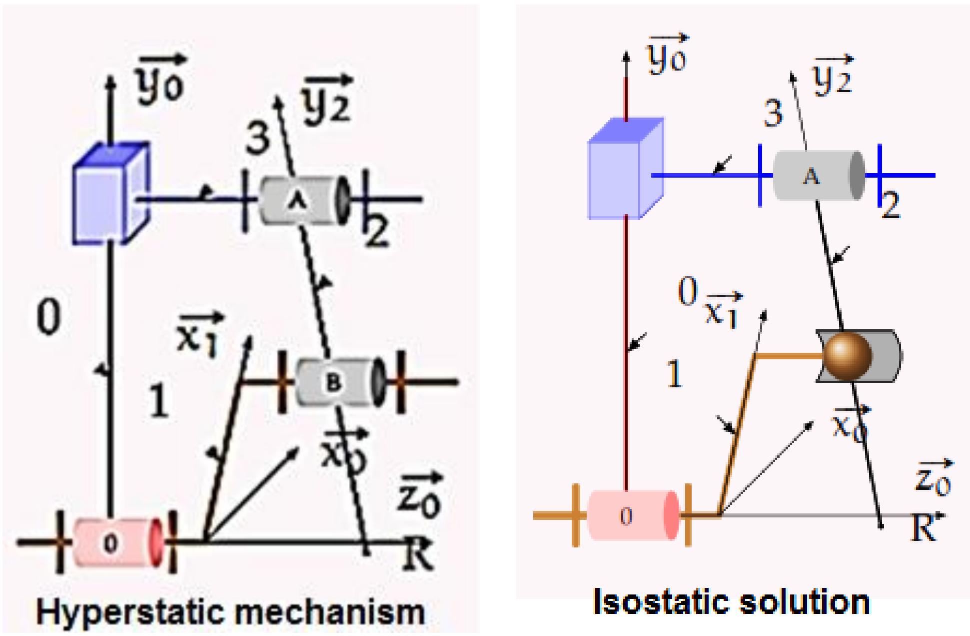


Figure III. 8 : **Isostatic solution (Case 2).**

Case 3 : $Z_{12} = 0\,, L_{23} = 0 \; and \;\; M_{23} = 0$

For the third case we obtain the following torsors:

$$[A_{2\to3}] = \begin{Bmatrix} X_{23} & L_{23} \\ Y_{23} & M_{23} \\ Z_{23} & 0 \end{Bmatrix}_{A(\vec{?},\vec{?},\overrightarrow{z_0})} \quad \text{becomes} \quad [A_{2\to3}] = \begin{Bmatrix} X_{23} & 0 \\ Y_{23} & 0 \\ Z_{23} & 0 \end{Bmatrix}_{A(\vec{?},\vec{?},\overrightarrow{z_0})} \quad \text{represent a spherical link at A.}$$

$$[A_{1\to2}] = \begin{Bmatrix} X_{12} & L_{12} \\ Y_{12} & M_{12} \\ Z_{12} & 0 \end{Bmatrix}_{B(\vec{?},\vec{?},\overrightarrow{z_0})} \quad \text{becomes} \quad [A_{1\to2}] = \begin{Bmatrix} X_{12} & L_{12} \\ Y_{12} & M_{12} \\ 0 & 0 \end{Bmatrix}_{B(\vec{?},\vec{?},\overrightarrow{z_0})} \quad \text{represent a sliding pivot at B.}$$

Which gives the equivalent isostatic diagram of the case 3, presented in the figure III.9.

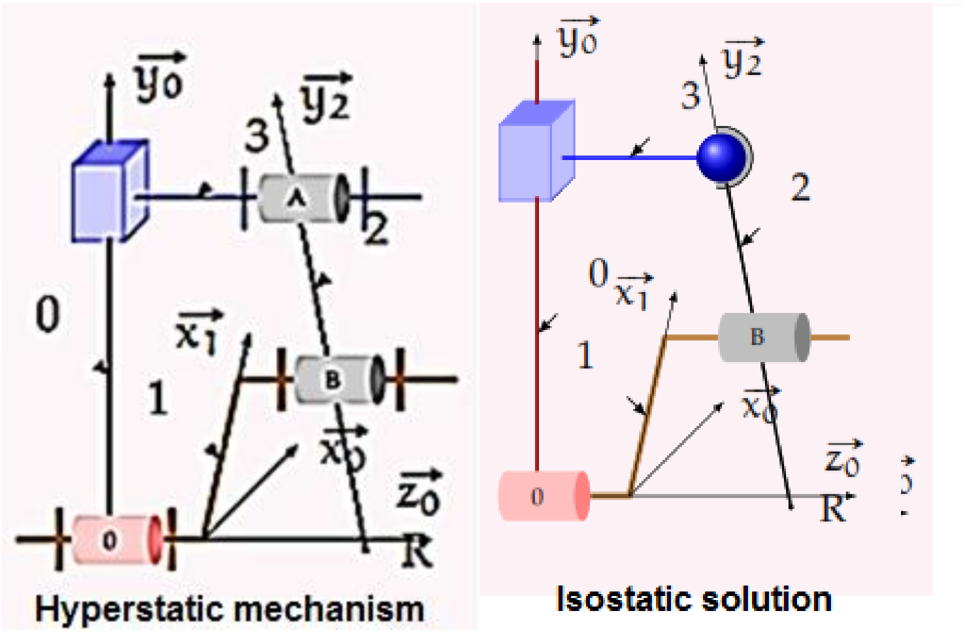


Figure III. 9 : Isostatic solution (Case 3).