University of Biskra
Mathematics Department
Module: Analysis 1

First year license
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Exam

Exercise 1 ***(./06pts)*** Let z_1 and z_2 be the complex solutions of the following equation:

$$z^2 - 2z + 2 = 0.$$

1. Find the values of z_1 and z_2. **(1.5pts)**
2. Rewrite z_1 and z_2 in their exponential form, $(z = re^{i\theta})$ **(1.5pts)**
3. Solve, in $\mathbb{C}$, the following equation: **(3pts)**

$$z^4 - 2z^2 + 2 = 0.$$

Exercise 2 ***(./07pts)***

1. Show that **(1pt)**

$$\forall\ a > 1 \text{ and } b > 1 \text{ we have } a + b < 1 + ab.$$

2. Let (U_n) be a real sequence defined by:

$$\begin{cases} u_0 = 3 \\ \forall n \in \mathbb{N} : u_{n+1} = \frac{1+\lambda u_n}{\lambda + u_n}, \quad \text{with } \lambda > 1. \end{cases}$$

   (a) Show that, $\forall n \in \mathbb{N} : u_n > 1$. **(2pts)**
   (b) Check that, $\forall n \in \mathbb{N} : u_{n+1} - u_n < 0$. **(1pt)**
   (c) Deduce that (U_n) is a convergent **(1.5pts)**
   sequence, then determine its limit. **(1.5pts)**

Exercise 3 ***(./07pts)*** Let f and g be two real functions defined by:

$$f(x) = \frac{e^x - 1}{x} - x^2 \qquad \text{and} \qquad g(x) = \frac{(e^x - 1)\left(\cos(x) - \frac{1}{2}\right)}{\sin(x)}.$$

1. Give the domain of f and g.**(1.5pts)**
2. Compute the limit of f and g at $x = 0$ (Without using Hopital's Rule).**(1.5pts)**
3. Do f and g have a removable discontinuity at $x = 0$?**(1pt)**
4. Check that the equation $g(x) = 0$ admits at least one solution on $\left[\frac{\pi}{4}; \frac{\pi}{2}\right]$.**(1.5pts)**
5. Check that f admits at least one extremum on $]0; 1[$.**(1.5pts)**

Solution of the Exam

Solution of the Exercise 1

Let z_1 and z_2 be the complex solutions of the following equation:

$$z^2 - 2z + 2 = 0.$$

1. we have

$$\Delta = 2^2 - 4 * 1 * 2 = -4 \Longrightarrow \sqrt{\Delta} = \pm 2i,$$ 0,1

so,

0,1 $$z_1 = \frac{2-2i}{2} = 1 - i \quad \text{and} \quad z_2 = \frac{2+2i}{2} = 1 + i.$$ 0,1

2. Rewrite z_1 and z_2 in their exponential form, ($z = re^{i\theta}$, with r is the module of z and θ is its argument). We have

$$|z_1| = |z_2| = \sqrt{1^2 + (\pm 1)^2} = \sqrt{2}$$ 0,1

then

$$\begin{cases} z_1 &= \sqrt{2}\left(\frac{\sqrt{2}}{2} - i\,\frac{\sqrt{2}}{2}\right) = \sqrt{2}\left(\cos\left(\frac{-\pi}{4}\right) + i \sin\left(\frac{-\pi}{4}\right)\right) = \sqrt{2}\, e^{\frac{-i\pi}{4}}. \\ z_2 &= \sqrt{2}\left(\frac{\sqrt{2}}{2} + i\,\frac{\sqrt{2}}{2}\right) = \sqrt{2}\left(\cos\left(\frac{\pi}{4}\right) + i \sin\left(\frac{\pi}{4}\right)\right) = \sqrt{2}\, e^{\frac{i\pi}{4}}. \end{cases}$$ 0,1 0,1

3. Deduce all the solutions of the following equation:

$$z^4 - 2z^2 + 2 = 0.$$

To solve this above equation let's put $z^2 = w$. The equation can be written as follows:

$$w^2 - 2w + 2 = 0.$$

From the first and second questions it's clear that the solutions of the new equation are:

$$w_1 = \frac{2-2i}{2} = 1 - i = \sqrt{2}\, e^{\frac{-i\pi}{4}} \quad \text{and} \quad w_2 = \frac{2+2i}{2} = 1 + i = \sqrt{2}\, e^{\frac{i\pi}{4}}.$$ 0,1

Since $z^2 = w$ then:

$$\begin{cases} |z^2| &= |w| \\ \arg(z^2) &= \arg(w) + 2k\pi,\ k \in \mathbb{Z}. \end{cases} \Longrightarrow \begin{cases} |z| &= \sqrt{|w|} \\ \arg(z) &= \frac{\arg(w)}{2} + k\pi,\ k \in \mathbb{Z}. \end{cases}$$ 0,1

Consequently, for w_1 and w_2 we get, respectively:

$$\begin{cases} |z| &= \sqrt{|w_1|} = \sqrt[4]{2} \\ \arg(z) &= \frac{\arg(w_1)}{2} + k\pi = \frac{-i\pi}{8} + k\pi,\ k \in \mathbb{Z}. \end{cases} \Longrightarrow \begin{cases} z_1 &= \sqrt[4]{2} e^{\frac{-i\pi}{8}} \\ z_2 &= \sqrt[4]{2} e^{\frac{7i\pi}{8}} \end{cases}$$ 0,1 0,1

$$\begin{cases} |z| &= \sqrt{|w_2|} = \sqrt[4]{2} \\ \arg(z) &= \frac{\arg(w_2)}{2} + k\pi = \frac{i\pi}{8} + k\pi,\ k \in \mathbb{Z}. \end{cases} \Longrightarrow \begin{cases} z_3 &= \sqrt[4]{2} e^{\frac{i\pi}{8}} \\ z_4 &= \sqrt[4]{2} e^{\frac{9i\pi}{8}} \end{cases}$$ 0,1 0,1

Solution of the Exercise 2

1.
$$a+b-1-ab = a(1-b)-(1-b) = (1-b)(a-1) = -(a-1)(b-1) < 0.$$

as $a+b-1-ab<0$ then $a+b<1+ab$.

2. To show that the proposition "$\forall n \in \mathbb{N} : u_n > 1$" is correct, we use the proof by induction.

   (a) We have $u_0 = 3 > 1$, so the proposition is true for $n=0$.

   (b) assumes that $u_n > 1$ is true.

   (c) Let's show that the proposition is also true for $n+1$.
   We have $u_n > 1$ (from the assumption) and $\lambda > 1$ so, using the result of the first question, we deduce that $1+\lambda u_n > \lambda + u_n$ so $\frac{1+\lambda u_n}{\lambda+u_n} > 1 \Longrightarrow u_{n+1} > 1$.

   We conclude that $\forall n \in \mathbb{N} : u_n > 1$.

3. Check that, $\forall n \in \mathbb{N} : u_{n+1} - u_n < 0$.

$$\begin{aligned} u_{n+1} - u_n &= \frac{1+\lambda u_n}{\lambda+u_n} - u_n \\ &= \frac{(1+\lambda u_n) - (\lambda+u_n)u_n}{\lambda+u_n} \\ &= \frac{1-(u_n)^2}{\lambda+u_n}. \end{aligned}$$

Since $u_n > 1 \Longrightarrow 1-(u_n)^2 < 0 \Longrightarrow \frac{1-(u_n)^2}{\lambda+u_n} < 0 \Longrightarrow u_{n+1} - u_n < 0$. This result means that (u_n) is a decreasing sequence.

4. Deduce that (U_n) is a convergent sequence, then determine its limit.

As $\left\{ \begin{array}{l} (u_n) \text{ is lower bounded} \\ (u_n) \text{ is a decreasing sequence} \end{array} \right. \Longleftrightarrow (u_n)$ is a convergent sequence

Let's assume that $\lim\limits_{n\to+\infty} u_n = l$ then

$$\begin{aligned} \lim_{n\to+\infty} u_{n+1} = \lim_{n\to+\infty} \frac{1+\lambda u_n}{\lambda+u_n} &\Longrightarrow l = \frac{1+\lambda l}{\lambda+l} \\ &\Longrightarrow l\,\lambda + l^2 = 1 + l\,\lambda \Longrightarrow l^2 = 1 \\ &\Longrightarrow \left\{ \begin{array}{lcl} l &=& 1 \ \text{(accepted)} \\ l &=& -1 \ \text{(rejected because } u_n > 1) \end{array} \right. \end{aligned}$$

we conclude that

$$\lim_{n\to+\infty} u_n = 1.$$

Solution of the Exercise 3

1. Give the domain of f and g.

$$\begin{aligned} D_f &= \{x \in \mathbb{R} : x \neq 0\} = \mathbb{R}^* = \mathbb{R}/\{0\}. \\ D_g &= \{x \in \mathbb{R} : \sin(x) \neq 0\} = \{x \in \mathbb{R} : x \neq k\pi,\ k \in \mathbb{Z}\} = \mathbb{R}/\{k\pi,\ k \in \mathbb{Z}\}. \end{aligned}$$

2. Calculation of the limits

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{e^x - 1}{x} - x^2 = \lim_{x\to 0} \frac{e^x - e^0}{x - 0} - x^2 = e^x|'_{x=0}\, 0 = e^0 - 0 = 1.$$ 0,1

$$\begin{aligned}
\lim_{x\to 0} g(x) &= \lim_{x\to 0} \frac{(e^x - 1)\left(\cos(x) - \frac{1}{2}\right)}{\sin(x)} \\
&= \lim_{x\to 0} \left(\frac{e^x - 1}{x}\right)\left(\frac{x}{\sin(x)}\right)\left(\cos(x) - \frac{1}{2}\right) \\
&= \lim_{x\to 0} \left(\frac{e^x - 1}{x}\right) \times \lim_{x\to 0} \left(\frac{x}{\sin(x)}\right) \times \lim_{x\to 0} \left(\cos(x) - \frac{1}{2}\right) \\
&= 1 * 1 * \left(1 - \frac{1}{2}\right) = \frac{1}{2}.
\end{aligned}$$ 0,1 0,1

3. Note that proving that a function has a removable discontinuity at x_0, amounts to checking if the limits in the neighborhood (at the left and the right) of x_0 are the same.

   From the second question we have

   (a) $\lim_{x\to 0} f(x) = 1$ so we can remove the discontinuity of f at $x = 0$ and we write:

   $$\tilde{f}(x) = \begin{cases} f(x), & \text{if } x \in D_f; \\ 1, & \text{if } x = 0. \end{cases}$$ 0,1

   (b) $\lim_{x\to 0} g(x) = \frac{1}{2}$ so we can remove the discontinuity of g at $x = 0$ and we write:

   $$\tilde{g}(x) = \begin{cases} g(x), & \text{if } x \in D_g; \\ \frac{1}{2}, & \text{if } x = 0. \end{cases}$$ 0,1

4. Check that g admit at least one solution on $\left[\frac{\pi}{4}; \frac{\pi}{2}\right]$. To check that g admit at least one solution we can use the intermediate values theorem.

   We have:

   (a) the function g is continuous on the interval $\left[\frac{\pi}{4}; \frac{\pi}{2}\right]$ 0,1

   (b) $g\left(\frac{\pi}{4}\right) = \frac{\left(e^{\frac{\pi}{4}}-1\right)\left(\cos(\frac{\pi}{4})-\frac{1}{2}\right)}{\sin(\frac{\pi}{4})} > 0$

   (because $\left(e^{\frac{\pi}{4}} - 1\right) > 0$, $\left(\cos(\frac{\pi}{4}) - \frac{1}{2}\right) = \frac{\sqrt{2}-1}{2} > 0$ and $\sin(\frac{\pi}{4}) = \frac{\sqrt{2}}{2} > 0$.

   (c) $g\left(\frac{\pi}{2}\right) = \frac{\left(e^{\frac{\pi}{2}}-1\right)\left(\cos(\frac{\pi}{2})-\frac{1}{2}\right)}{\sin(\frac{\pi}{2})} < 0$ 0,1

   (because $\left(e^{\frac{\pi}{2}} - 1\right) > 0$, $\left(\cos(\frac{\pi}{2}) - \frac{1}{2}\right) = \frac{0-1}{2} < 0$ and $\sin(\frac{\pi}{2}) = 1 > 0$

   From these three results, we conclude that $\exists c \in \left[\frac{\pi}{4}; \frac{\pi}{2}\right]$ such that $g(c) = 0$. 0,1

5. At first glance, it seems that we must use Roll's theorem to check the existence of an extremum in the interval $[0, 1]$. Unfortunately, this theorem cannot be used because $f(0^+) \neq f(1)$. Indeed, from the first question $f(0^+) = \frac{1}{2}$ and $f(1) = e^1 - 2 \neq \frac{1}{2}$.

To show the existence of the extremum we can use the intermediate values theorem. Indeed, note that, f admit an extremum on $]0,1[$ means that there exists $c \in]0,1[$ such that $f'(c) = 0$.

we have on the one hand

$$f'(x) = \frac{(x-1)e^x + 1}{x^2} - 2x.$$

0,5

on other hand

(a) As f' is defined and continuous on $]0,1[$. 0,5

(b) $\lim\limits_{x\to 1^-} f(x) = f'(1) = -1 < 0.$

(c) $\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+} \frac{(x-1)e^x+1}{x^2} - \lim\limits_{x\to 0^+} 2x = \lim\limits_{x\to 0^+} \frac{(x-1)e^x+e^x}{2x} = \frac{1}{2} > 0.$

From these three above points we conclude that exists $c \in]0,1[$ such that $f'(c) = 0$. consequently the existence of an extremum on $]0,1[$.

0,5