1.2.2 A mass on a vertical spring

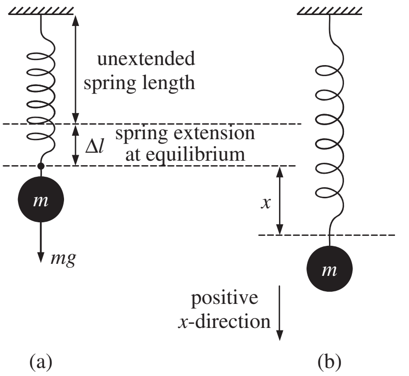

Figure 1.4 An oscillating mass on a vertical spring. (a) The mass at its equilibrium position. (b) The mass displaced by a distance x from its equilibrium position.

If we suspend a mass from a vertical spring, as shown in Figure 1.4, we have gravity also acting on the mass. When the mass is initially attached to the spring, the length of the spring increases by an amount Δl. Taking displacements in the downward direction as positive, the resultant force on the mass is equal to the gravitational force minus the force exerted upwards by the spring, i.e. the resultant force is given by $mg - k\Delta l$. The resultant force is equal to zero when the mass is at its equilibrium position. Hence

$$k\Delta l = mg.$$

When the mass is displaced downwards by an amount x, the resultant force is given by

$$F = m\frac{d^2x}{dt^2} = mg - k(\Delta l + x) = mg - k\Delta l - kx$$

i.e.

$$m\frac{d^2x}{dt^2} = -kx. \tag{1.8}$$

Perhaps not surprisingly, this result is identical to the equation of motion (1.5) of the horizontal spring: we simply need to measure displacements from the equilibrium position of the mass.

1.2.3 Displacement, velocity and acceleration in simple harmonic motion

To describe the harmonic oscillator, we need expressions for the displacement, velocity and acceleration as functions of time: $x(t)$, $v(t)$ and $a(t)$. These can be obtained by solving Equation (1.6) using standard mathematical methods. However, we will use our physical intuition to deduce them from the observed behaviour of a mass on a spring.

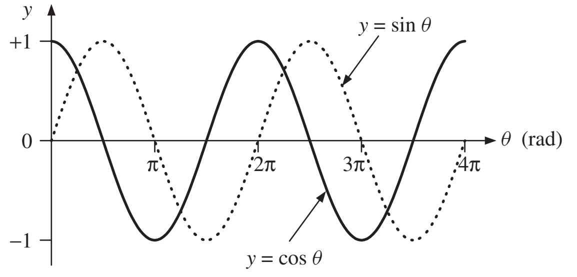

Figure 1.5 The functions $y = \cos\theta$ and $y = \sin\theta$ plotted over two complete cycles.

Observing the periodic motion shown in Figure 1.2, we look for a function $x(t)$ that also repeats periodically. Periodic functions that are familiar to us are $\sin\theta$ and $\cos\theta$. These are reproduced in Figure 1.5 over two complete cycles. Both functions repeat every time the angle θ changes by 2π. We can notice that the two functions are identical except for a shift of $\pi/2$ along the θ axis. We also note the initial condition that the displacement x of the mass equals A at $t = 0$. Comparison of the actual motion with the mathematical functions in Figure 1.5 suggests the choice of a cosine function for $x(t)$. We write it as

$$x = A\cos\left(\frac{2\pi t}{T}\right) \tag{1.9}$$

which has the correct form in that $(2\pi t/T)$ is an angle (in radians) that goes from 0 to 2π as t goes from 0 to T, and so repeats with the correct period. Moreover x equals A at $t = 0$ which matches the initial condition. We also require that $x = A\cos(2\pi t/T)$ is a solution to our differential equation (1.6). We define

$$\omega = \frac{2\pi}{T} \tag{1.10}$$

where ω is the *angular frequency* of the oscillator, with units of rad s^{-1}, to obtain

$$x = A\cos\omega t. \tag{1.11}$$

Then

$$\frac{dx}{dt} = v = -\omega A\sin\omega t, \tag{1.12}$$

and

$$\frac{d^2x}{dt^2} = a = -\omega^2 A\cos\omega t = -\omega^2 x. \tag{1.13}$$

So, the function $x = A\cos\omega t$ is a solution of Equation (1.6) and correctly describes the physical situation. The reason for writing the constant as ω^2 in Equation (1.6) is now apparent: the constant is equal to the square of the angular frequency of oscillation. We have also obtained expressions for the velocity v and acceleration a of the mass as functions of time. All three functions are plotted in Figure 1.6. Since they relate to different physical quantities, namely displacement, velocity and acceleration, they are plotted on separate sets of axes, although the time axes are aligned with respect to each other.

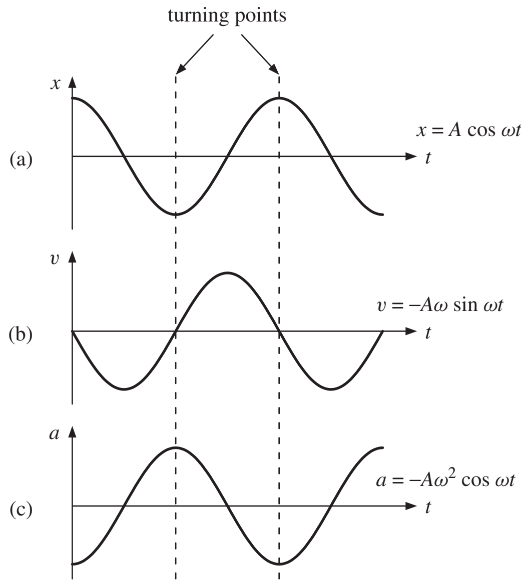

Figure 1.6 (a) The displacement x, (b) the velocity v and (c) the acceleration a of a mass undergoing SHM as a function of time t. The time axes of the three graphs are aligned.

Figure 1.6 shows that the behaviour of the three functions (1.11)–(1.13) agree with our observations. For example, when the displacement of the mass is greatest, which occurs at the *turning points* of the motion ($x = \pm A$), the velocity is zero. However, the velocity is at a maximum when the mass passes through its equilibrium position, i.e. $x = 0$. Looked at in a different way, we can see that the maximum in the velocity curve occurs before the maximum in the displacement curve by one quarter of a period which corresponds to an angle of $\pi/2$. We can understand at which points the maxima and minima of the acceleration occur by recalling that acceleration is directly proportional to the force. The force is maximum at the turning points of the motion but is of opposite sign to the displacement. The acceleration does indeed follow this same pattern, as is readily seen in Figure 1.6.

1.2.4 General solutions for simple harmonic motion and the phase angle ϕ

In the example above, we had the particular situation where the mass was released from rest with an initial displacement A, i.e. x equals A at $t = 0$. For the more

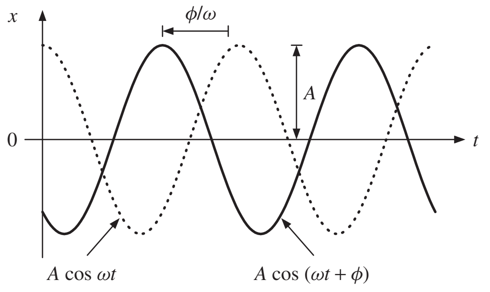

Figure 1.7 General solution for displacement x in SHM showing the phase angle ϕ, where $x = A\cos(\omega t + \phi)$.

general case, the motion of the oscillator will give rise to a displacement curve like that shown by the solid curve in Figure 1.7, where the displacement and velocity of the mass have arbitrary values at $t = 0$. This solid curve looks like the cosine function $x = A\cos\omega t$, that is shown by the dotted curve, but it is displaced horizontally to the left of it by a time interval $\phi/\omega = \phi T/2\pi$. The solid curve is described by

$$\boxed{x = A\cos(\omega t + \phi)} \tag{1.14}$$

where again A is the amplitude of the oscillation and ϕ is called the *phase angle* which has units of radians. [Note that changing ωt to $(\omega t - \phi)$ would shift the curve to the right in Figure 1.7.] Equation (1.14) is also a solution of the equation of motion of the mass, Equation (1.6), as the reader can readily verify. In fact Equation (1.14) is the *general solution* of Equation (1.6). We can state here a property of second-order differential equations that they always contain two arbitrary constants. In this case A and ϕ are the two constants which are determined from the initial conditions, i.e. from the position and velocity of the mass at time $t = 0$.

We can cast the general solution, Equation (1.14), in the alternative form

$$x = a\cos\omega t + b\sin\omega t, \tag{1.15}$$

where a and b are now the two constants. Equations (1.14) and (1.15) are entirely equivalent as we can show in the following way. Since

$$A\cos(\omega t + \phi) = A\cos\omega t\cos\phi - A\sin\omega t\sin\phi \tag{1.16}$$

and $\cos\phi$ and $\sin\phi$ have constant values, we can rewrite the right-hand side of this equation as

$$a\cos\omega t + b\sin\omega t,$$

where

$$a = A\cos\phi \text{ and } b = -A\sin\phi. \tag{1.17}$$

We see that if we add sine and cosine curves of the *same* angular frequency ω, we obtain another cosine (or corresponding sine curve) of angular frequency ω.

This is illustrated in Figure 1.8 where we plot $A\cos\omega t$ and $A\sin\omega t$, and also $(A\cos\omega t + A\sin\omega t)$ which is equal to $\sqrt{2}A\cos(\omega t - \pi/4)$. As the motion of a simple harmonic oscillator is described by sines and cosines it is called harmonic and because there is only a single frequency involved, it is called simple harmonic.

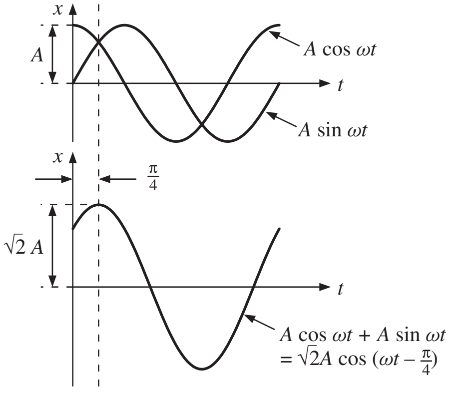

Figure 1.8 The addition of sine and cosine curves with the same angular frequency ω. The resultant curve also has angular frequency ω.

There is an important difference between the constants A and ϕ in the general solution for SHM given in Equation (1.14) and the angular frequency ω. The constants are determined by the initial conditions of the motion. However, the angular frequency of oscillation ω is determined only by the properties of the oscillator: the oscillator has a *natural frequency of oscillation* that is independent of the way in which we start the motion. This is reflected in the fact that the SHM equation, Equation (1.6), already contains ω which therefore has nothing to do with any particular solutions of the equation. This has important practical applications. It means, for example, that the period of a pendulum clock is independent of the amplitude of the pendulum so that it keeps time to a high degree of accuracy.[1] It means that the pitch of a note from a piano does not depend on how hard you strike the keys. For the example of the mass on a spring, $\omega = \sqrt{k/m}$. This expression tells us that the angular frequency becomes lower as the mass increases and becomes higher as the spring constant increases.

Worked example

In the example of a mass on a horizontal spring (cf. Figure 1.1) m has a value of 0.80 kg and the spring constant k is 180 N m^{-1}. At time $t = 0$ the mass is observed to be 0.04 m further from the wall than the equilibrium position and is moving away from the wall with a velocity of 0.50 m s^{-1}. Obtain an

[1] This assumes that the pendulum is operating as an ideal harmonic oscillator which is a good approximation for oscillations of small amplitude.