# Chapter II

# Analysis and modelling of mechanisms

# II Analysis and modelling of mechanisms

## II.1 Usual mechanical links

The mechanism is the kinematic chain having one fixed element (the base, the frame) and in which all the elements have pre-determinate motions. According to their functional role, the component elements of the mechanism can be:

- Driving (motor, input); the elements that receive the motion from outside the mechanism
- Driven (commanded, output); the elements whose motion depends on the motion of the driving elements.

## II.2 Kinematic diagram (SKELETON DIAGRAMS)

In analysing the motion of a machine, it is often difficult to visualize the movement of the components in a full assembly drawing. It is easier to represent the parts in skeleton form so that only the dimensions that influence the motion of the mechanism are shown. These “stripped-down” sketches of mechanisms are often referred to as SKELETON diagrams.

A skeleton diagram is a simplified drawing of a mechanism or machine that shows only the dimensions that affect its kinematics. Figure II.1 shows a connecting rod with its attached piston, from an internal combustion engine. The connecting rod and piston both have many geometric features, mostly associated with issues of strength and the size of the bearing at each joint. These features are kinematically unimportant.

The only geometric feature of either the connecting rod or the piston that is important to the kinematics of the mechanism containing them is the distance, l, between the centers of the connecting rod’s bearings. (These bearings are commonly known as the wrist pin bearing and the rod end bearing.) In a skeleton diagram of the engine’s mechanism, this connecting rod would be drawn as a stick whose length is l, as on the right hand side of Figure II.1. The piston would be shown as a square or rectangle whose dimensions are not specified, because they are kinematically unimportant. The bearings would be pin joints. Let us consider some examples of mechanisms and machines that are around us in our daily lives and examine their skeleton diagrams.

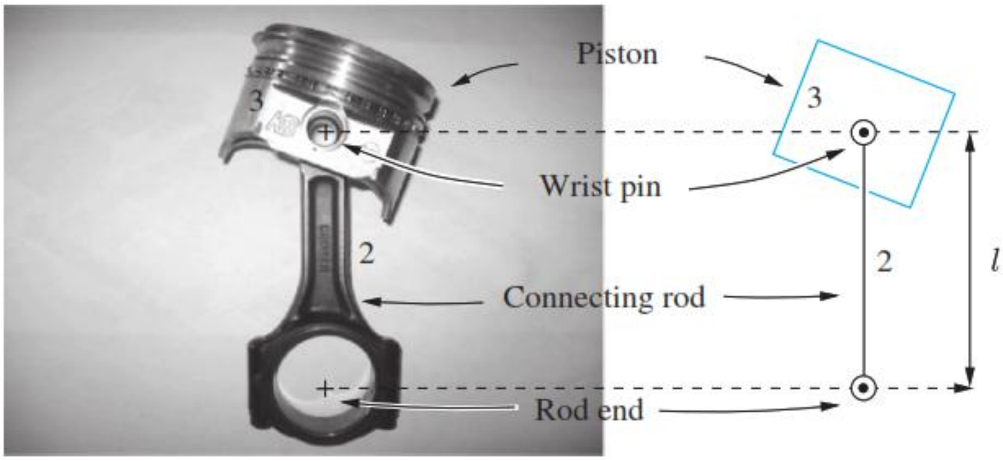



**Figure II. 1 :** SKELETON diagrams of connecting rod crank system.

Figure I.16 shows typical conventions used in creating kinematic diagrams. A kinematic diagram should be drawn to a scale proportional to the actual mechanism. For convenient reference, the links are numbered, starting with the frame as link number 1. To avoid confusion, the joints should be lettered.

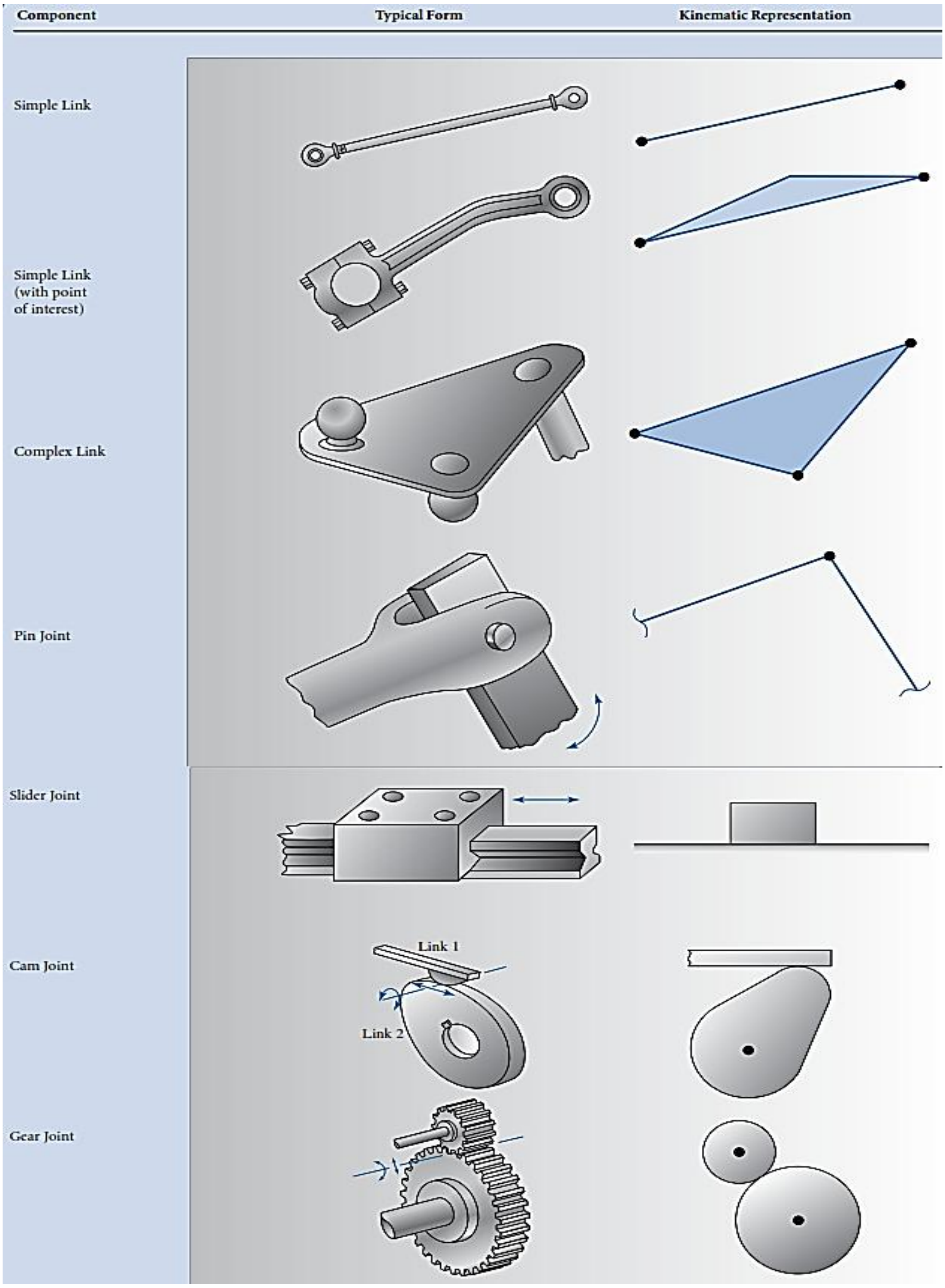
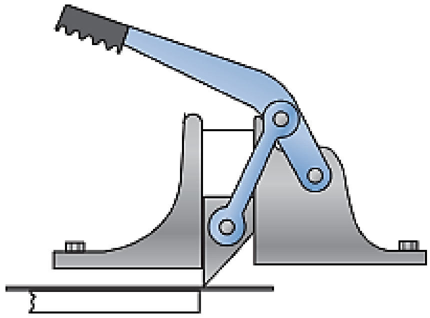



**Figure II. 2:** Typical conventions used in creating kinematic or Skeleton diagrams.

### II.2.1 Examples of Skeleton Diagrams

a) ***Example II.1***

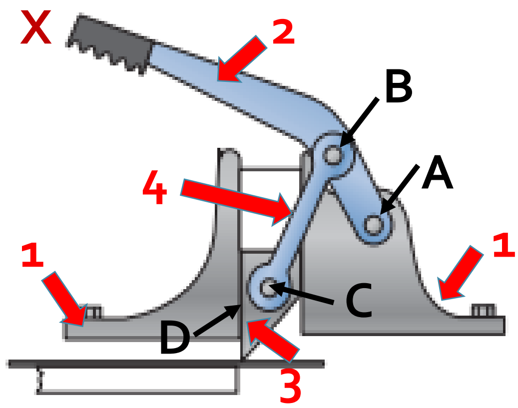
Draw a kinematic diagram of the shear used to cut and trim electronic circuit board laminates.



***Solution steps:***

***1. Identify the Frame:*** The first step in constructing a kinematic diagram is to decide the part that will be designated as the frame.

*The motion of all other links will be determined relative to the frame. In some cases, its selection is obvious as the frame is firmly attached to the ground. In this problem, the large base that is bolted to the table is designated as the frame. The motion of all other links is determined relative to the base. The base is numbered as link 1*




***2. Identify All Other Links:*** Careful observation reveals three other moving parts:

Link 2: Handle

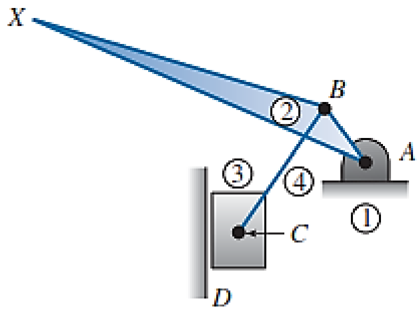
Link 3: Cutting blade Link

Link 4: Bar that connects the cutter with the handle

**3. *Identify the Joints:*** Pin joints are used to connect link 1 to 2, link 2 to 4, and link 4 to 3. These joints are lettered A, B and C. In addition, the cutter slides up and down, along the base. This sliding joint connects link 3 to 1, and is lettered D.

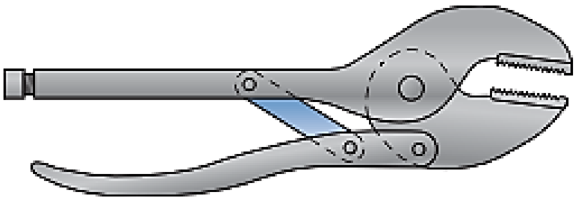
**4. *Identify Any Points of Interest:*** Finally, the motion of the end of the handle is desired. This is designated as point of interest X.

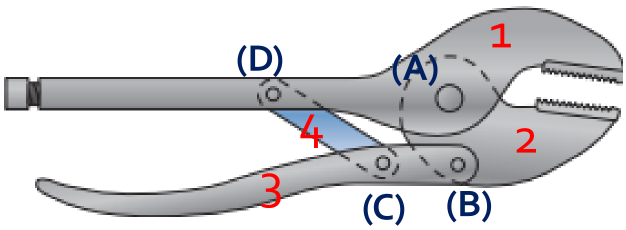
**5. *Draw the Kinematic Diagram:*** The kinematic diagram is given by the following graph :




b) ***Example II.2***

Draw a kinematic diagram of the pair of vise grips.






***Solution steps:***

**1. *Identify the Frame:*** The first step is to decide the part that will be designated as the frame. In this problem, no parts are attached to the ground. Therefore, the selection of the frame is rather arbitrary.

The top handle is designated as the frame. The motion of all other links is determined relative to the top handle. The top handle is numbered as link 1.

**2. *Identify All Other Links:*** Careful observation reveals three other moving parts:

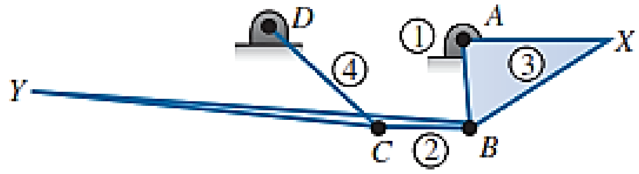
Link 2: Bottom handle

Link 3: Bottom jaw

Link 4: Bar that connects the top and bottom handle

**3. *Identify the Joints:*** Four pin joints are used to connect these different links (link 1 to 2, 2 to 3, 3 to 4, and 4 to 1). These joints are lettered A, B, C and D.

***4. Identify any Points of Interest:*** The motion of the end of the bottom jaw is desired. This is designated as point of interest X. Finally, the motion of the end of the lower handle is also desired. This is designated as point of interest Y.




***Example II.3***

The top left of Figure II.3 shows a six link mechanism that guides the hood of an automobile. The mechanism consists of six links, where link 1 is the body of the car (considered as the fixed link) and link 4 is the hood. The mechanism has a lot of features that are not important to the relative motion of the links. On the right side a skeleton diagram of the mechanism is overlayed on the actual mechanism, and at the bottom the skeleton diagram alone is shown. In the actual mechanism links 3 and 5 are curved.

In the skeleton diagram link 3 is shown as a stick whose length is the dimension of link 3. Link 5 is shown as a bent stick, where the distance between the joints and the angle of the bend are important dimensions. The length of the actual hood to which link 4 is attached is unimportant. What is significant with regard to link 4 is the distance between the two joints labeled points A and B. Referring to the skeleton diagram, you see it contains two "loops." One loop consists of links 1-2-6-5. You can imagine standing on 1, walking to 2, then to 6, then to 5, and finally back to 1, forming a closed loop. The other loop consists of 1-2-3-4-5. In the same way, you can go from 1 to 2 to 3 to 4 to 5 and back to 1. There is a third loop, 2-3-4-5-6; however, it exists as a consequence of the previous two loops. In other words, this third loop is not an independent loop. If either of the previous two loops is removed, this third loop no longer exists.

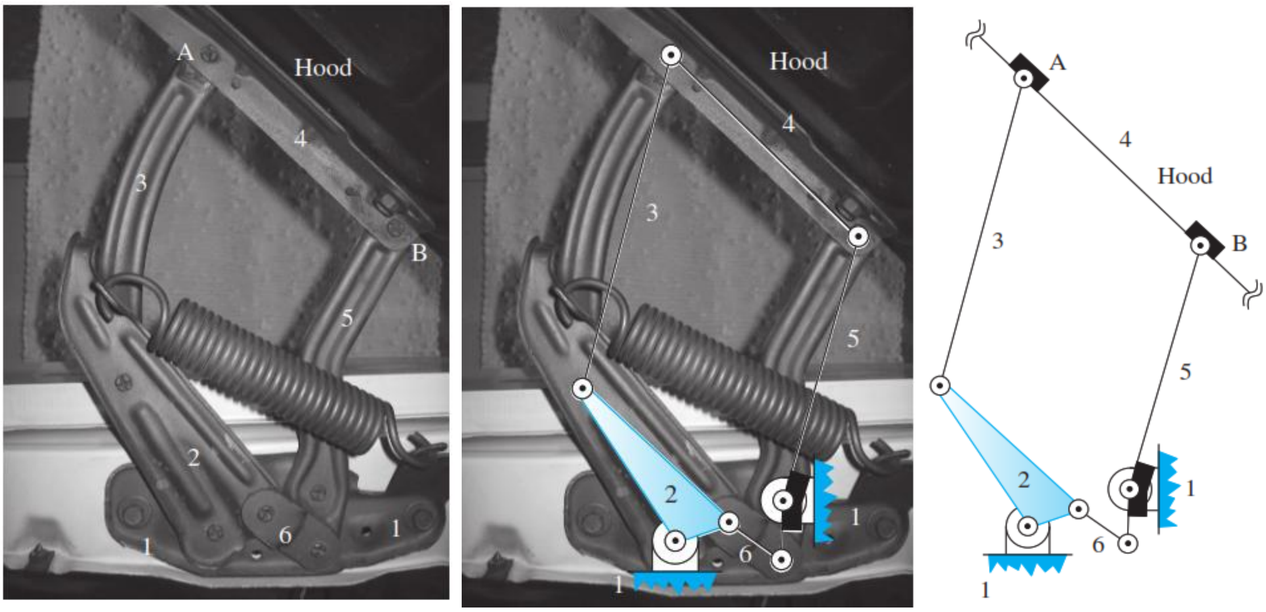



**Figure II. 3 :** An automobile hood mechanism.

***Example II.4:***

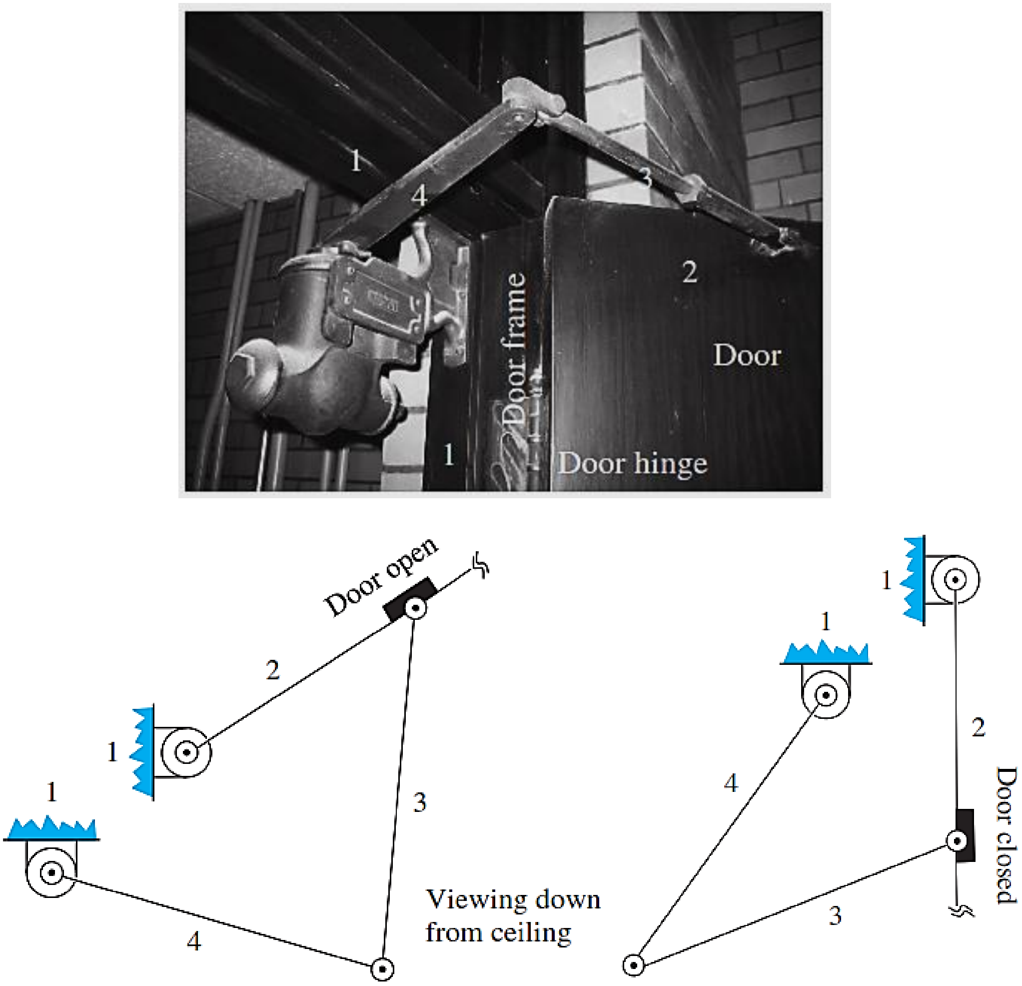


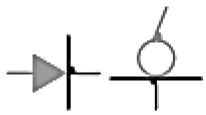
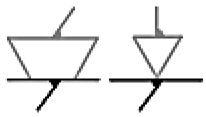
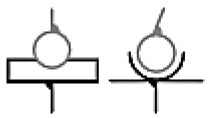
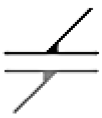
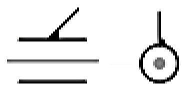
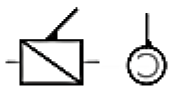
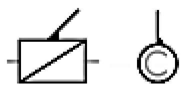
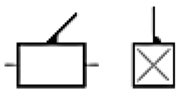
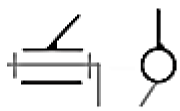
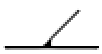

**Figure II. 4 :** A door damper mechanism.

## II.3 Kinematic diagram of a mechanism

The kinematic diagram makes it possible to model the kinematic interactions between the solids of the same mechanism. This type of representation makes it possible to decompose in detail a joint between two solids. For example, a joint pivot made by two ball bearings respectively ensuring a ball joint and an annular linear joint will be schematized by these two joints. Through these aspects, the kinematic diagram makes it possible, on the one hand, to help in the design of a mechanism by giving it the kinematic operating principle and, on the other hand, to help in the understanding of the system to be analyzed.

In practice, creating a kinematic diagram of a system simply consists of linking elementary joint diagrams together. These each represent an established kinematic, such as pivot linkage, sliding pivot, etc. A schematic description of these elements is proposed in the following table in two dimensions or three dimensions. Figure II.5-a and II.5-b shows respectively, the kinematic diagram of the manual pump and Kinematic diagram of the crank-connecting rod-piston mechanism.

**Table II. 1 :** Representation scheme of usual joints

| Joints | Representation scheme | Joints | Representation scheme |
|---|---|---|---|
| Punctual |  | Linear rectilinear |  |
| Annular linear |  | Ball joint |  |
| Plan on plan |  | Sliding pivot |  |
| Right hand helical |  | Left hand helical |  |
| Slide |  | Pivot |  |
| Fixed elements |  | | |

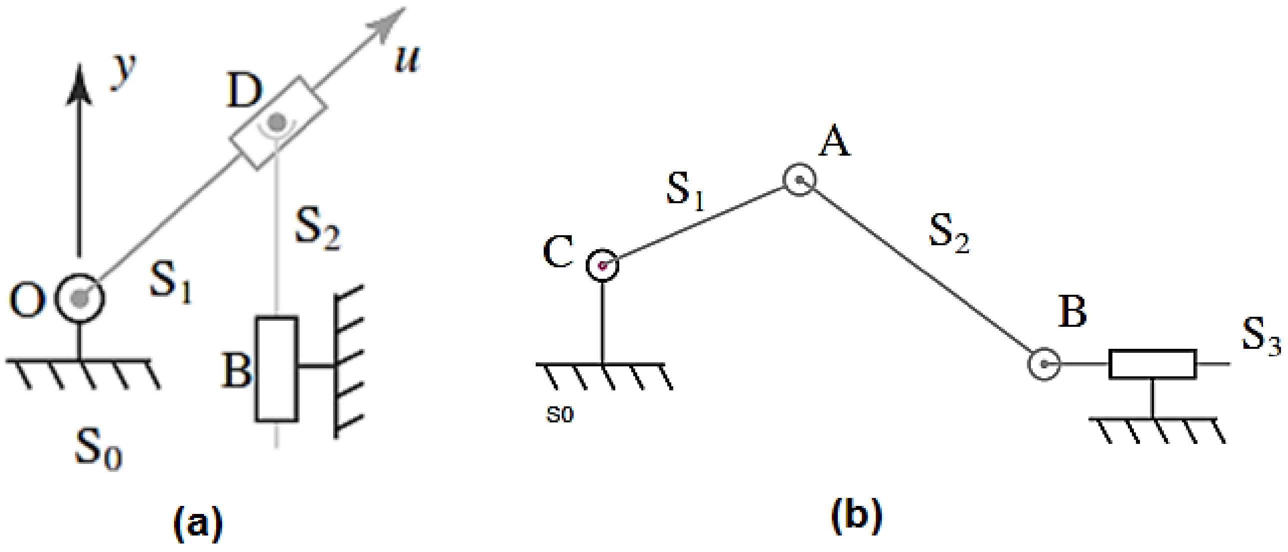



**Figure II. 5 :** Kinematic diagram of the manual pump.

## II.4 Static and kinematic torsors of a mechanism

### II.4.1 Torsor of a solid S1 on a solid S2

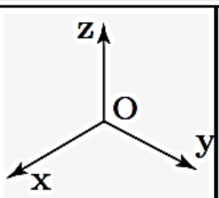
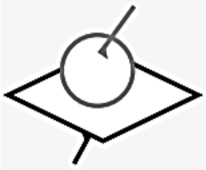
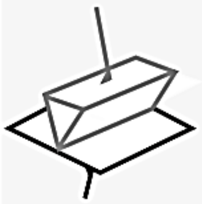
A static torsor of a joints is a torsor composed of a resultant (of components X, Y, Z) and a moment (of components L, M, N) in a reference R(x, y, z), which represent the force and the moment produced by the force at a point P, contact between the solids. On the other hand, a kinematic twister of a connection is a twister composed of a resultant representing the rotation speeds (of components $\Omega_x, \Omega_y$ $and$ $\Omega_z$) and a speed (of components $V_x, V_y$ $and$ $V_z$) in a reference R(x, y, z).

- Static torsor : $[T_s]_{\frac{S_1}{S_2}} = \begin{cases} \vec{R} \\ \vec{M} \end{cases} = \begin{cases} X.\vec{x} + Y.\vec{y} + Z.\vec{z} \\ L.\vec{x} + M.\vec{y} + N.\vec{z} \end{cases}$
- Kinematic torsor: $[T_K]_{S_1/S_2} = \begin{cases} \vec{R} \\ \vec{M} \end{cases} = \begin{cases} \Omega_x.\vec{x} + \Omega_y.\vec{y} + \Omega_z.\vec{z} \\ V_x.\vec{x} + V_y.\vec{y} + V_z.\vec{z} \end{cases}$

### II.4.2 Static and kinematic torsors of joins

The following table summarize the static and kinematic torsors of the mechanical usual joints.

**Table II. 2 :** Static and kinematic torsors of the mechanical usual joints.

| Joints | Kinematic scheme | Static torsor | Kinematic torsor |
|---|---|---|---|
| **Reference** (O, x, y, z) |  | $\{\ \mathcal{A}(S_1 \rightarrow S_2)\ \}_O$ | $\{\ \mathcal{V}(S_2/S_1)\ \}_O$ |
| **Punctual of normal** (O, z) $m = 5$ |  | $\left\{ \begin{array}{c} 0\,x + 0\,y + Z\,z \\ 0\,x + 0\,y + 0\,z \end{array} \right\}_O$ | $\left\{ \begin{array}{c} \Omega_x\,x + \Omega_y\,y + \Omega_z\,z \\ V_x\,x + V_y\,y + 0\,z \end{array} \right\}_O$ |
| **Rectilinear linear of axis** (O, x) **and normal** (O, z) $m = 4$ |  | $\left\{ \begin{array}{c} 0\,x + 0\,y + Z\,z \\ 0\,x + M\,y + 0\,z \end{array} \right\}_O$ | $\left\{ \begin{array}{c} \Omega_x\,x + 0\,y + \Omega_z\,z \\ V_x\,x + V_y\,y + 0\,z \end{array} \right\}_O$ |

| | | | |
|---|---|---|---|
| **Annular linear along the axis** (O, x) $m = 4$ |  | $\begin{Bmatrix} 0\,\mathbf{x} + Y\,\mathbf{y} + Z\,\mathbf{z} \\ 0\,\mathbf{x} + 0\,\mathbf{y} + 0\,\mathbf{z} \end{Bmatrix}_O$ | $\begin{Bmatrix} \Omega_x\,\mathbf{x} + \Omega_y\,\mathbf{y} + \Omega_z\,\mathbf{z} \\ V_x\,\mathbf{x} + 0\,\mathbf{y} + 0\,\mathbf{z} \end{Bmatrix}_O$ |
| **Ball joint of centre** O $m = 3$ |  | $\begin{Bmatrix} X\,\mathbf{x} + Y\,\mathbf{y} + Z\,\mathbf{z} \\ 0\,\mathbf{x} + 0\,\mathbf{y} + 0\,\mathbf{z} \end{Bmatrix}_O$ | $\begin{Bmatrix} \Omega_x\,\mathbf{x} + \Omega_y\,\mathbf{y} + \Omega_z\,\mathbf{z} \\ 0\,\mathbf{x} + 0\,\mathbf{y} + 0\,\mathbf{z} \end{Bmatrix}_O$ |
| **Plan/Plan** normal (O, z) $m = 3$ |  | $\begin{Bmatrix} 0\,\mathbf{x} + 0\,\mathbf{y} + Z\,\mathbf{z} \\ L\,\mathbf{x} + M\,\mathbf{y} + 0\,\mathbf{z} \end{Bmatrix}_O$ | $\begin{Bmatrix} 0\,\mathbf{x} + 0\,\mathbf{y} + \Omega_z\,\mathbf{z} \\ V_x\,\mathbf{x} + V_y\,\mathbf{y} + 0\,\mathbf{z} \end{Bmatrix}_O$ |
| **Sliding pivot axis** (O, z) $m = 2$ |  | $\begin{Bmatrix} X\,\mathbf{x} + Y\,\mathbf{y} + 0\,\mathbf{z} \\ L\,\mathbf{x} + M\,\mathbf{y} + 0\,\mathbf{z} \end{Bmatrix}_O$ | $\begin{Bmatrix} 0\,\mathbf{x} + 0\,\mathbf{y} + \Omega_z\,\mathbf{z} \\ 0\,\mathbf{x} + 0\,\mathbf{y} + V_z\,\mathbf{z} \end{Bmatrix}_O$ |
| **Helical slide axis** (O, z) $m = 1$ |  | $\begin{Bmatrix} X\,\mathbf{x} + Y\,\mathbf{y} + Z\,\mathbf{z} \\ L\,\mathbf{x} + M\,\mathbf{y} + N\,\mathbf{z} \end{Bmatrix}_O$ <br> $N = -pZ$ | $\begin{Bmatrix} 0\,\mathbf{x} + 0\,\mathbf{y} + \Omega_z\,\mathbf{z} \\ 0\,\mathbf{x} + 0\,\mathbf{y} + V_z\,\mathbf{z} \end{Bmatrix}_O$ <br> $V_z = p\Omega_z$ |
| **Slide axis** (O, z) $m = 1$ |  | $\begin{Bmatrix} X\,\mathbf{x} + Y\,\mathbf{y} + 0\,\mathbf{z} \\ L\,\mathbf{x} + M\,\mathbf{y} + N\,\mathbf{z} \end{Bmatrix}_O$ | $\begin{Bmatrix} 0\,\mathbf{x} + 0\,\mathbf{y} + 0\,\mathbf{z} \\ 0\,\mathbf{x} + 0\,\mathbf{y} + V_z\,\mathbf{z} \end{Bmatrix}_O$ |

| | | | |
|---|---|---|---|
| Pivot of axis (O, z) $m = 1$ |  | $\left\{\begin{matrix} X\,x + Y\,y + Z\,z \\ L\,x + M\,y + 0\,z \end{matrix}\right\}_O$ | $\left\{\begin{matrix} 0\,x + 0\,y + \Omega_z\,z \\ 0\,x + 0\,y + 0\,z \end{matrix}\right\}_O$ |
| Fixed element $m = 0$ |  | $\left\{\begin{matrix} X\,x + Y\,y + Z\,z \\ L\,x + M\,y + N\,z \end{matrix}\right\}_O$ | $\left\{\begin{matrix} 0\,x + 0\,y + 0\,z \\ 0\,x + 0\,y + 0\,z \end{matrix}\right\}_O$ |

## *II.5* Degrees of Freedom and the Mobility *M*

### II.5.1 Kutzbakh criterion

One of the first concerns in either the design or the analysis of a mechanism is the number of ***degrees of freedom***, also called ***Mobility*** of the device. The mobility ***M*** or ***DOF*** of a mechanism is the number of input parameters which must be independently controlled in order to bring the device into a particular position. Ignoring certain exception, it is possible to determine the mobility of the mechanism directly from a count of the number of links and the number and types of joints which it includes. If we consider;

- $J_1$ : Number of single degree of freedom pairs
- $J_2$ : Number of two degree of freedom pairs
- $N_L$: Number of elements or links

The mobility ***M*** of a planar n-link mechanism is given by the following formula:

$$M = DOF = 3.(N_L - 1) - 2.j_1 - j_2$$

This equation is called ***Kutzbach criterion*** for the mobility of planar mechanism. Its application is shown for several simple cases in figure II.4. If the Kutzbach criterion yiels ***m > 0***, the mechanism has ***m*** degrees of freedom. If ***m = 1***, the mechanism can be driven by a single input motion. If m = 2, then two separate input motions are necessary to produce constrained motion for the mechanism; such a case is shown if figure II.4-d.

If the Kutzbakh criterion yields ***m=0*** as in figure II.6-a, motion is impossible and the mechanism forms a structure. If the the Kutzbakh criterion gives ***m=-1*** or less, then there are redundant constraints in the chain and if forms a statically indeterminate structure, Examples are shown in figures II.7. Notes in these examples that when three links are joined by a single pin, two joints must be counted; such a connection is treated as two separate but concentric pairs.

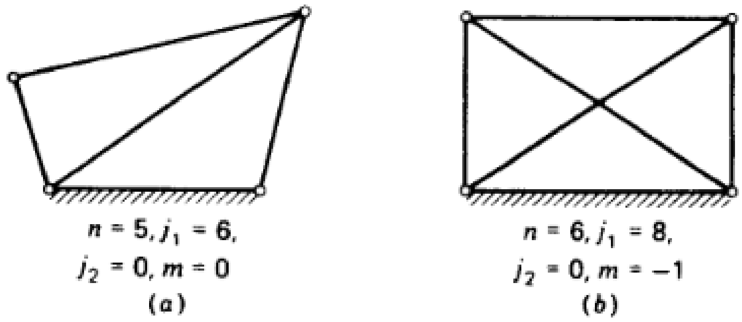



**Figure II. 6:** Example for Kutzbakh criterion to structures.

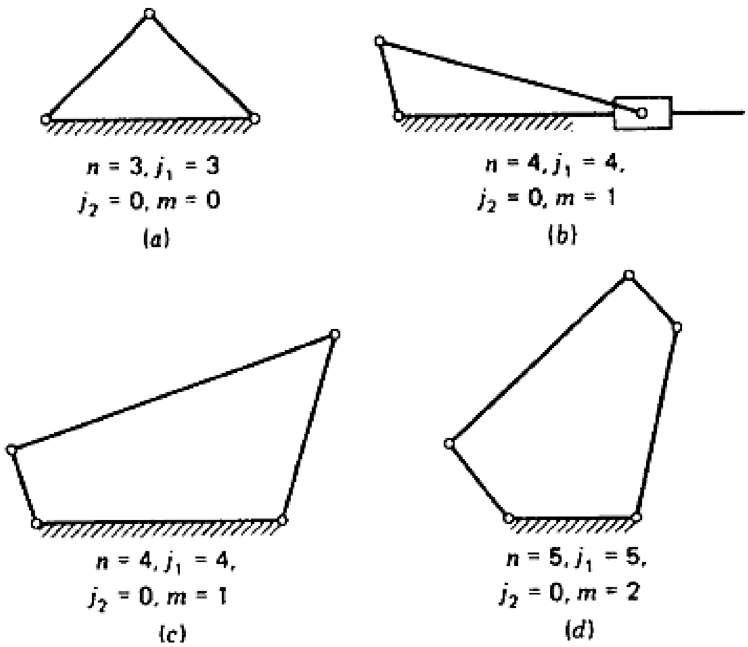



**Figure II. 7:** Example for Kutzbakh mobility criterion

### II.5.2 Gruebler's Criterion

It is important to realize that the F number is a theoretical result. Gruebler's Criterion can be fooled. Figure II.8 shows such a special case that occurs commonly. It is a pair of circular links pinned to ground at their respective centers and rolling upon one another.

This is an idealization of a pair of gears. The rolling circles are known as the "pitch circles" of the gears. For this system $N_L$ =3, $j_1$ = 3 (2 pin joints and 1 rolling joint), and $j_2$ = 0. From the equation of deegres of freedom we have :

$$DOF = 3.(N_L - 1) - 2.j_1 - j_2 = 3(3 - 1) - 2(3) - 1(0) = 0.$$

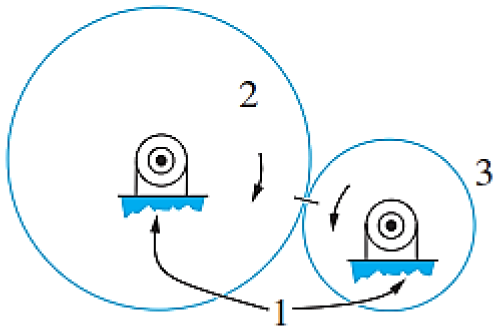



**Figure II. 8:** A pair of externally rolling gears

It is visually obvious that the gears counter-rotate and the system has one dof. The problem is that Gruebler's Criterion is unaware of two special geometric conditions:

- Links 2 and 3 are circular.

- Links 2 and 3 are pinned to ground at their respective centers.

Gruebler's Criterion thinks the system is the pair of random shapes pinned to ground with a rolling contact between them, as shown in Figure II.9. This mechanism is immovable and has zero DOF. If the joint between links 2 and 3 in Figure II.9 were a slipping joint instead of a rolling joint (which would have been communicated in the drawing by a lack of the hash mark shown where 2 contacts 3), then we would have had $j_1$ = 2 (2 pin joints), $j_2$ = 1 (1 slipping joint), and Gruebler's Criterion would have given DOF = 1. Figure II.10 shows two examples of over-constrained mechanisms that have DOF = 0 and are yet movable. Again, Gruebler's Criterion is unaware of the special geometry.

The actual number of DOF in a mechanism is known as the mobility, or M number, and it can be found only by inspection. M is always greater than or equal to DOF ($M \geq DOF$).

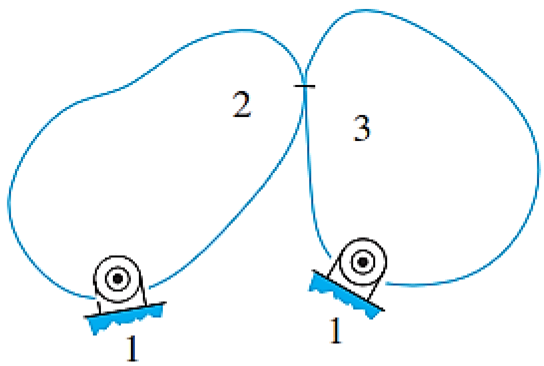



**Figure II. 9:** A pair of externally rolling eccentric gears

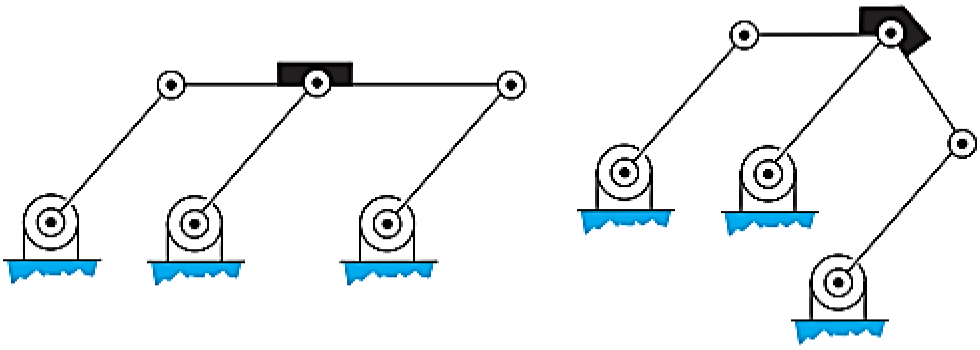


**Figure II. 10:** A pair of over-constrained mechanisms

### II.5.3 Statically indeterminate mechanism (Degree of hyperstaticity h)

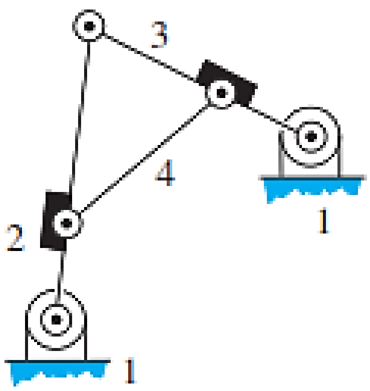
The degree of hyperstaticity ***h*** of a mechanism modeling is defined by the number of unknowns of mechanical action of joints that cannot be calculated by the fundamental principle of statics and therefore characterizes the overabundance of joints constituting the model of the system.

A system model is isostatic if ***h=0 (DOF=0)***. In this case it is possible to determine all of the joints unknowns by applying the fundamental principle of statics to each part. If ***h*** is negative, the structure is hyperstatic.

When DOF = 0 we have a determinate structure. When DOF < 0, however, the structure is statically indeterminate, and the degree of indeterminacy or the hyperstatism ***h*** is equal to the magnitude of DOF.

**Examlpe of a statically indeterminate mechanism**

Compute the theoretical number of degrees of freedom in the statically indeterminate structure shown in the following figure




This system has 4 links, $N_l$ = 4, and 5 pin joints, $j_1$ = 5. There are no point contacts, so there is no possibility of slipping joints and $j_2$ = 0. From the equation of degree of freedom we have :

$$M = DOF = 3.(N_l - 1) - 2.j_1 - j_2 = 3.(4 - 1) - 2.5 - 0 = -1$$

## II.6 Joint diagram/Structural graph of a mechanical system

The mechanical system includes $N_L$ parts and $N_j$ connection. We number all the parts of a mechanical system with consecutive numbers (1, 2, ....., $N_L$). The resulting connection diagram is then constructed as follows :

- Each piece is assigned a vertex (1, 2, ….., NL)
- To any connection or joint we assign an oriented arc (i,j), there are therefore $N_j$ arces.

In fact, if we refer to graph theory (C.Berge, graphs, graphs and algorithms), it is a multigraph, because there can exist several arces (links) between two vertices (pieces) given. In the theory of mechanical systems, we will only have to deal with connected graphs, because otherwise the mechanical systems would be disjoint.

Figure II.11 is an example of a connection diagram comprising 6 vertices (pieces) and 8 arcs (8 connections).

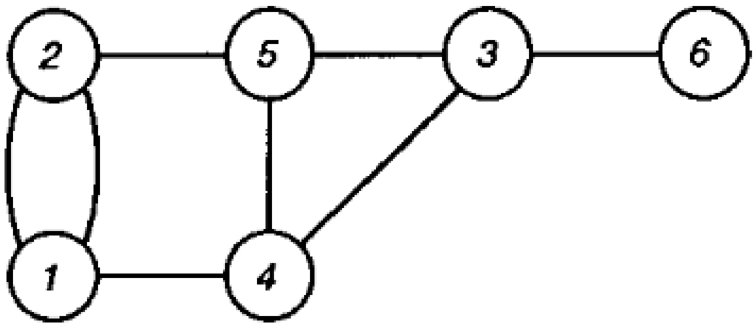



**Figure II. 11:** Joint diagram of a mechanical system.

### II.6.1 Tree

A tree is a path extracted from a graph such that when traversing it we do not encounter the same vertex twice. This path must not form a cycle. Figure II.12 represents a tree extracted from the previous graph (Figure II.11).

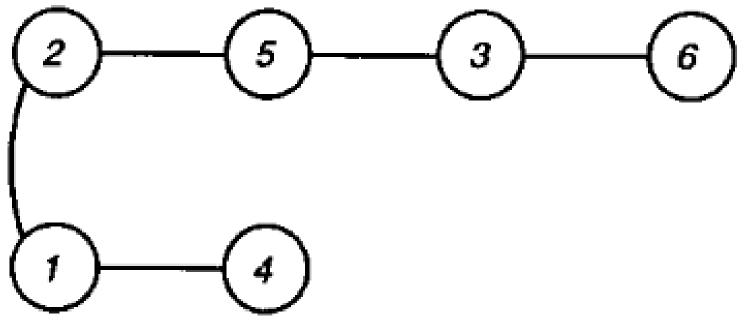



**Figure II. 12:** Extracted tree from the joint diagram (Figure II.11).

### II.6.2 Cycle

A cycle is a closed path extracted from a graph such that traversing it we do not encounter the same vertex twice (Figure II.13). From the cycles of a graph we can develop an algebra.

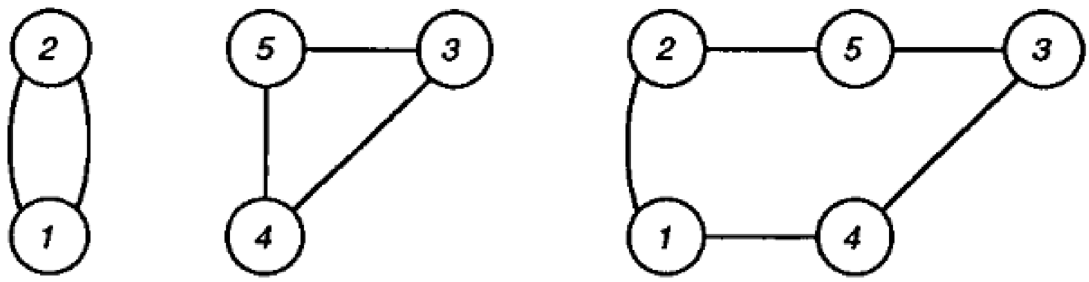



**Figure II. 13 :** Extracted cycles from the joint diagram (Figure II.11).

### II.6.3 Independent cycles

We will use the notion of independent cycles in geometry and kinematics of mechanical systems. This notion makes it possible in particular to clarify the notion of "***Loops***" existing in a mechanical system.

The independent cycles form a base of cycles (Figure II.14). All the cycles that can be constructed from the structure graph are therefore constructed from this base of cycles.

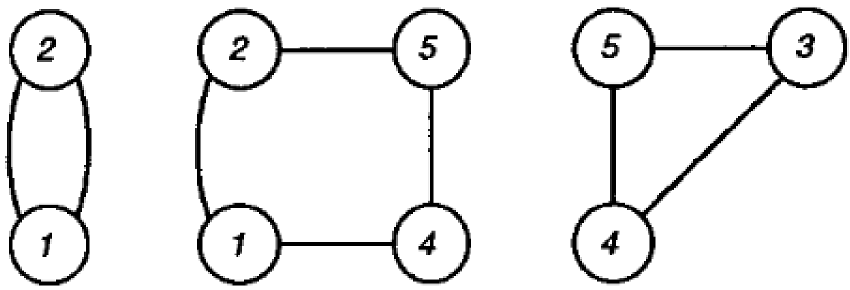
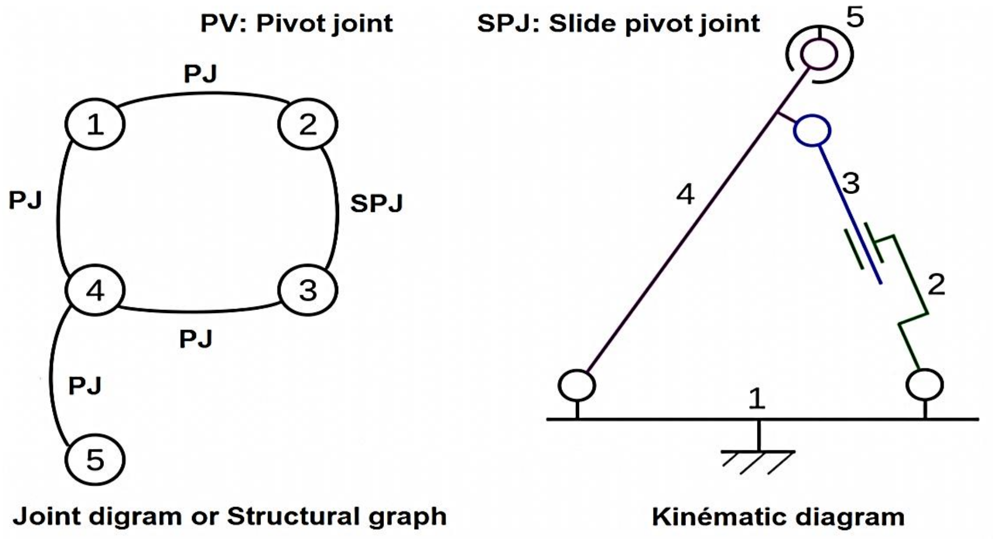



**Figure II. 14:** Cycle basis of Joint diagram of a mechanical system (Figure II.11).

***Example :***




## II.7 Cyclomatic number of a graph

This is the dimension of a cycle base. We can show that this number $N_c$ is directly calculable from the number of vertices and the number of arcs. For a connected graph (Figure II.11), we have:

$$N_c = N_j - N_L + 1$$

Where :

$N_c$ **:** Cyclomatic number of independent cycles

$N_j$**:** number of joints (Arcs).

$N_L$**:** Number of Links (parts)

From the graph of figure II.9 we have :

$$N_c = 8 - 6 + 1 = 3$$

The 1-4-3-5-2 cycle is not independent of the cycles in the cycle base.

### Examlpe (Nacelle support)

Figure II.15 shows an aerial platform support, and Figure II.16 the associated structure graph (Joint diagram). This structure graph includes:

$N_j = 13$ and $N_L = 11$

so $\boldsymbol{N_c = 13 - 11 + 1 = 3}$ (Three independent cycles).

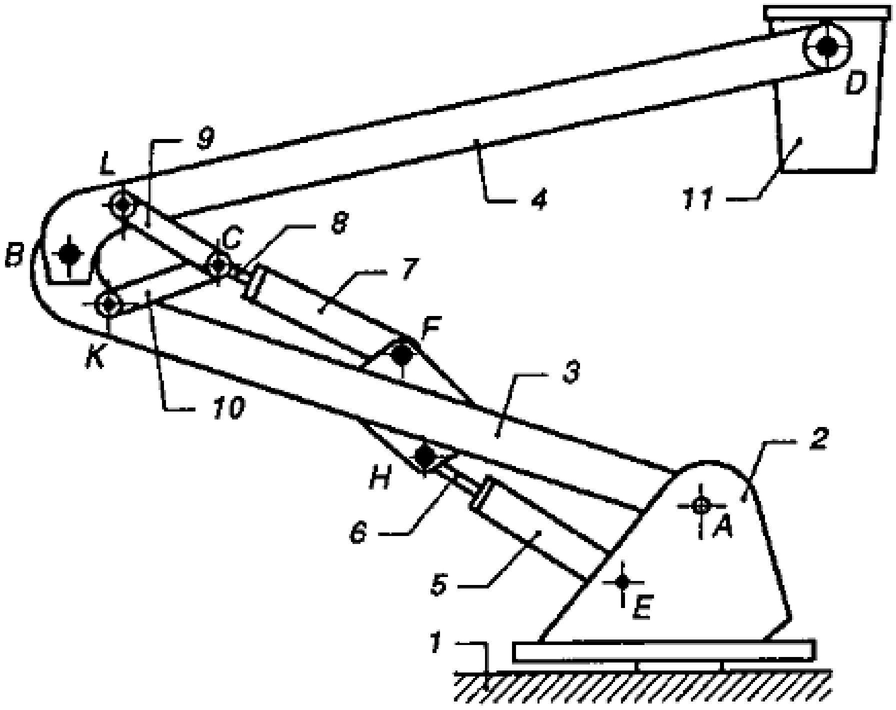



**Figure II. 15:** Nacelle support mechanism.

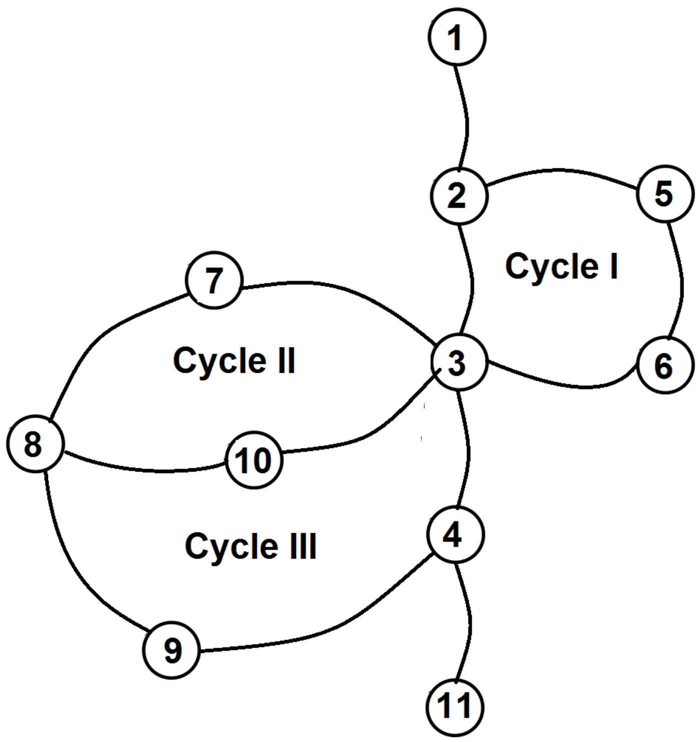



**Figure II. 16 :** Joint diagram of nacelle mechanism.

The three cycles are :

- Cycle I : 2-3-6-5 Cycle II : 3-7-8-10 Cycle III : 3-10-8-9-4

Cycle I (2-3-6-5) is not independent of the three previous cycles composing the cycle base. We can read the structure graph as follows:

- The main framework of the mechanical system is made up of the chain: 1-2-3-4-11
- Cycle I, in parallel with joints 2-3, serves to motorize this connection
- Likewise, cycle II sets cycle III in motion. This cycle, in parallel with connection 2-4, is an intermediate for its motorization.

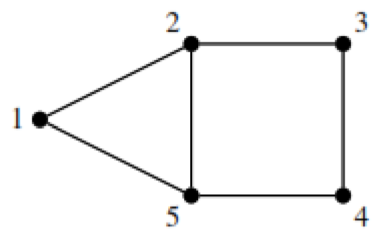
### II.7.1 Number of equations

Once the independent closed chains have been counted, it is possible to evaluate the number of scalar equations available for solving the problem. Let ***Eq*** be this number which results from the application of the law of composition of movements on each of the independent chains.

- $E_q = 6.N_c$ For the kinematic study
- $E_s = 6.(N_L - 1)$ For statically study

**Example**

Consider a mechanism whose structure graph is given below:




The number of cycle :

$$\boldsymbol{N_c = N_j - N_L + 1 = 6 - 5 + 1 = 2}$$

Cycle I : 1-2-5-1 Cycle II : 2-3-4-5-2

The cycle III : 1-2-3-4-5-1 is also a closed cycle, but it is deduced from the previous two.

The two torsorial equations to consider are, for example:

For the cycle 1-2-5-1 : $T_{(1/2)} + T_{(2/5)} + T_{(5/1)} = 0$ closed cycle

For the cycle 2-3-4-5-2 : $T_{(2/3)} + T_{(3/4)} + T_{(4/5)} + T_{(5/2)} = 0$ closed cycle

We obtain 12 scalar equations. We also see that if we add the two previous equations, we obtain :

$$T_{(1/2)} + T_{(5/1)} + T_{(2/3)} + T_{(3/4)} + T_{(4/5)} = 0$$

This equation corresponds well to the route of the third loop 1-2-3-4-5-1.

### II.7.2 Number of unknowns

We note ***Ic*** the number of scalar kinematic unknowns. This number is determined by simple sum of the degrees of freedom of each of the ***Nj*** joints.

The number of scalar kinematic unknowns depends on the nature of the models adopted for the connections.

### II.7.3 Mobility Index

The mechanic's problem is thus to treat, or even solve, a system of ***Eq*** equations with unknown ***Ic***. This system is a homogeneous linear system that is written in matrix form.

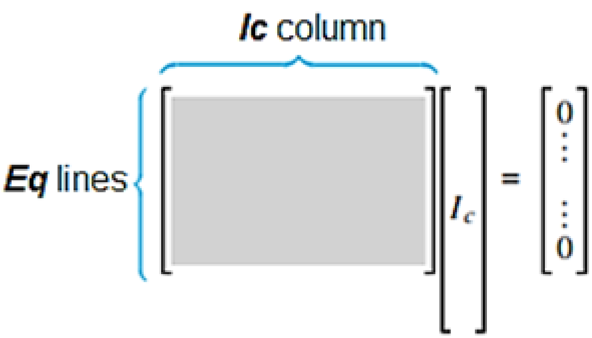



**Figure II. 17:** Linear equation system of a mechanism problem.

The mobility index is calculated by the following equation:

$$\boldsymbol{M_{index} = I_c - E_q}$$

The second member includes:

- The components of external mechanical actions other than the connection components, such as the components due to gravity, to a deformable element, to a receiver or to a motor, etc.
- The dynamic components.

By taking up the definition of the mobility index seen in kinematics and taking into account the duality between kinematics and mechanical actions which is expressed by the equality:

$$I_c + I_s = 6.N_j \quad \text{and} \quad E_q = 6.N_c = 6.(\boldsymbol{N_j - N_L + 1})$$

Moreover ,

$$M_{index} = I_c - E_q = (6.N_j - I_s) - (6.(N_j - N_L + 1))$$

$$I_c - E_q = 6.(N_L - 1) - I_s$$

With : $6.(N_L - 1) = E_s$ : represent the number of equation in the dynamic study.

$I_s$ columns: represent the number of unknowns.

### II.7.4 Degree of mobility m

The resolution of the previous system of equations (Figure II.17) takes into account its rank, denoted $r_c$.

- In the case where $r_c = I_c$, the only solution is the nullity of all the unknowns, therefore of all the kinematic parameters. The mechanism then defines a rigid structure, no movement is possible.
- Otherwise, we assume that the rank of the system and the equations arranged are known.

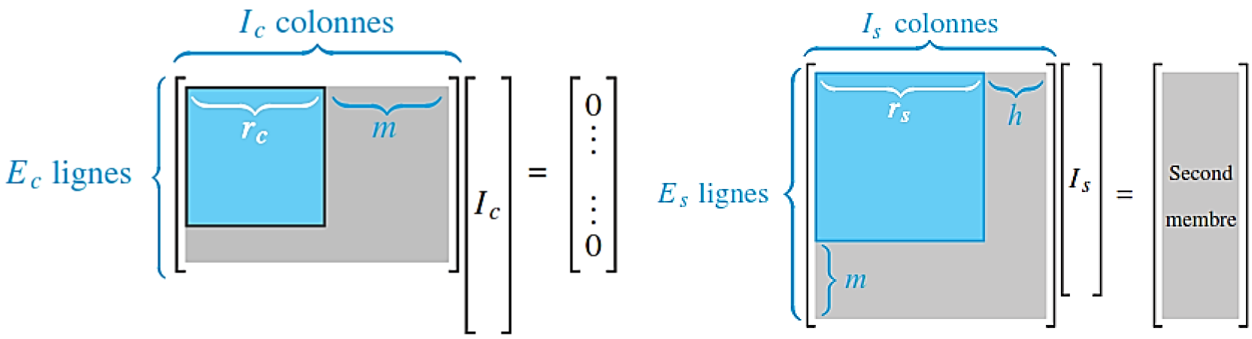



**Figure II. 18 :** Parameters of the linear equation system of a mechanism problem (kinematic and dynamic aaproachs).

The degree of mobility of a mechanism ***m*** is the number of possible independent movements (Figure II.18). It is a natural number denoted ***m*** and calculated by:

$$\boldsymbol{m = I_c - r_c} \quad \text{for the kinematic approach}$$

$$\boldsymbol{m = E_s - r_s} \quad \text{for the dynamic approach}$$

The degree of mobility ***m*** is always positive or zero. Indeed, the rank of a system of $E_c$ equations with $I_c$ unknowns is less than or equal to the smaller of these two numbers, which means that the rank is always less than or equal to the number of unknowns $I_c$.

$$\boldsymbol{r_c \leq \min(I_c, E_c) \leq I_c}$$

Finding the rank of the system of equations is very informative, because it makes it possible to differentiate the unknowns which can become main unknowns from those which cannot.

The degree of mobility ***m*** represents the number of unknowns that must be passed in the second member. All the other unknowns of the problem are then expressed in terms of these main unknowns.

This is how we call the ***input-output law*** of a mechanism any relationship between kinematic unknowns which can be interpreted as an unknown expressed as a function of one or more main unknowns. A mechanism admits at most $r_c$ input-output laws.

### II.7.5 The degree of static *h*

The degree of static of a mechanism is the number of main unknowns in the system of homogeneous equations comprising only the unknowns of mechanical actions transmissible by perfect connections or joints. It is a natural number denoted ***h*** and calculated by :

$$\boldsymbol{h = E_c - r_c} \quad \text{for the kinematic study}$$

$$\boldsymbol{h = I_s - r_s} \quad \text{for the dynamic study}$$

### II.7.6 Input-Output law

The analysis of the mechanism in its industrial environment makes it possible to choose the input parameters whose value is imposed as well as the output parameters.

We call ***input-output law*** the implicit relationships linking the kinematic input parameters $\boldsymbol{e_i}$, the kinematic output parameters $\boldsymbol{s_i}$ and the geometric data $\boldsymbol{G_i}$.

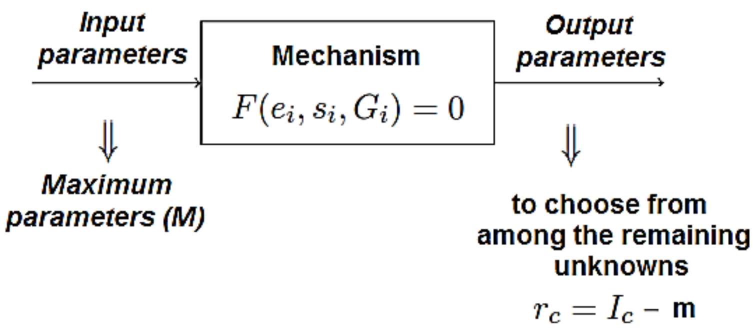
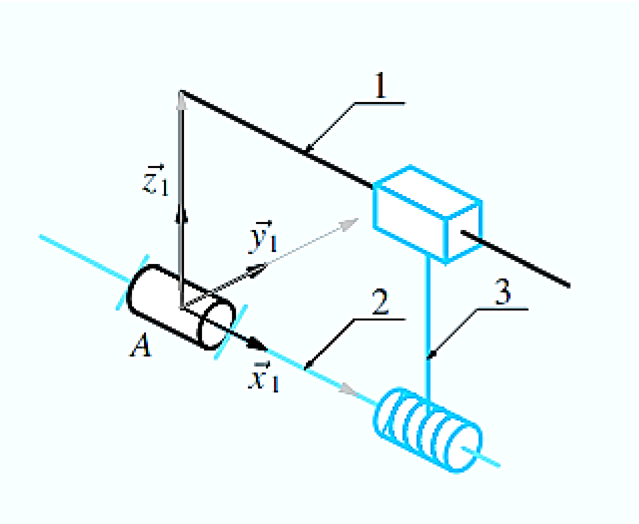



**Figure II. 19 :** Input/Output law.

**Example 1**

We propose to analyze a system of transformation of movement using the combination of a screw and a nut. This mechanism has three solids:




• A support 1, to which we associate a reference ($A, x_1, y_1, z_1$);

• A nut 3, guided in rectilinear translation relative to the support by a steering slide $x_1$;

• A screw 2, in pivot axis joint ($A, x_1$) with the support and in helical joint with the same axis with the nut.

1. Configure this mechanism.

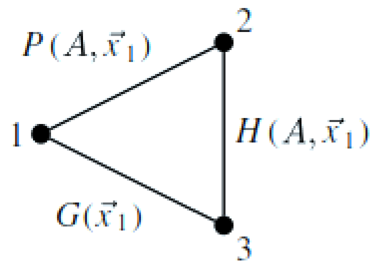
2. A motor drives the screw relative to the bracket and the nut is hooked to a receiver. Determine the input-output law.

3. We want a displacement following $+x_1$ of the receiver during positive rotation of the motor. Determine the direction to impose on the helix of the helical connection.

4. Evaluate the degree of static of this structure.

***Solution***

1- This mechanism involves a closed chain of three solids.




With : - $P(A, \vec{x}_1)$ is pivot along axis $x_1$

- $G(\vec{x}_1)$ is slide along axis $x_1$

- $H(A, \vec{x}_1)$ Helical along axis $x_1$

We write the three kinematic torsors to pose the three kinematic variables:

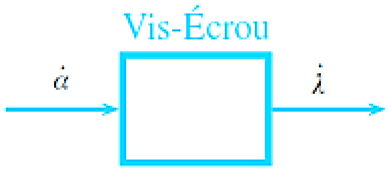
$$T_P(2/1) = \begin{Bmatrix} \dot{\alpha}.\vec{x}_1 \\ \vec{0} \end{Bmatrix} \qquad T_G(3/1) = \begin{Bmatrix} \vec{0} \\ \dot{\lambda}.\vec{x}_1 \end{Bmatrix} \qquad T_H(2/3) = \begin{Bmatrix} \omega_{23}.\vec{x}_1 \\ u_{23}.\vec{x}_1 \end{Bmatrix} \text{ with } u_{23} = p.\omega_{23}$$

The analysis of the vertices makes it possible to obtain the geometric properties specific to this structure:

- Screw 2 has two merging lines and a helix of pitch p;
- On frame 1 a parallel line and direction are defined;
- Nut 3 also has a parallel line and direction, as well as a helix of pitch p.

The pitch of the propeller is the only non-zero value characteristic of the geometry of the mechanism.

2- The desired input-output law concerns two of the three kinematic unknowns. The composition of the movements on the closed chain gives four scalar equations of the form 0=0 and two non-zero equations:




$$T_P(2/1) + T_G(3/1) + T_H(2/3) = \begin{Bmatrix} \dot{\alpha}.\vec{x}_1 \\ \vec{0} \end{Bmatrix} + \begin{Bmatrix} \vec{0} \\ \dot{\lambda}.\vec{x}_1 \end{Bmatrix} + \begin{Bmatrix} -\omega_{23}.\vec{x}_1 \\ p.\omega_{23}.\vec{x}_1 \end{Bmatrix} = \begin{Bmatrix} \vec{0} \\ \vec{0} \end{Bmatrix}$$

We obtain : $\begin{cases} \dot{\alpha}.\vec{x}_1 - \omega_{23}.\vec{x}_1 = \vec{0} \rightarrow \\ \dot{\lambda}.\vec{x}_1 + p.\omega_{23}.\vec{x}_1 = \vec{0} \rightarrow \end{cases}$

- scalar resultant equation $\vec{x}_1$;

$$\dot{\alpha} - \omega_{23} = 0 \qquad \text{so} \qquad \dot{\alpha} = \omega_{23}$$

- equation of scalar moment resultant $\vec{x}_1$;

$$\dot{\lambda} + p.\omega_{23} = 0$$

- S : $\dot{\lambda} = -p.\omega_{23} = -p.\dot{\alpha}$

3- We want $\dot{\lambda} \geq 0$ for $\dot{\alpha} \geq 0$, so it is necessary to use a left-handed helix for the helical joint. Remember that a left-handed propeller has a negative pitch $p < 0$.

4- The calculations carried out in question 2 allow us to answer with certainty:

- We have six equations for three unknowns, so the structure admits a mobility index

$$\boldsymbol{M_{index} = I_c - E_q = 6 - 3 = 3}$$

- Four 04 equations are of the form 0=0, therefore the rank is less than or equal to 2 and the structure admits at least one degree of mobility;
- The two written equations allow us to affirm that the rank is worth Rc=2 and we can give the values of the degrees of mobility and stasis.

$$m = 1 \; and \; h = 4$$

In conclusion, the structure is hyperstatic of degree 4.

**Example 2 (Dynamic approach)**

We consider the intermediate axis marked 2 of a gear reducer:

- It is guided in relation to a frame denoted 1 by two angular contact ball bearings, whose contacts are modelled by spherical type joints, with respective centers A and B, parameterized by :

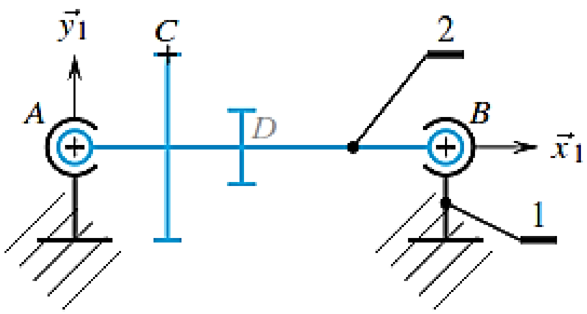
$$\overrightarrow{AB} = L.\vec{x}_1$$

- It includes a pinion which meshes with a motor shaft m at point ***C***, located by:

$$\overrightarrow{AC} = c.\vec{x}_1 + R.\vec{y}_1$$

- It also includes a toothed wheel which meshes with a receiver shaft ***r*** at a point ***D*** located by:

$$\overrightarrow{AC} = d.\vec{x}_1 - r.\vec{z}_1$$

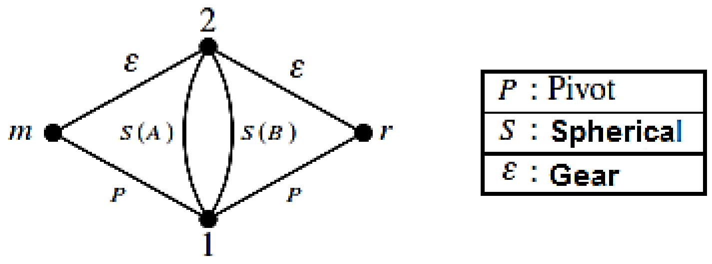


We neglect the mass and inertia of the tree 2 and the surrounding environment 2 retained for the study then includes:

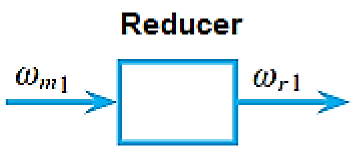
• Frame 1; • The motor shaft noted ***m***; • The receiver tree noted ***r***;

**Solution of example 2 (Dynamic approach)**

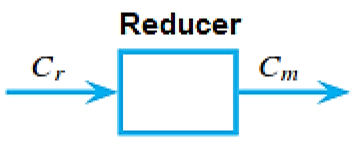
We draw the graph of the joints of the entire reducer to highlight the difference between the joints unknowns to keep in the left side and the unknowns to pass into the second side.




From a kinematic view, it is the motor which imposes the movement and would have to determine an input-output law $\omega_{r1} = f(\omega_{m1})$ by calling $\omega_{m1}$ and $\omega_{r1}$ the kinematic variables associated with the two pivot links.

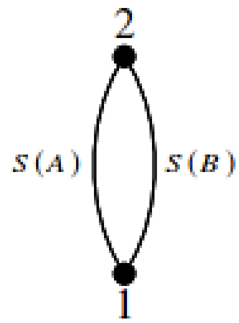



From a dynamic point of view, it is the receiver which requires power and would have to determine an input-output law $C_m = f_1(C_r)$, by calling $C_r$ and $C_m$ respectively the receiver and motor torques.




For the mechanism as a whole, the mechanical actions transmissible in the gears are joints actions.

Now, the object of the study here is the only closed chain 1-2-1, for which the mechanical linking actions are only at the level of the spherical type links S(A) an S(B).

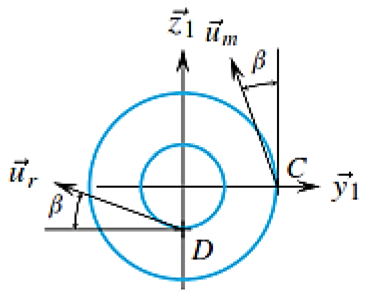



We write the torsors associated with the two connections and we model by sliders the mechanical actions transmissible by the gears.

$$\vec{F}_{(1a\to 2)} = \vec{F}_A = \begin{cases}\vec{R}_A \\ \vec{0}\end{cases} \qquad \vec{F}_{(1b\to 2)} = \vec{F}_B = \begin{cases}\vec{R}_B \\ \vec{0}\end{cases}$$

$$\vec{F}_{(m\to 2)} = \vec{F}_C = \begin{cases}F_m.\vec{u}_m \\ \vec{0}\end{cases} \qquad \vec{F}_{(r\to 2)} = \vec{F}_D = \begin{cases}F_r.\vec{u}_r \\ \vec{0}\end{cases}$$

The two directions $\vec{u}_m$ and $\vec{u}_r$ are in the plane $(\vec{y}_1, \vec{z}_1)$ and we orient them in the figure below :




In order to write the system of equations, we pose the components of any resultants:

$$\vec{R}_A = X_A.\vec{x}_1 + Y_A.\vec{y}_1 + Z_A.\vec{z}_1 \qquad \text{and} \qquad \vec{R}_B = X_B.\vec{x}_1 + Y_B.\vec{y}_1 + Z_B.\vec{z}_1$$

As the mass and the inertia of 2 are neglected, we apply the equilibrium theorem to the shaft 2 with respect to the supposedly Galilean reference frame 1 and we write the moment equation for example at point ***A*** to obtain the system of six scalar equations searched. All terms relating to gears are passed in the second member.

$$\begin{cases} X_A + X_B = 0 \\ Y_A + Y_B = F_m.\sin(\beta) + F_r.\cos(\beta) \\ Z_A + Z_B = -F_m.\cos(\beta) - F_r.\sin(\beta) \\ 0 = -R.F_m.\cos(\beta) + r.F_r.\sin(\beta) \\ -L.Z_B = c.F_m.\cos(\beta) + d.F_r.\sin(\beta) \\ L.Y_B = c.F_m.\sin(\beta) + d.F_r.\cos(\beta) \end{cases}$$

The search for degrees of mobility and static is carried out using the associated homogeneous system. In static:

$$\begin{cases} X_A + X_B = 0 \\ Y_A + Y_B = 0 \\ Z_A + Z_B = 0 \\ 0 = 0 \\ -L.Z_B = 0 \\ L.Y_B = 0 \end{cases}$$

The resolution is immediate and we deduce the different results without needing to use matrix writing:

• In the absence of solicitation ($F_m = F_r = 0$), the joints unknowns are zero, except the two components $X_A$ and $X_B$ which remain undetermined;

$$Y_B = Z_B = Z_A = Y_A = 0$$

So :

$$\begin{cases} X_A + X_B = 0 \\ 0 = 0 \\ 0 = 0 \\ 0 = 0 \\ 0 = 0 \\ 0 = 0 \end{cases}$$

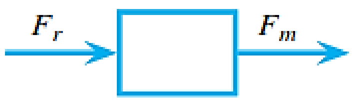
• The rank ***r_s*** is therefore equal to 5;

• The degree of static ***h*** is equal to 1, with $X_A$ and $X_B$ as the possible main unknown;

• The degree of mobility ***m*** is equal to 1, with an equation of the form 0=0 for the moment equation at point ***A*** scalar $\vec{x}_1$. We obviously find the results of the kinematic approach, with a mounting constraint in translation along x1 and a possible movement in rotation around the axis (A,x1).

When the structure is analyzed, the mechanic can have two centers of interest:

- He wishes to determine the input-output law, in this case in the form $F_m = f_2(Fr)$.




- He wishes to know the values of the components of mechanical linking actions according to external stresses.

He then works with a system of equations where all the unknowns are put on the left hand side.

$$\begin{cases} X_A + X_B = 0 \\ Y_A + Y_B - F_m.\sin(\beta) - F_r.\cos(\beta) = 0 \\ Z_A + Z_B + F_m.\cos(\beta) + F_r.\sin(\beta) = 0 \\ R.F_m.\cos(\beta) - r.F_r.\sin(\beta) = 0 \\ -L.Z_B - c.F_m.\cos(\beta) - d.F_r.\sin(\beta) = 0 \\ L.Y_B - c.F_m.\sin(\beta) - d.F_r.\cos(\beta) = 0 \end{cases}$$

It is a homogeneous system of 6 equations with 8 unknowns, of rank equal to 6.

We can therefore express six unknowns as a function of two main unknowns. We can show that we must take $X_A$ and $X_B$ for the first and that the choice of $F_r$ is suitable for the second:

• The first equation retains the indeterminacy highlighted previously;

$$X_A + X_B = 0$$

• The last five equations form a system of five equations with six unknowns of rank equal to five, with $F_r$ as the possible main unknown;

$$\begin{cases} Y_A + Y_B - F_m.\sin(\beta) = F_r.\cos(\beta) \\ Z_A + Z_B + F_m.\cos(\beta) = -F_r.\sin(\beta) \\ \boldsymbol{R.F_m.\cos(\beta) = r.F_r.\sin(\beta)} \\ -L.Z_B - c.F_m.\cos(\beta) = d.F_r.\sin(\beta) \\ L.Y_B - c.F_m.\sin(\beta) = d.F_r.\cos(\beta) \end{cases}$$

Among these five equations, the third provides the ***input-output law***.