Theory of mechanisms

Course and corrected exercises

3rd year Mechanical construction

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Preface

This handout "***Theory of mechanisms: Course and corrected exercises***" is a course intended for third-year mechanical engineering students, specializing in mechanical construction. This handout is developed in accordance with the official program set by the Ministry of Higher Education and Scientific Research.

This course allows students to learn the fundamental principles of the theory of mechanisms, such as degrees of freedom, the different mechanical connections and their calculations, kinematic diagrams, etc. This course is reinforced by application exercises to facilitate the understanding of the concepts for students.

This handout is structured in four chapters :

The first chapter represents mathematical reminders on vector and torsorial calculations where these notions will be used in the kinematic and static calculations of mechanisms as well as the equivalent connections of a mechanism.

The second part deals with the analysis and modeling of mechanisms. This part includes the usual links of mechanisms and their kinematic and static torsors followed by the principle of the construction of kinematic diagrams of a mechanism. In addition, it also includes the calculations of mobility, the number of cycles, the number of equations and the unknowns of a mechanism.

The third chapter presents a kinematic and static analysis of different types of mechanisms. This part contains the concept of the methods used to determine the kinematic and static torsors of the equivalent link of several links linked in series or in parallel. Followed by a kinematic and static study of closed chains and the resolution of hyperstatic systems.

The last chapter represents a study on planar mechanisms and their theory with graphical and analytical methods. In this part the analyses of the mechanisms will be based on Grashoff's law on 4-bar mechanisms to determine the type of mechanism to be obtained.

Contents

Chapter I

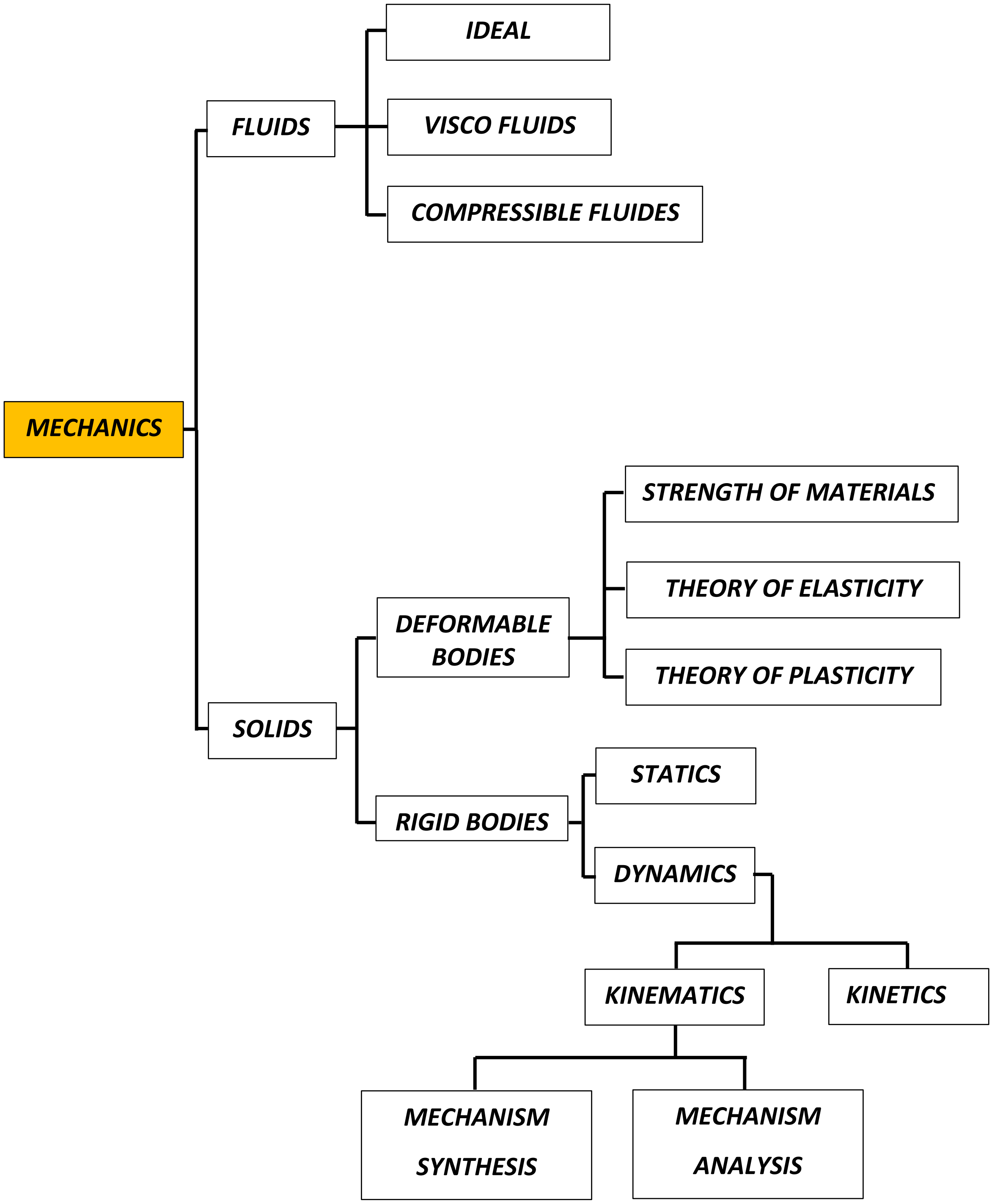
Introduction

Introduction

Mechanism analysis (or kinematics of machines) is inherently a vital part in the design of a new machine or in studying the design of an existing machine. For this reason, the subject has always been of considerable importance to the mechanical engineer. Moreover, if we consider the tremendous advances that have been made within recent years in the design of high-speed machines, computers, complex instruments, automatic controls, and mechanical robots, it is not surprising that the study of mechanisms has continued to attract greater attention and emphasis than ever before.

Mechanism analysis may be defined as a systematic analysis of a mechanism based on principles of kinematics, or the study of motion of machine components without regard to the forces that cause the motion. To better appreciate the role of mechanism analysis in the overall design process, consider the following. Typically, the design of a new machine begins when there is a need for a mechanical device to perform a specific function. To fulfil this need, a conceptual or inventive phase of the design process is required to establish the general form of the device. Having arrived at a concept, the designer usually prepares a preliminary geometric layout of the machine or mechanism for a complete kinematic analysis. Here the designer is concerned not only that all components of the machine are properly proportioned so that the desired motions can be achieved (synthesis phase), but also with the analysis of the components themselves to determine such characteristics as displacements, velocities, and accelerations (analysis phase). At the completion of this analysis, the designer is ready to proceed to the next logical step in the design process: kinetic analysis, in which individual machine members are analyzed further to determine the forces resulting from the motion.

Mechanism analysis therefore serves as a necessary prerequisite for the proper sizing of machine members, so that they can withstand the loads and stresses to which they will be subjected. The following flowchart shows the relationship of mechanism analysis to other branches of mechanics.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FLUIDS
IDEAL
VISCO FLUIDS
COMPRESSIBLE FLUIDES
SOLIDS
DEFORMABLE BODIES
STRENGTH OF MATERIALS
THEORY OF ELASTICITY
THEORY OF PLASTICITY
RIGID BODIES
STATICS
DYNAMICS
KINEMATICS
KINETICS
MECHANISM SYNTHESIS
MECHANISM ANALYSIS

Chapter I

Preliminary and recalls

I Preliminary and recalls

I.1 Vectors and Torsors

I.1.1 The vector

A vector is an element of a vector space. The mechanic is more particularly interested in vectors in the three-dimensional Euclidean vector space. These vectors have the particularity of being represented by arrows, which is convenient for drawing speeds or forces.

One of the remarkable properties of these “arrow” vectors is that we can add them by drawing them end to end.

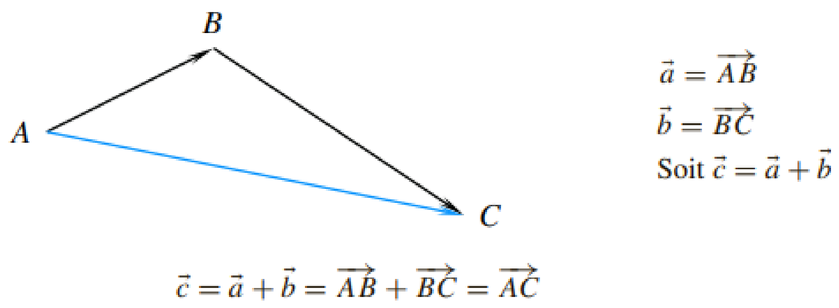


Figure I. 1: Vector Representation and sum of vectors.

I.1.1.1 Scalar product

The scalar product is an application which associates a pair of vectors with a real number equal to the product of the norms and the cosine of the angle oriented from the first vector to the second vector.

$$\begin{aligned} E \times E &\longrightarrow R \\ (\vec{V_1}, \vec{V_2}) &\longrightarrow \vec{V_1}.\vec{V_2} = \|\vec{V_1}\|, \|\vec{V_2}\|.\cos(\vec{V_1}, \vec{V_2}) \end{aligned}$$

I.1.1.2 Vector product

The vector product is an application which associates a pair of vectors with a vector carried by the unit vector directly orthogonal to these two vectors and with a module equal to the product of the norms and the sine of the angle oriented from the first vector towards the second vector.

$$\begin{aligned} E \times E &\longrightarrow E \\ (\vec{V_1}, \vec{V_2}) &\longrightarrow \vec{V} = \vec{V_1} \wedge \vec{V_2} = \|\vec{V_1}\|, \|\vec{V_2}\|.\sin(\vec{V_1}, \vec{V_2}).\vec{n} \end{aligned}$$

With $\vec{V}$ perpendicular to the plane formed by $\vec{V_1}$ and $\vec{V_2}$ ($\vec{V} \perp \vec{V_1}$ and $\vec{V} \perp \vec{V_2}$).

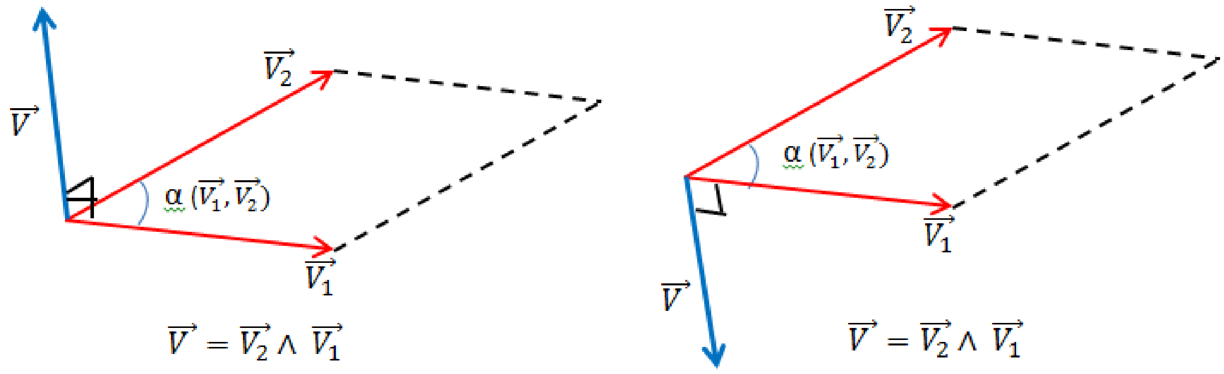


Figure I. 2 : Vector product representation.

I.1.1.3 The mixed product

Mixed product is a triple vector product that combines the concept of scalar product and vector products to yield a scalar.

$$E \times E \times E \longrightarrow R$$
$$(\vec{V_1}, \vec{V_2}, \vec{V_3}) \longrightarrow V = (\vec{V_1}, \vec{V_2}, \vec{V_3}) = \vec{V_1}.(\vec{V_2} \wedge \vec{V_3})$$

The absolute value of the mixed product is the volume of the parallelepiped formed by the 3 vectors.

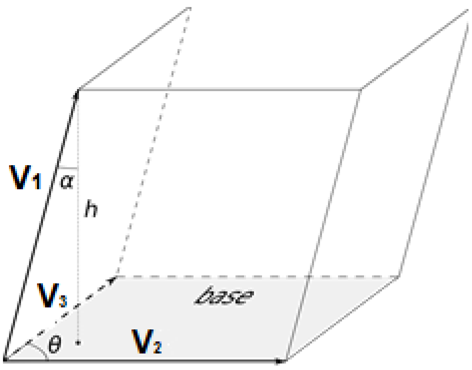


Figure I. 3 : Graphic representation of the mixed product.

The important property of the mixed product is its invariance by circular permutation on the vectors.

$$(\vec{V_1}, \vec{V_2}, \vec{V_3}) = (\vec{V_2}, \vec{V_3}, \vec{V_1}) = (\vec{V_3}, \vec{V_1}, \vec{V_2})$$

Either : $\vec{V_1}.(\vec{V_2} \wedge \vec{V_3}) = \vec{V_2}.(\vec{V_3} \wedge \vec{V_1}) = \vec{V_3}.(\vec{V_1} \wedge \vec{V_2})$

The mixed product is a tool to favour during future calculations. Many of the input-output laws sought are expressed in the form of an equality of two mixed products, and its use not only simplifies the writing, but limits many calculations.

I.1.1.4 Double vector product

To find it quickly, simply realize that the result of the double vector product $\overrightarrow{V_1} \wedge (\overrightarrow{V_2} \wedge \overrightarrow{V_3})$ is a vector defined in the plane $(\overrightarrow{V_2} \wedge \overrightarrow{V_3})$ and respect the order in which these two vectors appear.

$$\overrightarrow{V_1} \wedge \left(\overrightarrow{V_2} \wedge \overrightarrow{V_3}\right) = \left(\overrightarrow{V_1}.\overrightarrow{V_3}\right) * \overrightarrow{V_2}\big) - \big(\overrightarrow{V_1}.\overrightarrow{V_2}\big) * \overrightarrow{V_3}$$

The double vector product is useful to find:

- The decomposition of a vector.
- Solving a linear equation of the form : $\vec{a} \wedge \vec{b} = \vec{c}$

a) ***Decomposition of a vector***

We consider any vector$\vec{U}$, any plane (P) and $\vec{n}$ a vector orthogonal to this plane. We wish to decompose the vector $\vec{U}$ into the sum of a vector carried by the normal $\vec{n}$ and a vector of the plane (P).

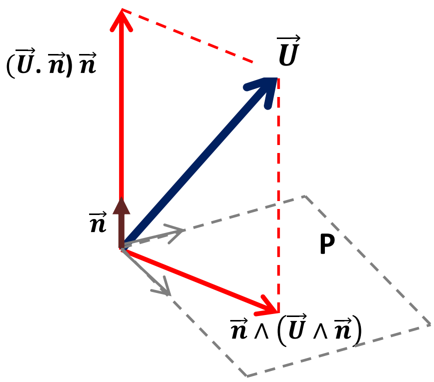


Figure I. 4 : Decomposition of a vector.

b) ***Solving the equation $\vec{a} \wedge \vec{b} = \vec{c}$***

Equation $\vec{a} \wedge \vec{b} = \vec{c}$ admits only a solution if vector $\vec{a}$ is non-zero and orthogonal to vector $\vec{c}$. The solution is then written:

$$\vec{x} = -\frac{\vec{a} \wedge \vec{c}}{\vec{a}^2} + \lambda.\vec{a} \ \text{ with } \lambda \in R$$

I.1.2 Vector derivation

I.1.2.1 Definition

The derivative of a vector $\overrightarrow{\boldsymbol{U}}$ in a base $\boldsymbol{k}$ is a vector whose components in the base $\boldsymbol{k}$ are the respective derivatives of the coordinates of $\overrightarrow{\boldsymbol{U}}$ in this same base. This definition clearly introduces the fact that deriving a vector implies an observation base. A notation specifying this basis is essential.

We note that the derivative of a vector $\overrightarrow{U}$ with respect to time only has meaning relative to a vector base ***k*** and is written : $\left[\frac{\overrightarrow{U}}{dt}\right]_k$.

Either $\overrightarrow{U}(t) = \lambda(t).\overrightarrow{u}(t)$ any vector, with $u(t)$ represent the unit vector. All quantities depend on time, so the variable (t) is rarely explained and we are content to write:

$$\overrightarrow{U} = \lambda.\overrightarrow{u}$$

Over time and in relation to an observation base, the vector $\overrightarrow{U}$ can change module or change direction, which is expressed very well by the formula for deriving a product of functions.

$$\left[\frac{\overrightarrow{U}}{dt}\right]_k = \left[\frac{\lambda.\overrightarrow{u}}{dt}\right]_k = \frac{d\lambda}{dt}.\overrightarrow{u} + \lambda.\left[\frac{d\overrightarrow{u}}{dt}\right]_k$$

With : $\frac{d\lambda}{dt}.\overrightarrow{u}$ represent the rate of variation of module with constant direction

$\lambda.\left[\frac{d\overrightarrow{u}}{dt}\right]_k$ represent the velocity of direction variation at constant module

I.1.2.2 Vectorial derivation properties

- Derivative of a sum of two vectors

$$\left[\frac{d(\vec{U} + \vec{V})}{dt}\right]_k = \left[\frac{d\vec{U}}{dt}\right]_k + \left[\frac{d\vec{V}}{dt}\right]_k$$

- Derivative of a product of two vectors

$$\left[\frac{d(\lambda.\vec{U})}{dt}\right]_k = \left[\frac{d\vec{\lambda}}{dt}\right]_k.\vec{U} + \lambda.\left[\frac{d\vec{U}}{dt}\right]_k$$

- Derivative of a dot product

$$\left[\frac{d(\vec{U}.\vec{V})}{dt}\right]_k = \left[\frac{d\vec{U}}{dt}\right]_k.\vec{V} + \vec{U}.\left[\frac{d\vec{V}}{dt}\right]_k$$

- Derivative of a vector product

$$\left[\frac{d(\vec{U}\wedge\vec{V})}{dt}\right]_k = \left[\frac{d\vec{U}}{dt}\right]_k \wedge \vec{V} + \vec{U}\wedge\left[\frac{d\vec{V}}{dt}\right]_k$$

I.1.2.3 Variation of a unit vector

A unit vector is a vector with a norm equal to 1, therefore constant. The only possible evolution is a change of direction which can only be appreciated in relation to a vector observation base. When we want to change the observation base, we use a formula for changing bases, called the vector derivation formula. This is the fundamental formula to use for all vector derivative calculations. It is expressed by :

$$\left[\frac{d\vec{u}}{dt}\right]_k = \left[\frac{d\vec{u}}{dt}\right]_i + \vec{\Omega}(i/k)\wedge\vec{u}$$

Where $\vec{\Omega}(i/k)$ is called the ***rotation vector*** of the vector base (***i***) relative to the vector base (***k***), and this formula makes it possible to determine the derivative of a vector in a base (***k***) from the knowledge of its derivative in a base (***i***). For practical calculations, it immediately appears judicious to look for each time a base (***i***) in which the vector $\vec{u}$ does not

I.1.2.4 Rotation vector

The rotation vector is obviously related to the angles that we define to pass from one vector base to another. It is constructed in two steps, starting with the case where two vector bases have a common vector.

Consider two vector bases (***i***) and (***k***) having a common vector. We call the rotation vector of the base (***i***) relative to the base (***k***) the vector denoted $\vec{\Omega}(i/k)$, carried by the direction common to the two bases and of algebraic value the scalar derivative of the angle defined between the two bases. The rotation vector is defined by :

$$\vec{\Omega}(i/k) = \left[\frac{d\alpha_{ik}}{dt}\right]_k = \dot{\alpha}_{ik}\,.\vec{z}_k$$

Where is $\dot{\alpha}_{ik}$ called rotation rate, sometimes rotation speed, and has as unit radian per second.

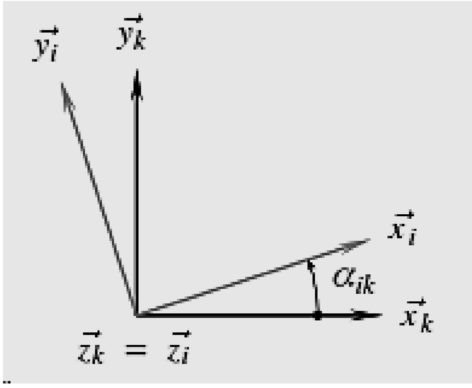


Figure I. 5 : Change of base by rotation around the z axis.

I.1.2.5 Composition of rotations

Composition of rotations When the vector bases do not have common vectors, we end eavor to chain together elementary rotations, as for Euler angles, so as to find ourselves in the known situation where two vector bases have a common vector .

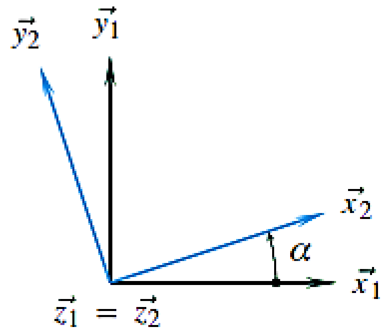
Assuming we can describe a rotation (i/k) by linking two rotations (i/j) and (j/k), we obtain the rotation vector $\vec{\Omega}(i/k)$, by the sum of the two rotation vectors $\vec{\Omega}(i/j)$, and $\vec{\Omega}(j/k)$.

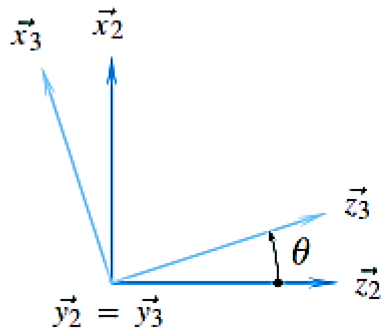
$$\vec{\Omega}(i/k) = \vec{\Omega}(i/j) + \vec{\Omega}(j/k$$

Example

These two figures make it possible to pose:

$$\vec{\Omega}(2/1) = \dot{\alpha}\,.\vec{z}_1 \; and \; \vec{\Omega}(3/2) = \dot{\theta}\,.\vec{y}_2 \quad with \; \vec{z}_1 = \vec{z}_2 \; and \; \vec{y}_1 = \vec{y}_2$$



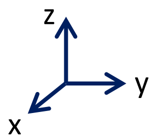


The rotation vector $\vec{\Omega}(3/1)$ is obtained by the composition of the two elementary rotations $3/2$ and $2/1$ and we instantly obtain its expression by the sum of the two previous rotation vectors.

$$\vec{\Omega}(3/1) = \vec{\Omega}(3/2) + \vec{\Omega}(2/1) = \dot{\theta}\,.\vec{y}_2 + \dot{\alpha}\,.\vec{z}_1$$

I.1.3 Euler's angles

One of the applications of the vector product is the construction of direct trihedral. Let (x,y,z) be a direct orthonormal base, we have the three immediate equalities :



$\vec{z} = \vec{x} \wedge \vec{y} \qquad \vec{z} = \vec{y} \wedge \vec{z} \qquad \vec{y} = \vec{z} \wedge \vec{x}$

We cannot finish the section on the different locating systems without looking at the recurring problem in mechanics of positioning any vector base in relation to another, knowing that we do not know how to define an angle to go from one basis to another when there is no common vector between the two. Euler's angle principle solves this problem simply and effectively. This involves carrying out three successive rotations, which can be easily understood by presenting them in a different order:

- The first, called ***precession***, around one of the three vectors of the first base, no matter which one;
- The third, called ***proper rotation***, around one of the vectors of the second base;
- Finally the second, called ***nutation***, around one of the two vectors orthogonal to the two vectors previously chosen. This vector is called a nodal vector.

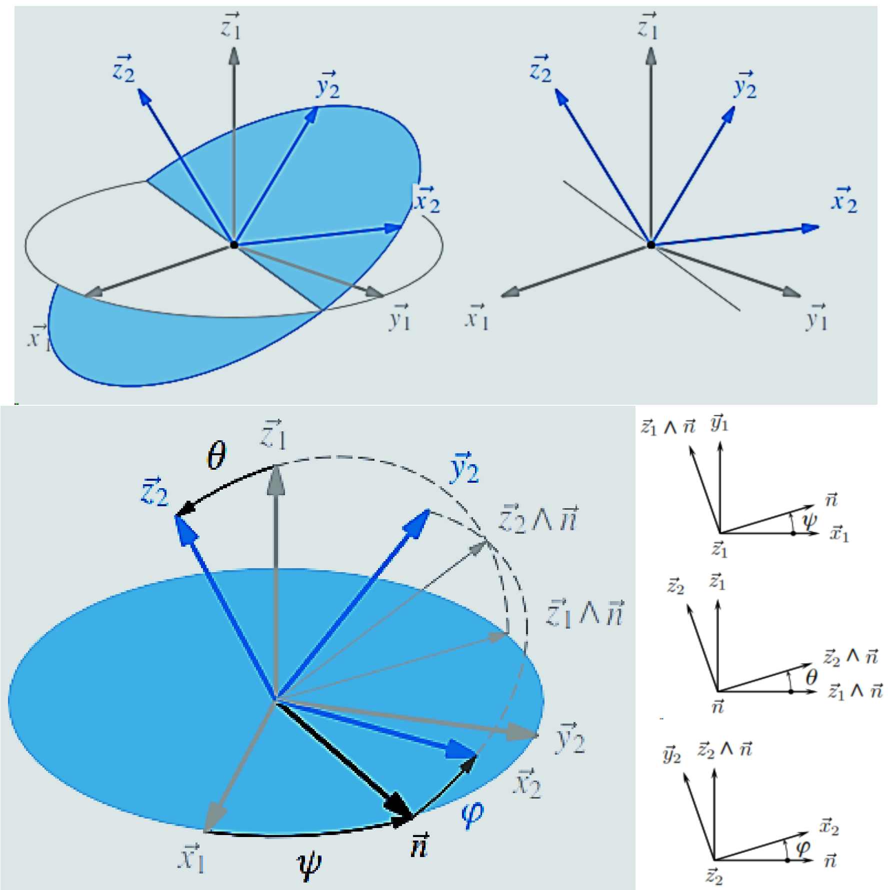


Figure I. 6 : Euler's angle : The precession is around z_1, the proper rotation around z_2 and the nodal vector is following the line of intersection of the two planes illustrating the rotations.

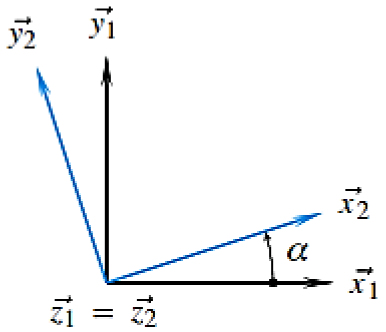
The nodal vector (n) is placed orthogonal to the vectors (z1) and (z2). Assuming the latter to be different from each other, it suffices to put.

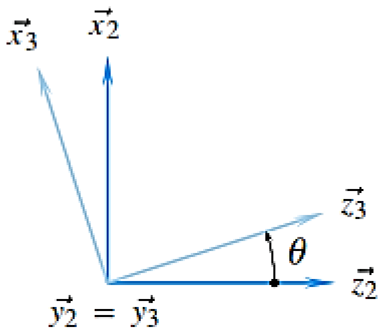
$$\vec{n} = \frac{\vec{z}_1 \wedge \vec{z}_2}{|\vec{z}_1 \wedge \vec{z}_2|}$$

a) ***Application example:***

We define the vector $\overrightarrow{AP} = \rho.\vec{x}_2 + b.\vec{x}_3$, with length b remaining constant over time, with :

$$\vec{\Omega}(2/1) = \dot{\alpha}\,.\vec{z}_1 \; and \; \vec{\Omega}(3/2) = \dot{\theta}\,.\vec{y}_2 \quad with \; \vec{z}_1 = \vec{z}_2 \; and \; \vec{y}_1 = \vec{y}_2$$





The derivative of this vector in base 1 is calculated by :

$$\left[\frac{\overrightarrow{AP}}{dt}\right]_1 = \dot{\rho}.\vec{x}_2 + \rho.\left[\frac{d\vec{x}_2}{dt}\right]_1 + b.\left[\frac{d\vec{x}_3}{dt}\right]_1$$

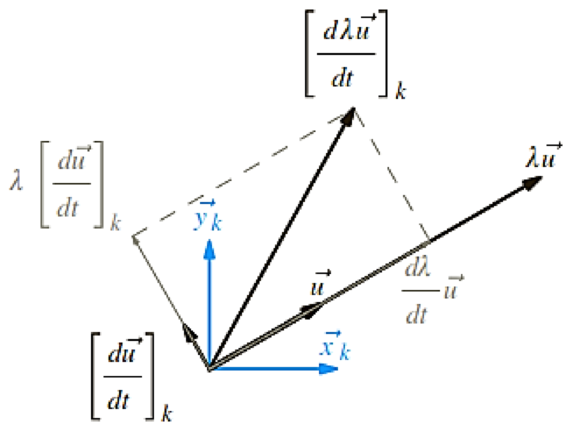


Figure I. 7 : Representation of the vector derivative.

Two independent calculations were carried out, using the base change formula then the angle definition figures.

- $\left[\frac{d\vec{x}_2}{dt}\right]_1 = \left[\frac{d\vec{x}_2}{dt}\right]_2 + \vec{\Omega}_{(2/1)} \wedge \vec{x}_2$

- The derivate of vector $\vec{x}_2$ in its base is null $\left[\frac{d\vec{x}_2}{dt}\right]_2 = \vec{0}$
- $\vec{\Omega}_{(2/1)} = \left[\frac{d\vec{\alpha}_{(2/1)}}{dt}\right]_1 = \dot{\alpha}.\vec{z}_1$

We obtain : $\left[\frac{d\vec{x}_2}{dt}\right]_1 = \dot{\alpha}.\vec{z}_1 \wedge \vec{x}_2 = \dot{\alpha}.\vec{y}_2$

- $\left[\frac{d\vec{x}_3}{dt}\right]_1 = \left[\frac{d\vec{x}_3}{dt}\right]_3 + \vec{\Omega}_{(3/1)} \wedge \vec{x}_3$
- The derivate of vector $\vec{x}_3$ in its base is null $\left[\frac{d\vec{x}_3}{dt}\right]_3 = \vec{0}$
- $\vec{\Omega}_{(3/1)} = \vec{\Omega}_{(3/2)} + \vec{\Omega}_{(2/1)} = \left[\frac{d\vec{\theta}_{(3/2)}}{dt}\right]_2 + \dot{\alpha}.\vec{z}_1 = \dot{\theta}.\vec{y}_2 + \dot{\alpha}.\vec{z}_1$

We obtain : $\left[\frac{d\vec{x}_3}{dt}\right]_1 = \left(\dot{\theta}.\vec{y}_2 + \dot{\alpha}.\vec{z}_1\right) \wedge \vec{x}_3 = (\dot{\theta}.\vec{y}_2 \wedge \vec{x}_3) + (\dot{\alpha}.\vec{z}_1 \wedge \vec{x}_3)$

$$\left[\frac{d\vec{x}_3}{dt}\right]_1 = -\dot{\theta}.\vec{z}_3 + (\dot{\alpha}.\vec{z}_1 \wedge \cos(\theta).\vec{x}_2)$$

$$\left[\frac{d\vec{x}_3}{dt}\right]_1 = -\dot{\theta}.\vec{z}_3 + \dot{\alpha}.\cos(\theta).\vec{y}_2$$

So: $\left[\frac{\overrightarrow{AP}}{dt}\right]_1 = \dot{\rho}.\vec{x}_2 + \dot{\alpha}.(\rho + b.\cos(\theta)).\vec{y}_2 - b.\dot{\theta}.\vec{z}_3$

I.1.3.1 Exercises on the vector calculation

Exercise I.2

Consider the following vectors: $\vec{A} = A_x.\vec{\imath} + A_y.\vec{\jmath} + A_z.\vec{k}$ and $\vec{B} = B_x.\vec{\imath} + B_y.\vec{\jmath} + B_z.\vec{k}$

1- Calculate scalar products : $\vec{A}.\vec{B}$, $\vec{A}.\vec{A}$ and $\vec{B}.\vec{B}$

We consider the three following vectors :

$\vec{V}_1 = 2.\vec{\imath} - \vec{\jmath} + 5.\vec{k}$, $\vec{V}_2 = -3.\vec{\imath} + 1{,}5.\vec{\jmath} - 7{,}5.\vec{k}$ and $\vec{V}_3 = -5.\vec{\imath} + 4.\vec{\jmath} + \vec{k}$

2- Calculate : $\vec{V}_1.\vec{V}_2$ and $\vec{V}_1 \wedge \vec{V}_2$

3- Calculate the following products : $\vec{V}_1.\overrightarrow{(V_2} \wedge \vec{V}_3)$ and $\vec{V}_1 \wedge \overrightarrow{(V_2} \wedge \vec{V}_3)$

4- Determine the area of the triangle formed by the vectors $\vec{V}_2$ and $\vec{V}_3$

Solution I.1

1- $\vec{A}.\vec{B} = A_x.B_{x+}A_y.B_{y+}A_z.B_z$

$\vec{A}.\vec{A} = {A_x}^2 + {A_y}^2 + {A_z}^2$

$$\vec{B}.\vec{B} = {B_x}^2 + {B_y}^2 + {B_z}^2$$

2- $\vec{V}_1.\vec{V}_2 = \big(2.(-3)\big) + \big((-1).1{,}5\big) + \big(5.(-7.5)\big) = -45$

$$\vec{V}_1 \wedge \vec{V}_2 = \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix} \wedge \begin{pmatrix} -3 \\ 1.5 \\ -7.5 \end{pmatrix}$$

$$\vec{V}_1 \wedge \vec{V}_2 = \big((-1).(-7.5)\big) - (1{,}5\,.5)).\vec{\imath} - ((2.(-7{,}5)) - ((-3).5)).\vec{\jmath} + (2.1{,}5) - ((-3).(-1)).\vec{k}$$

$$\vec{V}_1 \wedge \vec{V}_2 = 0.\vec{\imath} + 0.\vec{\jmath} + 0.\vec{k} = \vec{0} \qquad \text{so}: \ \vec{V}_1 \parallel \vec{V}_2$$

In addition, $\vec{V}_1.\vec{V}_2 = -45$ is negative, means that the two vectors have opposite directions.

3- $\vec{V}_1.\big(\vec{V}_2 \wedge \vec{V}_3\big) = \vec{V}_1.\begin{pmatrix} -3 \\ 1.5 \\ -7.5 \end{pmatrix} \wedge \begin{pmatrix} -5 \\ 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix}.\begin{pmatrix} 31{,}5 \\ 40{,}5 \\ -4{,}5 \end{pmatrix} = 0$

$$\vec{V}_1 \wedge \overline{(\vec{V}_2} \wedge \vec{V}_3) = \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix} \wedge \begin{pmatrix} 31{,}5 \\ 40{,}5 \\ -4{,}5 \end{pmatrix} = -198.\vec{\imath} + 166{,}5.\vec{\jmath} + 112{,}5.\vec{k}$$

4- The surface of the triangle formed by the vectors $\vec{V}_2$ and $\vec{V}_3$ is given by half the modulus of the vector product of the two vectors:

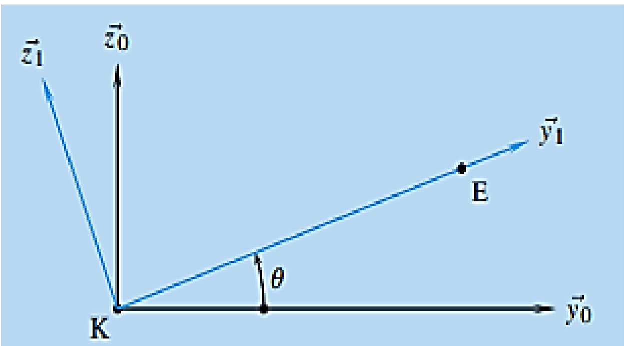
$$\big(\vec{V}_2 \wedge \vec{V}_3\big) = 31{,}5.\vec{\imath} + 40{,}5.\vec{\jmath} - 4{,}5.\vec{k}$$

$$S = \frac{\left|\big(\vec{V}_2 \wedge \vec{V}_3\big)\right|}{2} = \frac{\sqrt{(31{,}5)^2 + (40{,}5)^2 + (4{,}5)^2}}{2} = \frac{51{,}50}{2} = 25{,}75\ (u^2)$$

Exercise I.2

A mechanism has an lever of variable length rotating around an axis. The plane of evolution of the lever is a plane (K, $\overrightarrow{\boldsymbol{y_0}}, \overrightarrow{\boldsymbol{z_0}}$). A motor activates the axis rotation (K, $\overrightarrow{\boldsymbol{x_0}}$) and a cylinder varies the length $\boldsymbol{\rho}$ of the arm defined by $\overrightarrow{\boldsymbol{KE}} = \boldsymbol{\rho}.\overrightarrow{\boldsymbol{y_1}}$.

- Calculate $\left[\frac{d\overrightarrow{KE}}{dt}\right]_0$ and $\left[\frac{d^2\overrightarrow{KE}}{dt^2}\right]_0$



Solution I.2

- $\left[\frac{d\overrightarrow{KE}}{dt}\right]_0 = \left[\frac{d\rho.\overrightarrow{y_1}}{dt}\right]_0 = \frac{d\rho}{dt}.\overrightarrow{y_1} + \rho.\left[\frac{d\overrightarrow{y_1}}{dt}\right]_0$

We have : $\left[\frac{d\vec{y_1}}{dt}\right]_0 = \left[\frac{d\vec{y_1}}{dt}\right]_1 + \vec{\Omega}_{(1/0)} \wedge \vec{y}_1 = \left[\frac{\boldsymbol{d\theta}}{\boldsymbol{dt}}\right]_{\boldsymbol{0}} . \vec{x}_0 \wedge \vec{y}_1 = \dot{\theta}.\vec{x}_0 \wedge \vec{y}_1$ *, with :* $\left[\frac{d\vec{y_1}}{dt}\right]_1 = \vec{0}$

So : $\left[\frac{\boldsymbol{d\vec{y_1}}}{\boldsymbol{dt}}\right]_{\boldsymbol{0}} = \dot{\theta}.\vec{z}_1$

$$\left[\frac{d\overrightarrow{KE}}{dt}\right]_0 = \dot{\rho}.\vec{y_1} + \rho.\dot{\theta}.\vec{z}_1$$

- $\left[\frac{d^2\overrightarrow{KE}}{dt^2}\right]_0 = \left[\frac{d\ \ (\dot{\rho}.\vec{y_1}+\rho.\dot{\theta}.\vec{z}_1)}{dt}\right]_0 = \left[\frac{d\ \ (\dot{\rho}.\vec{y_1})}{dt}\right]_0 + \left[\frac{d\ \ (\rho.\dot{\theta}.\vec{z}_1)}{dt}\right]_0$

$$\left[\frac{d^2\overrightarrow{KE}}{dt^2}\right]_0 = \frac{d\dot{\rho}}{dt}.\vec{y_1} + \dot{\rho}.\left[\frac{d\vec{y_1}}{dt}\right]_0 + \left[\frac{d\ \ (\rho.\dot{\theta})}{dt}\right]_0 .\vec{z}_1 + \rho.\dot{\theta}.\left[\frac{d\vec{z}_1}{dt}\right]_0$$

$$\left[\frac{d^2\overrightarrow{KE}}{dt^2}\right]_0 = \ddot{\rho}.\vec{y_1} + \dot{\rho}.\dot{\theta}.\vec{z}_1 + (\dot{\rho}.\dot{\theta} + \rho.\ddot{\theta}).\vec{z}_1 + \rho.\dot{\theta}.\left[\frac{d\vec{z}_1}{dt}\right]_0$$

With : $\left[\frac{d\vec{z}_1}{dt}\right]_{\boldsymbol{0}} = \left[\frac{d\vec{z}_1}{dt}\right]_{\boldsymbol{1}} + \vec{\Omega}_{\left(\frac{1}{0}\right)} \wedge \vec{z}_1 = \dot{\theta}.\vec{x}_0 \wedge \vec{z}_1 = -\dot{\theta}.\vec{y}_1$

So : $\left[\frac{d^2\overrightarrow{KE}}{dt^2}\right]_0 = \ddot{\rho}.\vec{y_1} + 2.\dot{\rho}.\dot{\theta}.\vec{z}_1 + \rho.\ddot{\theta}.\vec{z}_1 - \rho.\dot{\theta}^2.\vec{y}_1$

I.1.4 The torsors

I.1.4.1 Vector field

We call a vector field an application which associates a vector with each point in the affine space.

$$\begin{array}{ccc} \mathbb{C} & \longrightarrow & E \\ P & \longrightarrow & \vec{U}(P) \end{array}$$

Among the possible particularities of a vector field, we retain two:

- ***Uniform vector field***

$$\forall A \in \mathbb{C}, \qquad \forall B \in \mathbb{C}, \ \vec{U}(A) = \vec{U}(B)$$

The vectors are identical at each moment, which does not assume their evolution over time.

- ***Equiprojective vector field***

$$\forall A \in \mathbb{C}, \qquad \forall B \in \mathbb{C}, \ \vec{U}(A).\overrightarrow{AB} = \vec{U}(B).\overrightarrow{AB}$$

The two vectors $\vec{U}(A)$ and $\vec{U}(B)$ have the same orthogonal projection on the direction $\overrightarrow{AB}$.

I.1.4.2 The torsor

A torsor is an equiprojective vector field. The word "***Torsor***", from the same family as torsade, comes from the remarkable shape of an equiprojective vector field.

An equiprojective vector field has the remarkable property of corresponding to an antisymmetric linear application. There is a vector that we denote ($\vec{r}$) and which allows you to easily change points on the vector field :

$$\forall A \in \mathbb{C}, \qquad \forall B \in \mathbb{C}, \; \vec{U}(B) = \vec{U}(A) + \vec{r} \wedge \overrightarrow{AB}$$

With :

$\vec{r}$: The resultant vector of torsor $\boldsymbol{T}$.

$\vec{U}(P)$: The moment vector at point $\boldsymbol{P}$ of the torsor.

The two vectors $\vec{r}$ and $\vec{U}(P)$ are called reduction elements of the torsor at the point **P**. They are usually presented behind an opening brace, always starting with the resultant vector.

$$\mathrm{T} = \begin{cases} \vec{r} & \text{The resultant of torsor} \\ \vec{U}(P) & \text{Moment vector at point } \boldsymbol{P} \end{cases}$$

I.1.4.3 Torsor Properties

a) ***Sum of two torsors***

Consider two equiprojective vector fields $\vec{U}_1(P)$ and $\vec{U}_2(P)$. We construct a new vector field $\vec{U}(P)$ defined by:

$$\forall \mathrm{P}, \; \vec{U}(P) = \vec{U}_1(P) + \vec{U}_2(P) \quad \text{With :} \quad \mathrm{T}_1 = \begin{cases} \overrightarrow{r_1} \\ \overrightarrow{U_1}(A) \end{cases} \text{and } \mathrm{T}_2 = \begin{cases} \overrightarrow{r_2} \\ \overrightarrow{U_2}(A) \end{cases}$$

This field $\vec{U}(P)$ is also an equiprojective vector field, whose reduction elements $\vec{r}$ and $\vec{U}(A)$ are simply determined by:

$$\begin{cases} \vec{r} = \overrightarrow{r_1} + \overrightarrow{r_2} \\ \vec{U}(A) = \vec{U}_1(A) + \vec{U}_2(A) \end{cases}$$

The torsorial writing of the sum of two equiprojective vector fields is elementary.

$$\mathrm{T} = \mathrm{T}_1 + \mathrm{T}_2$$

b) ***Comoment – Automoment***

Consider two equiprojective vector fields $\vec{U}_1(P)$ and $\vec{U}_2(P)$ whose reduction elements are determined at the same point A.

$$\mathrm{T}_1 = \begin{cases} \overrightarrow{r_1} \\ \overrightarrow{U_1}(A) \end{cases} \text{and } \mathrm{T}_2 = \begin{cases} \overrightarrow{r_2} \\ \overrightarrow{U_2}(A) \end{cases}$$

c) ***Co-moment***

We call the comoment of the two torsors T_1 and T_2, which we note $T_1 \otimes T_2$ the scalar number obtained from the reduction elements in the following way:

$$T_1 \otimes T_2 = \overrightarrow{r_1}.\overrightarrow{U_2}(A) + \overrightarrow{U_1}(A).\overrightarrow{r_2}$$

- The comoment is independent of the calculation point.
- The comoment of a torsor with itself gives double scalar product of the resultant and the moment at any point.

d) ***Automoment***

We call the automoment of a torsor the scalar invariant obtained by the scalar product of the resultant vector and the moment vector at any point.

e) ***Central axis***

For a torsor with a non-zero resultant, there exists a particular line in space for which the moment vector is collinear with the resultant vector.

Let a torso T be known by its reduction elements $\vec{r}$ and $\vec{U}(A)$. We search for the set of points ***P*** verifying :

$$\vec{r} \wedge \vec{U}(P) = \vec{0}$$

By using the following formula for change of point to introduce the known moment $\vec{U}(A)$:

$$\vec{U}(P) = \vec{U}(A) + \vec{r} \wedge \overrightarrow{AP}$$

It's necessary to search for all the points ***P*** verifying :

$$\vec{r} \wedge \left(\vec{U}(A) + \vec{r} \wedge \overrightarrow{AP}\right) = \vec{0} \quad \Rightarrow \quad \vec{r} \wedge \vec{U}(A) + \vec{r} \wedge \left(\vec{r} \wedge \overrightarrow{AP}\right) = \vec{0}$$

we have : $\vec{a} \wedge \left(\vec{b} \wedge \vec{c}\right) = (\vec{c}\,.\vec{a}\,) \times \vec{b} - (\vec{b}\,.\vec{a}\,) \times \vec{c}$

So : $\vec{r} \wedge \left(\vec{r} \wedge \overrightarrow{AP}\right) = \left(\overrightarrow{AP}.\vec{r}\right) \times \vec{r} - (\vec{r}.\vec{r}) \times \overrightarrow{AP}$

we can write $\overrightarrow{AP}$ in the form: $\overrightarrow{AP} = \overrightarrow{AH} + \overrightarrow{HP}$ with $\overrightarrow{AH} \perp \vec{r}$

So : $\vec{r} \wedge \left(\vec{r} \wedge \overrightarrow{AP}\right) = \left(\left(\overrightarrow{AH} + \overrightarrow{HP}\right).\vec{r}\right) \times \vec{r} - (\vec{r}^{\,2}) \times \overrightarrow{AP}$

$= \left[\left(\overrightarrow{AH}\,.\vec{r}\right) + \left(\overrightarrow{HP}.\vec{r}\right)\right] \times \vec{r} - (\vec{r}^{\,2}) \times \overrightarrow{AP}$ We have : $\overrightarrow{AH}\,.\vec{r} = 0$ $(\overrightarrow{AH} \perp \vec{r})$

$= (\overrightarrow{HP}.\vec{r}) \times \vec{r} - (\vec{r}^{\,2}) \times \overrightarrow{AP}$

The vector $\overrightarrow{HP}$ is parallel to $\vec{r}$ so we can write $\overrightarrow{HP}$ in the form: $\overrightarrow{HP} = \lambda\,.\vec{r}$

With : λ is a constant.

$$\vec{r} \wedge \left(\vec{r} \wedge \overrightarrow{AP}\right) = (\lambda\,.\vec{r}.\vec{r}) \times \vec{r} - (\vec{r}^2) \times \overrightarrow{AP}$$

$$\vec{r} \wedge \left(\vec{r} \wedge \overrightarrow{AP}\right) = (\lambda\,.\vec{r}^2) \times \vec{r} - (\vec{r}^2) \times \overrightarrow{AP}$$

$$\vec{r} \wedge \vec{U}(A) + (\lambda\,.\vec{r}^2) \times \vec{r} - (\vec{r}^2) \times \overrightarrow{AP} = \vec{0}$$

$$\vec{r} \wedge \vec{U}(A) + (\lambda\,.\vec{r}^2) \times \vec{r} - (\vec{r}^2) \times \left(\overrightarrow{AH} + \overrightarrow{HP}\right) = \vec{0}$$

$$\vec{r} \wedge \vec{U}(A) + (\lambda\,.\vec{r}^2) \times \vec{r} - (\vec{r}^2 \times \ \overrightarrow{AH} + \vec{r}^2 \times \overrightarrow{HP}) = \vec{0}$$

$$\vec{r} \wedge \vec{U}(A) + (\lambda\,.\vec{r}^2) \times \vec{r} - (\vec{r}^2 \times \ \overrightarrow{AH} + \vec{r}^2 \times \lambda\,.\vec{r}) = \vec{0}$$

$$\vec{r} \wedge \vec{U}(A) - \vec{r}^2 \times \ \overrightarrow{AH} = \vec{0}$$

$$\overrightarrow{AH} = \frac{\vec{r} \wedge \vec{U}(A)}{\vec{r}^2}$$

$$\overrightarrow{AP} = \overrightarrow{AH} + \overrightarrow{HP} = \frac{\vec{r} \wedge \vec{U}(A)}{\vec{r}^2} + \overrightarrow{HP} \quad with \quad \overrightarrow{HP} = \lambda\,.\vec{r}$$

Therefore the formula which defines the central axis is written in the form:

$$\overrightarrow{AP} = \frac{\vec{r} \wedge \vec{U}(A)}{\vec{r}^2} + \lambda\,.\vec{r} \qquad\qquad ,\lambda \in \mathbb{R}$$

With P is the central point of the torsor.

Definition: *The central axis of a torso is the line for which the resultant vector and the moment vector are collinear. This central axis is defined as soon as the resulting vector is not zero.*

The most remarkable graphic representation of a torso is that built around the central axis:

- The central axis is oriented by the resulting vector.
- All points on the central axis have the same moment, called central moment.
- The central moment is collinear with the resultant.
- The moment vector is invariant along any line parallel to the central axis.
- Two planes perpendicular to the central axis admit the same distribution of moment vectors. For a plane perpendicular to the central axis:
    - A moment vector at any point $\boldsymbol{Q}$ is equal to the central moment increased by a component orthogonal to the resultant and proportional to the distance from the central axis to the point considered.
    - For any point on a circle centered on the central axis, the additional component has the same module and is orthogonal to the radius.

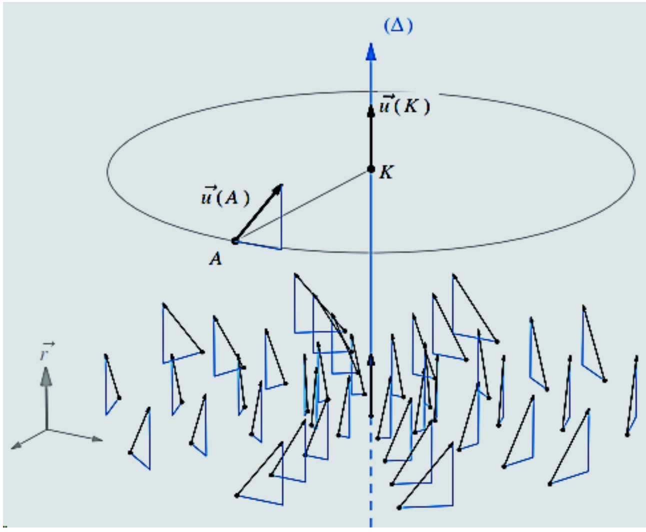


Figure I. 8 : The distribution of moment vectors around the central axis.

I.1.4.4 Particular torsors

a) ***Null Torsor***

A null torsor is a zero vector field. Its reduction elements are the zero vectors and we denote it simply $\vec{0}$. $T = \begin{cases} \vec{0} \\ \vec{0} \end{cases}$.

b) ***Couple torsor***

A couple torsor is a torsor with uniform vector field not null and null resultant. $T = \begin{cases} \vec{0} \\ \vec{C} \end{cases}$

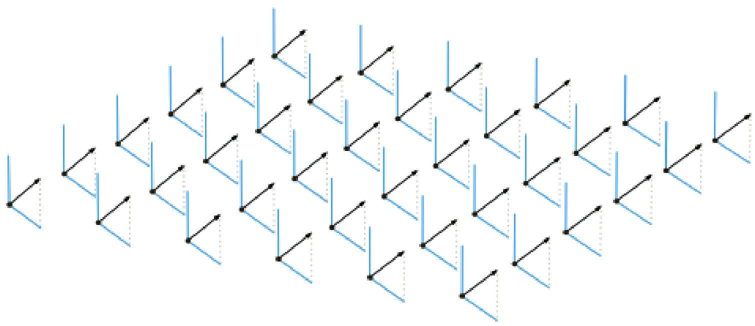


Figure I. 9 : A non-zero uniform vector field.

c) ***Sliding torsor***

A sliding torsor is a non-zero vector field with automoment null. The moment is zero at any point on the central axis. $T = \begin{cases} \vec{r} \\ \vec{0} \end{cases}$

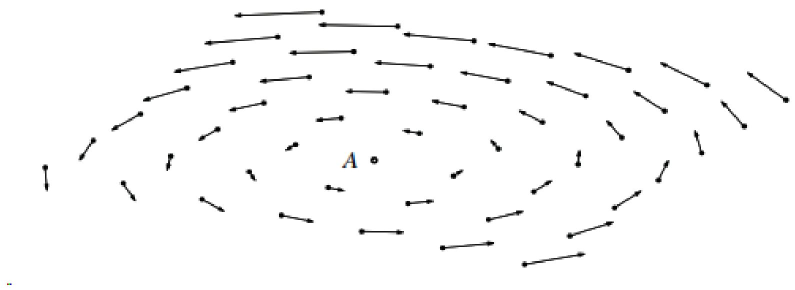


Figure I. 10 : A non-zero vector field with automoment equal to zero.

I.1.4.5 Application exercises on the torsors

Exercice I.3

Consider the three vectors $\vec{V_1} = -\vec{\imath} + \vec{\jmath} + \vec{k}$, $\vec{V_2} = \vec{\jmath} + 2\vec{k}$, $\vec{V_3} = \vec{\imath} - \vec{\jmath}$, defined in an orthonormal coordinate system $R(O, \vec{\imath}, \vec{\jmath}, \vec{k})$, and linked respectively to the points : A(0,1,2) , B(1,0,2) , C(1,2,2).

1- Construct the torsor $[\vec{T}]_0$, associated with the system of three vectors.
2- Deduce the automoment
3- Calculate the pitch of the torso
4- Determine the central axis of the torsor.

Solution I.3

1- Construction of the torsor $[\vec{T}]_0$

The torsor T $= \begin{cases} \vec{R} \\ \vec{M} \end{cases}$

- The resultant $\vec{R}$ of the torsor :

$$\vec{R} = \sum \vec{V_i} = \vec{V_1} + \vec{V_2} + \vec{V_3} = (= -\vec{\imath} + \vec{\jmath} + \vec{k}) + (\vec{\jmath} + 2\vec{k}) + (\vec{\imath} - \vec{\jmath})$$

$$\vec{R} = \vec{\jmath} + 3.\vec{k}$$

- The moment $\vec{M}$:

$$\vec{M} = \overrightarrow{OA} \wedge \vec{V_1} + \overrightarrow{OB} \wedge \vec{V_2} + \overrightarrow{OC} \wedge \vec{V_3}$$

$$\vec{M} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \wedge \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \wedge \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \wedge \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix}$$

$$\vec{M} = -\vec{\imath} - 2.\vec{\jmath} - \vec{k}$$

2- Automoment **A**

$$A = \vec{R}.\vec{M} = (\vec{\jmath} + 3.\vec{k}).(-\vec{\imath} - 2.\vec{\jmath} - \vec{k}) = -5$$

3- Pitch of the torsor

$$P = \frac{\vec{R}.\vec{M}}{R^2} = \frac{A}{R^2} = \frac{-5}{\sqrt{1^2 + 3^2}} = -\frac{5}{\sqrt{10}}$$

4- Central axes of the torsor

$$\overrightarrow{OP} = \frac{\vec{R}\wedge\vec{M}}{R^2} + \lambda\,.\overrightarrow{R} \qquad , \lambda \in \mathbb{R}$$

We have : $\vec{R} \wedge \vec{M} = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} \wedge \begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix} = 5.\vec{\imath} - 3.\vec{\jmath} + \vec{k}$

$$\overrightarrow{OP} = \frac{1}{2}.\vec{\imath} - \frac{3}{10}.\vec{\jmath} + \frac{1}{10}.\vec{k} + \lambda.\left(\vec{\jmath} + 3.\vec{k}\right)$$

$$\overrightarrow{OP} = \frac{1}{2}.\vec{\imath} + (\lambda - \frac{3}{10}).\vec{\jmath} + (3\lambda + \frac{1}{10}).\vec{k}$$

$\overrightarrow{OP} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \lambda - \frac{3}{10} \\ 3\lambda + \frac{1}{10} \end{pmatrix}$, with $\lambda = y + \frac{3}{10}$, so : $z = 3\left(y + \frac{3}{10}\right) + \frac{1}{10}$

We obtain: $z = 3.y + 1$

The central axis is straight in a plan parallel to the plane (yoz) located at:

$$x = \frac{1}{2} \text{ and } z = 3.y + 1$$

Exercice I.2

Consider the torsor $\left[\overrightarrow{T_1}\right]_0$ be defined by the three vectors:

$$\overrightarrow{V_1} = -2\vec{\imath} + 3\vec{\jmath} - 7\vec{k}\,, \qquad \overrightarrow{V_2} = 3\vec{\imath} - \vec{\jmath} - \vec{k}\,, \qquad \overrightarrow{V_3} = -\vec{\imath} - 2\vec{\jmath} + 8\vec{k}$$

Defined in an orthonormal coordinate system $R(O,\vec{\imath},\vec{\jmath},\vec{k})$, and linked respectively to the points : A(1,0,0) , B(0,1,0) , C(0,0,1).

Either the Torsor: $\left[\overrightarrow{T_2}\right]_0 = \begin{cases} \overrightarrow{R_2} \\ \overrightarrow{M_{20}} \end{cases}$ with $\overrightarrow{R_2} = 2\vec{\imath} + \vec{\jmath} + 3\vec{k}$ and $\overrightarrow{M}_{20} = -3\vec{\imath} + 2\vec{\jmath} - 7\vec{k}$

1- Determine the reduction elements of the torsor $\left[\overrightarrow{T_1}\right]_0$
2- Calculate the pitch and the central axis of the torsor $\left[\overrightarrow{T_2}\right]_0$
3- Calculate the sum and product of the two torsors
4- Calculate the auto-moment of the sum torsor $\left[\overrightarrow{T}\right]_0 = \left[\overrightarrow{T_1}\right]_0 + \left[\overrightarrow{T_2}\right]_0$.

I.2 Definitions and assumptions

I.2.1 Machines

A machine is a mechanism or group of mechanisms used to perform useful work. A machine transmits forces.

The term “Machine” should not be confused with “mechanism” even though in actuality, both may refer to the same device. The difference in terminology is related primarily to function. Whereas the function of a machine is to transmit energy, that of a mechanism is to transmit motion. Stated in other words, the term “mechanism” applies to the geometric arrangement that imparts definite motions to parts of a machine.

Figure I.11 illustrates an adjustable height platform that is driven by hydraulic cylinders. Although the entire device could be called a machine, the parts that take the power from the cylinders and drive the raising and lowering of the platform comprise the mechanism.

Figure I. 11 : Adjustable height platform.

I.2.2 A mechanism

A mechanism is a combination of rigid bodies so connected that the motion of one will produce a definite and predictable motion of others, according to a physical law. Alternatively, a mechanism is considered to be a kinematic chain in which one of the rigid bodies is fixed. An example of a mechanism is the slider crank shown in Figure I.12. Instruments, watches, and governors provide other examples of mechanisms.

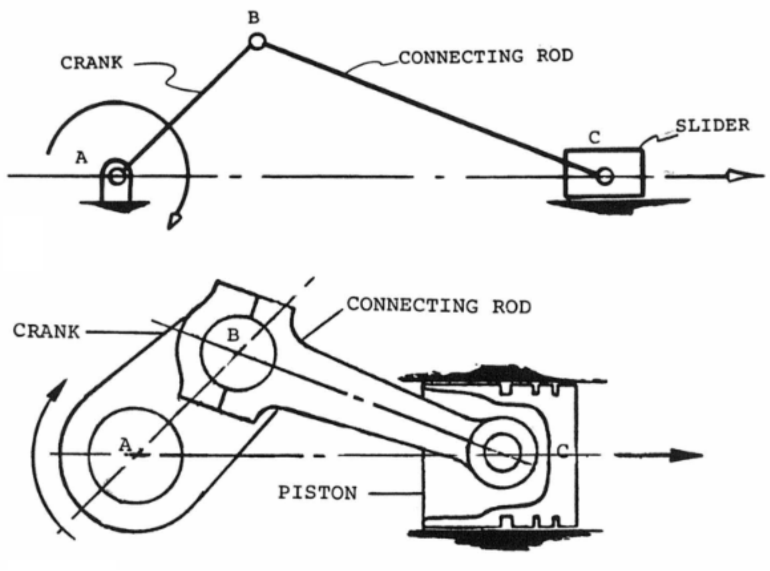


Figure I. 12 : Slider-crank mechanism.

I.2.3 Linkage

A linkage is a mechanism where rigid parts are connected together to form a chain. One part is designated the *frame* because it serves as the frame of reference for the motion of all other parts. The *frame* is typically a part that exhibits no motion. For example the chain saw, the mechanism takes power from a small engine and delivers it to the cutting edge of the chain.

The purpose of the mechanism in Figure I.12 is to lift the platform and any objects that are placed upon it. Synthesis is the process of developing a mechanism to satisfy a set of performance requirements for the machine. Analysis ensures that the mechanism will exhibit motion that will accomplish the set of requirements.

I.2.4 Links

Are the individual parts of the mechanism (Figures I.13 and I.14). They are considered rigid bodies and are connected with other links to transmit motion and forces. Theoretically, a true rigid body does not change shape during motion. Although a true rigid body does not exist, mechanism links are designed to minimally deform and are considered rigid.

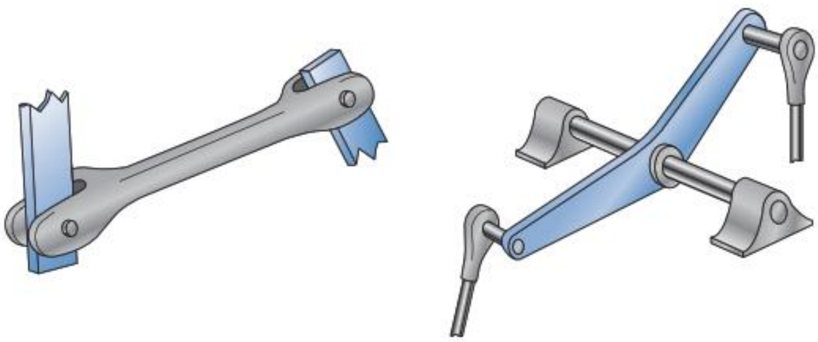
Elastic parts, such as springs, are not rigid and, therefore, are not considered links. They have no effect on the kinematics of a mechanism and are usually ignored during kinematic analysis. They do supply forces and must be included during the dynamic force portion of analysis.

a) ***Simple link***

Is a rigid body that contains only two joints, which connect it to other links (Figure I.13). Crank link is considered as a simple link that is able to complete. Also, rocker is a simple link that oscillates through an angle, reversing its direction at certain intervals.

b) ***Complex link***

Is a rigid body that contains more than two joints. Figure I.13 illustrates the simple and complex link. A crank is a simple link that is able to complete.



(a) Simple link (b) Complex link

Figure I. 13: Simple and complex link

- **Rocker arm** is a complex link, containing three joints, that is pivoted near its center.
- **Bellcrank** link is similar to a rocker arm, but is bent in the center (Figure I.13-b).

I.2.5 Joint

Is a movable connection between links and allows relative motion between the links. The two primary joints, also called full joints, are the revolute and sliding joints.

The revolute joint is also called a ***pin*** or hinge joint. It allows pure rotation between the two links that it connects.

The sliding joint is also called a piston or prismatic joint. It allows linear sliding between the links that it connects. Figure I.14 illustrates these two primary joints.

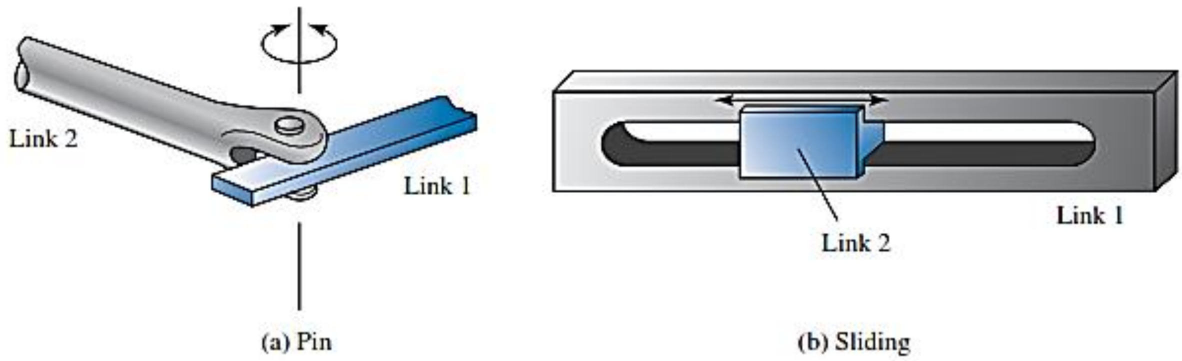


Figure I. 14 : Individual parts of the mechanism.

*A **cam joint*** is shown in Figure I.15-a. It allows for both rotation and sliding between the two links that it connects. Because of the complex motion permitted, the cam connection is called a higher-order joint, also called half joint.

*A **gear connection*** also allows rotation and sliding between two gears as their teeth mesh. This arrangement is shown in Figure I.15-b. The gear connection is also a higher-order joint.

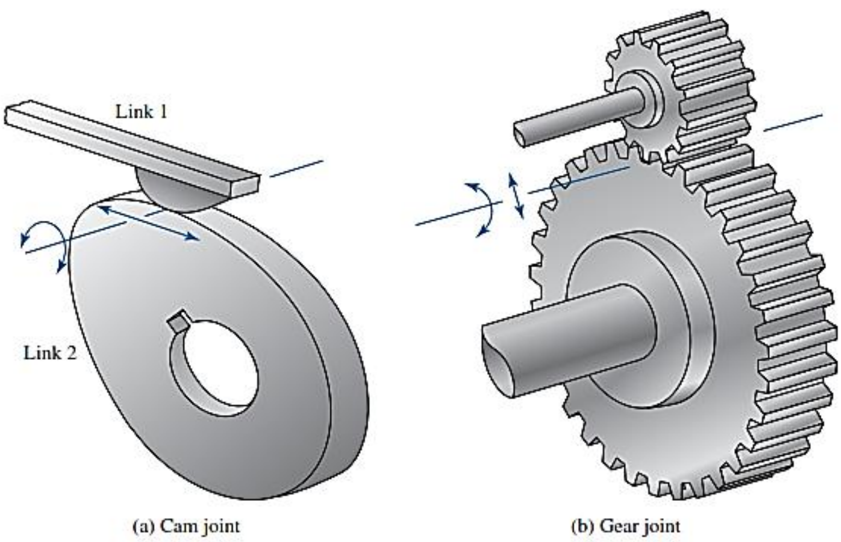


Figure I. 15: Higher-order joint (a) Cam joint, (b) Gear joint.

I.2.6 Planar mechanism

A mechanism is considered a planar mechanism if all the trajectories of the points of the moving elements remain in parallel planes during the movement (Figure I.16).

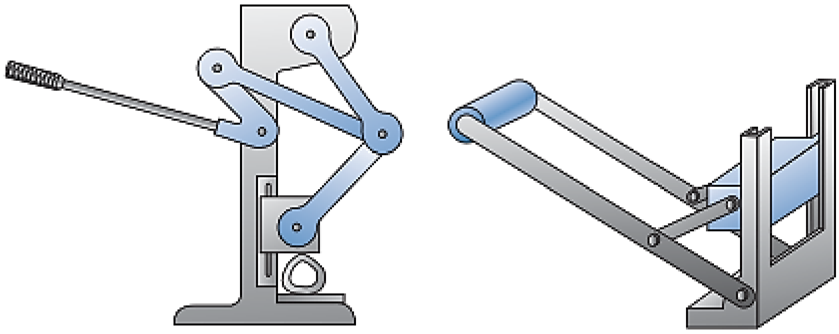


Figure I. 16 : Example for a planar mechanism.

I.2.7 Spherical mechanisms

Spherical mechanisms are pin-jointed spatial linkages which are used to move an object along a three-dimensional path in space. A spherical mechanism differs from a planar mechanism in the orientation of its axes of revolution. In a planar mechanism all the axes of the pin joints are parallel. In a spherical mechanism, however, all of the axes of revolution intersect at a single point. This constrains each of the pin joints to lie on concentric spheres about this point.

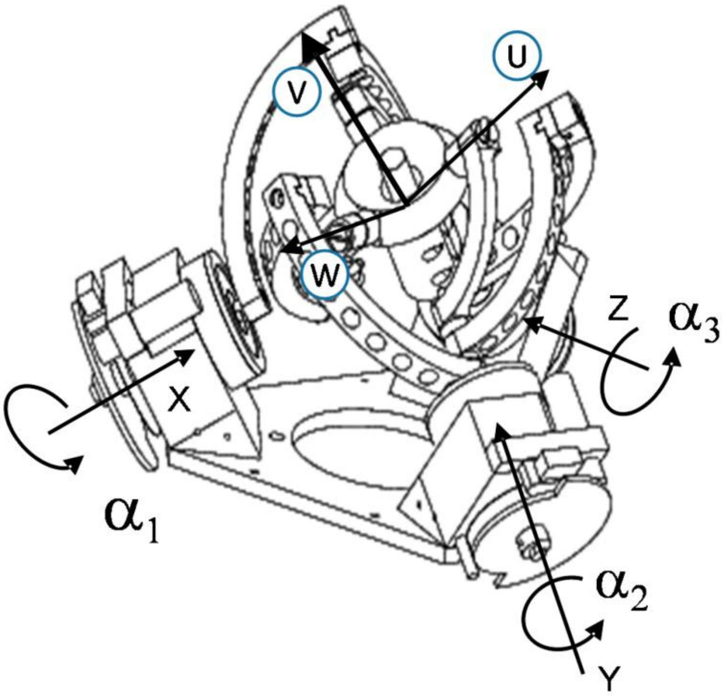


Figure I. 17: Example of the spherical mechanism

I.2.8 Spatial mechanisms

Spatial mechanisms have no restrictions on the movements of their constituent elements. Spatial mechanisms are essentially three-dimensional mechanisms such that one or more members of such a mechanism move in a plane different to others.

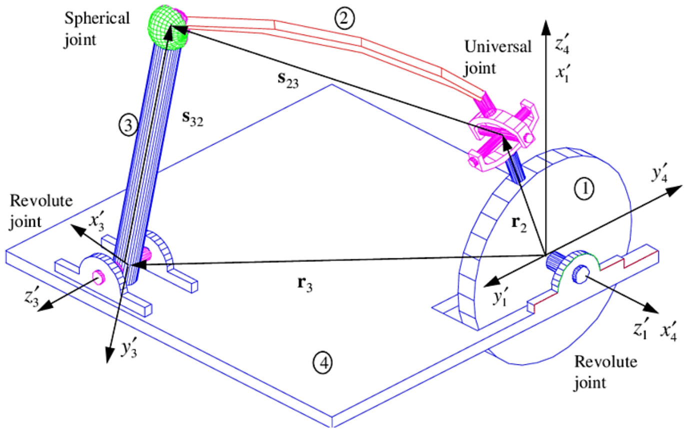


Figure I. 18 : Example for special mechanism