2

The Damped Harmonic Oscillator

In our description of an apple swinging back and forth at the end of a string (Section 1.1) we noted that this oscillating system is not ideal. After we set the apple in motion, the amplitude of oscillation steadily reduces and the apple eventually comes to rest. This is because there are dissipative forces acting and the system steadily loses energy. For example, the apple will experience a frictional force as it moves through the air. The motion is damped and such damped oscillations are the subject of this chapter. All real oscillating systems are subject to damping forces and will cease to oscillate if energy is not fed back into them. Often these damping forces are linearly proportional to velocity. Fortunately, this linear dependence leads to an equation of motion that can be readily solved to obtain solutions that describe the motion for various degrees of damping. Clearly the rate at which the oscillator loses energy will depend on the degree of damping and this is described by the *quality* of the oscillator. At first sight, damping in an oscillator may be thought undesirable. However, there are many examples where a controlled amount of damping is used to quench unwanted oscillations. Damping is added to the suspension system of a car to stop it bouncing up and down long after it has passed over a bump in the road. Additional damping was installed on London's Millennium Bridge shortly after it opened because it suffered from undesirable oscillations.

2.1 PHYSICAL CHARACTERISTICS OF THE DAMPED HARMONIC OSCILLATOR

A tuning fork is an example of a damped harmonic oscillator. Indeed we hear the note because some of the energy of oscillation is converted into sound. After it is struck the intensity of the sound, which is proportional to the energy of the tuning fork, steadily decreases. However, the frequency of the note does not change. The

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ends of the tuning fork make thousands of oscillations before the sound disappears and so we can reasonably assume that the degree of damping is small. We may suspect, therefore, that the frequency of oscillation would not be very different if there were no damping. Thus we infer that the displacement x of an end of the tuning fork is described by a relationship of the form

$$x = (\text{amplitude that reduces with } t) \times \cos \omega t$$

where the angular frequency ω is about but not necessarily the same as would be obtained if there were no damping. We shall assume that the amplitude of oscillation decays exponentially with time. The displacement of an end of the tuning fork will therefore vary according to

$$x = A_0 \exp(-\beta t) \cos \omega t \tag{2.1}$$

where A_0 is the initial value of the amplitude and β is a measure of the degree of damping. The minus sign indicates that the amplitude reduces with time. As we shall see, this expression correctly describes the motion of a damped harmonic oscillator when the degree of damping is small and so the assumptions we have made above are reasonable.

2.2 THE EQUATION OF MOTION FOR A DAMPED HARMONIC OSCILLATOR

An example of a damped harmonic oscillator is shown in Figure 2.1. It is similar to the simple harmonic oscillator described in Section (1.2.2) but now the mass is immersed in a viscous fluid. When an object moves through a viscous fluid it experiences a frictional force. This force dampens the motion: the higher the velocity the greater the frictional force. So as a car travels faster the frictional force increases thereby reducing the fuel economy, while the velocity of a falling raindrop reaches a limiting value because of the frictional force. The damping force F_d acting on the mass in Figure 2.1 is proportional to its velocity v so long as v is not too large, i.e.

$$F_d = -bv \tag{2.2}$$

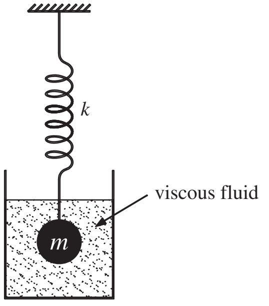

Figure 2.1 An example of a damped mechanical oscillator showing an oscillating mass immersed in a viscous fluid.

where the minus sign indicates that the force always acts in the opposite direction to the motion. The constant b depends on the shape of the mass and the viscosity of the fluid and has the units of force per unit velocity. When the mass is displaced from its equilibrium position there will be the restoring force due to the spring and in addition the damping force $-bv$ due to the fluid. The resulting equation of motion is

$$ma = -kx - bv \tag{2.3a}$$

or

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + b\frac{\mathrm{d}x}{\mathrm{d}t} + kx = 0. \tag{2.3b}$$

We introduce the parameters

$$\omega_o^2 = k/m, \quad \gamma = b/m. \tag{2.4}$$

In terms of these, Equation (2.3b) becomes

$$\boxed{\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \gamma\frac{\mathrm{d}x}{\mathrm{d}t} + \omega_o^2 x = 0.} \tag{2.5}$$

This is the equation of a damped harmonic oscillator. The relationship $k/m = \omega_o^2$ is familiar from our discussion of the simple harmonic oscillator. Now we designate this angular frequency ω_o and describe it as the *natural frequency of oscillation*, i.e. the oscillation frequency if there were no damping. This allows the possibility that the damping does change the frequency of oscillation. In the present example the damping force is linearly proportional to velocity. This linear dependence is very convenient as it has led to an equation that we can readily solve. A damping force proportional to, say, v^2 would be much more difficult to handle. Fortunately, this linear dependence is a good approximation for many other oscillating systems when the velocity is small. Equation (2.5) has different solutions depending on the degree of damping involved, corresponding to the cases of (i) *light damping*, (ii) *heavy* or *over damping* and (iii) *critical damping*. Light damping is the most important case for us because it involves oscillatory motion whereas the other two cases do not.

2.2.1 Light damping

This condition corresponds to the mass in Figure 2.1 being immersed in a fluid of low viscosity like thin oil or even just air. In our previous, qualitative discussion of a lightly damped oscillator, Section 2.1, we suggested an expression for the displacement that had the form $x = A_0 \exp(-\beta t)\cos\omega t$. We adopt a similar functional form here. Then

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -A_0 \exp(-\beta t)(\omega \sin\omega t + \beta\cos\omega t)$$

and

$$\frac{d^2x}{dt^2} = A_0 \exp(-\beta t)[2\beta\omega \sin \omega t + (\beta^2 - \omega^2) \cos \omega t].$$

Substituting these into Equation (2.5) and collecting terms in $\sin \omega t$ and $\cos \omega t$ gives

$$A_0 \exp(-\beta t)[(2\beta\omega - \gamma\omega) \sin \omega t + (\beta^2 - \omega^2 - \gamma\beta + \omega_o^2) \cos \omega t] = 0.$$

This can only be true for all times if the $\sin \omega t$ and $\cos \omega t$ terms are both equal to zero. Therefore,

$$2\beta\omega - \gamma\omega = 0$$

giving $\beta = \gamma/2$ and

$$\beta^2 - \omega^2 - \gamma\beta + \omega_o^2 = 0.$$

Substituting for β we obtain

$$\omega^2 = \omega_o^2 - \gamma^2/4. \tag{2.6}$$

So our solution for the equation of the lightly damped oscillator is

$$\boxed{x = A_0 \exp(-\gamma t/2) \cos \omega t} \tag{2.7}$$

where $\omega = (\omega_o^2 - \gamma^2/4)^{1/2}$. Equation (2.7) represents oscillatory motion if ω is real, i.e. $\gamma^2/4 < \omega_o^2$ is the condition for light damping. Equation (2.6) shows that the angular frequency of oscillation ω is approximately equal to the undamped value ω_o when $\gamma^2/4 \ll \omega_o^2$. To obtain the general solution of Equation (2.5) we need to include a phase angle ϕ giving

$$x = A_0 \exp(-\gamma t/2) \cos(\omega t + \phi). \tag{2.8}$$

The parameters γ and ω are determined solely by the properties of the oscillator while the constants A_0 and ϕ are determined by the initial conditions. For convenience in our following discussion we will take $\phi = 0$. If we let $\gamma = 0$ we obtain, as expected, our previous results for the simple harmonic oscillator.

A graph of $x = A_0 \exp(-\gamma t/2) \cos \omega t$ is shown in Figure 2.2 where the steady decrease in the amplitude of oscillation is apparent. The dotted lines represent the $\exp(-\gamma t/2)$ term which forms an *envelope* for the oscillations. The zeros in x occur when $\cos \omega t$ is zero and so are separated by π/ω. Therefore the period of the oscillation T, equal to twice this separation, is $2\pi/\omega$. Successive maxima are also separated by T. We consider successive maxima A_n and A_{n+1}. If A_n occurs at time t_o then

$$A_n = x(t_o) = A_0 \exp(-\gamma t_o/2) \cos \omega t_o$$

and

$$A_{n+1} = x(t_o + T) = A_0 \exp[-\gamma(t_o + T)/2] \cos \omega(t_o + T).$$

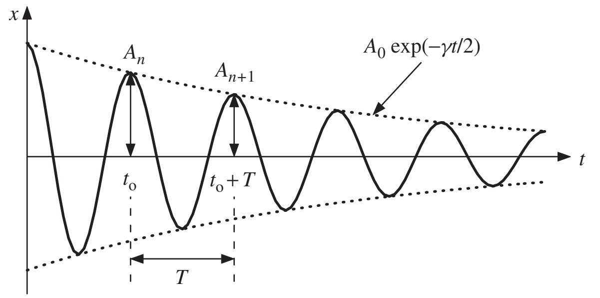

Figure 2.2 A graph of $x = A_0 \exp(-\gamma t/2) \cos \omega t$ illustrating the decay in amplitude of a damped harmonic oscillator. The dotted lines represent the $\exp(-\gamma t/2)$ term of Equation (2.8), which forms an *envelope* of the oscillations.

Since $\cos \omega t_o = \cos \omega (t_o + T)$ we have

$$\frac{A_n}{A_{n+1}} = \exp\left(\frac{\gamma T}{2}\right). \tag{2.9}$$

We see that successive maxima decrease by the same fractional amount. The natural logarithm of A_n/A_{n+1}, i.e.

$$\ln\left(\frac{A_n}{A_{n+1}}\right) = \frac{\gamma T}{2},$$

is called the *logarithmic decrement* and is a measure of this decrease. Note that the larger amplitude occurs in the numerator of this expression.

2.2.2 Heavy damping

Heavy damping occurs when the degree of damping is sufficiently large that the system returns sluggishly to its equilibrium position without making any oscillations at all. This corresponds to the mass in Figure 2.1 being immersed in a fluid of large viscosity like syrup. For this case the oscillatory part of our solution, $\cos \omega t$ in Equation (2.1), is no longer appropriate. Instead we replace it with the general function $f(t)$, i.e.

$$x = \exp(-\beta t) f(t). \tag{2.10}$$

Substituting x and its derivatives into Equation (2.5) and letting $\beta = \gamma/2$ gives

$$\frac{d^2 f}{dt^2} + (\omega_o^2 - \gamma^2/4) f = 0 \tag{2.11}$$

or

$$\frac{d^2 f}{dt^2} = \alpha^2 f \tag{2.12}$$

where $\alpha^2 = (\gamma^2/4 - \omega_0^2)$. The solutions to Equation (2.12) depend dramatically on the sign of α^2. The α^2 term is negative when $\gamma^2/4 < \omega_0^2$ and this leads to an oscillatory solution with the complex form $f(t) = A \exp i(\alpha t + \phi)$. This solution is not appropriate for the case of heavy damping where there is no oscillation. In fact it corresponds to the case of light damping, discussed in Section 2.2.1. The α^2 term is positive when $\gamma^2/4 > \omega_0^2$. In this case Equation (2.12) has the general solution

$$f(t) = A \exp(\alpha t) + B \exp(-\alpha t),$$

giving

$$\begin{aligned} x &= \exp(-\gamma t/2)[A \exp(\alpha t) + B \exp(-\alpha t)] \\ &= A \exp[-\gamma/2 + (\gamma^2/4 - \omega_0^2)^{1/2}]t + B \exp[-\gamma/2 - (\gamma^2/4 - \omega_0^2)^{1/2}]t. \quad (2.13) \end{aligned}$$

This is the non-oscillatory solution that we require. The term $(\gamma^2/4 - \omega_0^2)^{1/2}$ is clearly less than $\gamma/2$ and so the exponents of both exponential terms are negative in sign. Hence the displacement reduces to zero with time and there is no oscillation.

2.2.3 Critical damping

An interesting situation occurs when $\gamma^2/4 = \omega_0^2$. Then Equation (2.12) becomes

$$\frac{\mathrm{d}^2 f}{\mathrm{d}t^2} = 0. \quad (2.14)$$

This equation has the general solution

$$f = A + Bt, \quad (2.15)$$

leading to

$$x = A \exp(-\gamma t/2) + Bt \exp(-\gamma t/2) \quad (2.16)$$

where A has the dimension of length and B has the dimensions of velocity. This is the case of critical damping. Here the mass returns to its equilibrium position in the shortest possible time without oscillating. Critical damping has many important practical applications. For example, a spring may be fitted to a door to return it to its closed position after it has been opened. In practice, however, critical damping is applied to the spring mechanism so that the door returns quickly to its closed position without oscillating. Similarly, critical damping is applied to analogue meters for electrical measurements. This ensures that the needle of the meter moves smoothly to its final position without oscillating or overshooting so that a rapid reading can be taken. Springs are used in motor cars to provide a smooth ride. However damping is also applied in the form of shock absorbers as illustrated schematically in Figure 2.3. Without these the car would continue to bounce up and down long after it went over a bump in the road. A shock

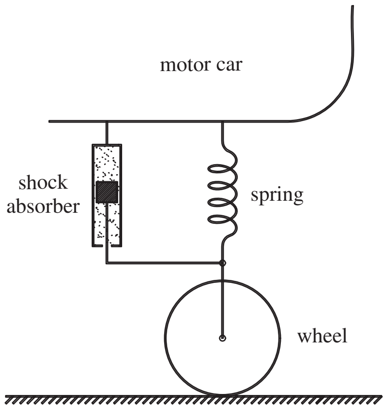

Figure 2.3 Schematic diagram of a car suspension system showing the spring and shock absorber.

absorber consists essentially of a piston that moves in a cylinder containing a viscous fluid. Holes in the piston allow it to move up and down in a damped manner and the damping constant is adjusted so that the suspension system is close to the condition of critical damping. You can see the effect of a shock absorber by pushing down on the front of a car, just above a wheel. The car quickly returns to equilibrium with little or no oscillation. You may also notice that the resistance is greater when you push down quickly than when you push down slowly. This reflects the dependence of the damping force on velocity.

In summary we find three types of damped motion and these are illustrated in Figure 2.4. They correspond to the conditions:

(i) $(\gamma^2/4 < \omega_0^2)$ Light damping; damped oscillations.

(ii) $(\gamma^2/4 > \omega_0^2)$ Heavy damping; exponential decay of displacement.

(iii) $(\gamma^2/4 = \omega_0^2)$ Critical damping; quickest return to equilibrium position without oscillation.

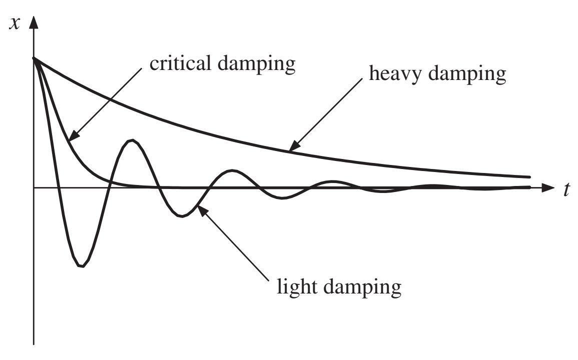

Figure 2.4 The motion of a damped oscillator for the cases of light damping, heavy damping and critical damping.

To appreciate the physical origin of these different types of motion, we recall that $\gamma^2/4$ is the damping term while ω_o^2 is proportional to the spring constant k through $\omega_o^2 = k/m$. When the damping term is small compared with k/m, the motion is governed by the restoring force of the spring and we have damped oscillatory motion. Conversely, when the damping term is large compared with k/m the damping force dominates and there is no oscillation. At the point of critical damping the two forces balance. We finally note that the relative size of $\gamma^2/4$ compared with ω_o^2 also determines the response of the oscillator to an applied periodic driving force, as we shall see in Chapter 3.

Worked example

A mass of 2.5 kg is attached to a spring that has a value of k equal to 600 N m^{-1}. (a) Determine the value of the damping constant b that is required to produce critical damping. (b) The mass receives an impulse that gives it a velocity of $v_i = 1.5$ m s^{-1} at $t = 0$. Determine the maximum value of the resultant displacement and the time at which this occurs.

Solution

(a) For critical damping, $\gamma^2/4 = b^2/4m^2 = \omega_o^2 = k/m$. Therefore,

$$b = \sqrt{4mk} = \sqrt{4 \times 2.5 \times 600} = 77.5 \text{ kg s}^{-1}.$$

(b) General solution for critical damping is

$$x = A \exp(-\gamma t/2) + Bt \exp(-\gamma t/2).$$

Therefore

$$v = \frac{\mathrm{d}x}{\mathrm{d}t} = \exp(-\gamma t/2)(B - \gamma Bt/2 - \gamma A/2).$$

Initial conditions, $x = 0$ and $v = v_i$ at $t = 0$, give $A = 0$ and $B = v_i$. Therefore,

$$x(t) = v_i t \exp(-\gamma t/2).$$

Maximum displacement occurs when $\mathrm{d}x/\mathrm{d}t = 0$, giving

$$v_i \exp(-\gamma t/2)(1 - \gamma t/2) = 0.$$

Hence

$$t = \frac{2}{\gamma} = \frac{2m}{b} = \frac{2 \times 2.5}{77.5} = 6.5 \times 10^{-2} \text{ s}$$

and

$$x = \frac{2v_i}{\mathrm{e}\gamma} = \frac{2mv_i}{\mathrm{e}b} = \frac{2 \times 2.5 \times 1.5}{\mathrm{e} \times 77.5} = 3.6 \times 10^{-2} \text{ m}.$$

TABLE 2.1 Typical values of Q for a variety of damped oscillators.

Damped oscillator system	Typical value of Q
Paper weight suspended on a rubber band	10
Clock pendulum	75
Electrical *LCR* circuit	200
Plucked violin string	10^3
Microwave cavity oscillator	10^4
Quartz crystal	10^6

Typical values of Q for a variety of damped oscillators are presented in Table 2.1.

2.4 DAMPED ELECTRICAL OSCILLATIONS

In our mechanical example of a mass moving through a fluid we saw that the fluid offered a resistance that damped the motion. In the case of an electrical oscillator it is the resistance in the circuit that impedes the flow of current. An electrical oscillator is shown in Figure 2.8. It consists of an inductor L and capacitor C

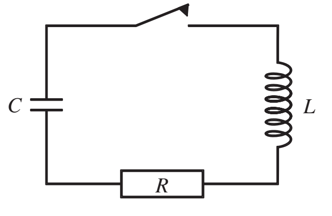

Figure 2.8 The circuit of a damped electrical oscillator consisting of an inductor L, a capacitor C and a resistor R connected in series.

as before (see Figure 1.21) but now there is also the resistor R. We charge the capacitor to voltage $V_C = q/C$, and then close the switch. Kirchoff's law gives

$$L\frac{\mathrm{d}I}{\mathrm{d}t} + RI + \frac{q}{C} = 0$$

or

$$L\frac{\mathrm{d}^2q}{\mathrm{d}t^2} + R\frac{\mathrm{d}q}{\mathrm{d}t} + \frac{q}{C} = 0. \tag{2.26}$$

This has the identical form to Equation (2.3b),

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + b\frac{\mathrm{d}x}{\mathrm{d}t} + kx = 0, \tag{2.3b}$$

and we recognise the analogous quantities we met before: q is equivalent to x, L to m and k to $1/C$. However, we see that R is analogous to the mechanical damping constant b and so R/L is the equivalent of γ $(= b/m)$. Since the above differential equations have identical forms, their solutions also have identical forms. The importance of this is that we can use our results for the mechanical oscillator to immediately write down the corresponding results for the electrical case. Thus from Equation (2.7) it follows that

$$q = q_0 \exp(-Rt/2L) \cos[(1/LC - R^2/4L^2)^{1/2} t] \tag{2.27}$$

where q_0 is the initial charge on the capacitor. This corresponds to the case of light damping which now means that $R^2/4L^2 < 1/LC$. Since the voltage V_C across the capacitor is equal to q/C

$$V_C = V_0 \exp(-Rt/2L) \cos[(1/LC - R^2/4L^2)^{1/2} t] \tag{2.28}$$

where V_0 is the initial value of the voltage. This is an oscillating voltage at an angular frequency ω given by

$$\omega^2 = \frac{1}{LC} - \frac{R^2}{4L^2} \tag{2.29}$$

which is essentially equal to $1/LC$ when $R^2/4L^2 \ll 1/LC$. The amplitude of the oscillating voltage decays exponentially with a time constant of $R/2L$ and so R/L has the dimensions of $[\text{time}]^{-1}$. For $R^2/4L^2 > 1/LC$ we have the case of heavy damping and for $R^2/4L^2 = 1/LC$ we have critical damping. Similarly we find that the quality factor Q of the circuit is given by

$$Q = \frac{1}{R}\sqrt{\frac{L}{C}}. \tag{2.30}$$

For example, with $L = 10$ mH, $C = 2.5$ nF and $R = 10\,\Omega$, $Q = 200$, which is a typical value for an electrical oscillator. Again we emphasise the exact correspondence between the equations and solutions that describe the mechanical and electrical systems, so that mechanical systems can be simulated by electrical circuits. Such analogue computers can greatly facilitate the design and development of mechanical systems.

PROBLEMS 2

2.1 A spring balance consists of a pan that hangs from a spring. A damping force $F_{\rm d} = -bv$ is applied to the balance so that when an object is placed in the pan it comes to rest in the minimum time without overshoot. Determine the required value of b for an object of mass 2.5 kg that extends the spring by 6.0 cm. (Assume $g = 9.81$ m s^{-2}.)

2.2 A mass of 0.50 kg hangs from the end of a light spring. The system is damped by a light sail attached to the mass so that the ratio of amplitudes of consecutive oscillations is equal to 0.90. It is found that 10 complete oscillations takes 25 s. Obtain a quantitative expression for the damping force and determine the damping factor γ of the system.