University of Biskra
Mathematics Department
Module: Analysis 1

First year license
2023/2024

Solution of the Worksheet N° 4

Solution of the Exercise 1

1. Prove that the derivative of an even differentiable function is odd, and the derivative of an odd differentiable function is even.

Case of f is an even function: We know that $f(g(x))' = g'(x)f'(g(x))$ and if f is an even function then $f(-x) = f(x)$. Using these two notions, we can show that:
$[f(x)]' = [f(-x)]' = (-x)'f'(-x) = -f'(-x) \Longrightarrow f'(-x) = -f'(x)$, so $f'(x)$ is an odd function.

Case of f is an odd function We know that $f(g(x))' = g'(x)f'(g(x))$ and if f is an odd function then $f(-x) = -f(x)$. Using these two notions, we can show that:
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2. What about the nth derivative of an even and an odd function?

Using the above results and the definition that $f^{(n)} = \left[f^{(n-1)}\right]'$ then with the same reasoning as the first derivative, we will have the following:

$$f \text{ is an even function then } \Longleftrightarrow \begin{cases} f^{(n)}, \text{ is an even function,} & \text{if } n \text{ is an even number;} \\ f^{(n)}, \text{ isan odd function,} & \text{if } n \text{ is an odd number;} \end{cases}$$

$$f \text{ is an odd function then } \Longleftrightarrow \begin{cases} f^{(n)} \text{ is an even function,} & \text{if } n \text{ is an odd number;} \\ f^{(n)} \text{ is an odd function,} & \text{if } n \text{ is an even number;} \end{cases}$$

Solution of the Exercise 2

Before answering the exercise, let's recall that:

$$\cos^2(x) + \sin^2(x) = 1.$$

$$\begin{cases} \cos(-x) &= \cos(x) & \text{is an even function} \\ \sin(-x) &= -\sin(x) & \text{is an odd function} \end{cases} \qquad \begin{cases} \cos(x+2k\pi) &= \cos(x) \\ \sin(x+2k\pi) &= \sin(x) \end{cases} \quad \forall k \in \mathbb{Z}.$$

$$\begin{cases} \cos(x+y) &= \cos(x)\cos(y) - \sin(x)\sin(y) \\ \sin(x+y) &= \sin(x)\cos(y) + \sin(y)\cos(x) \end{cases} \Longrightarrow \begin{cases} \cos(2x) &= \cos^2(x) - \sin^2(x) \\ \sin(2x) &= 2\sin(x)\cos(y) \end{cases}$$

1. We have f is defined by:

$$f(x) = \left(\frac{\sin(2x)}{2\sqrt{1-\cos(x)}}\right)^m, \quad m \in \mathbb{N}^*,$$

then,

$$D_f = \{x \in \mathbb{R} : \; 1 - \cos(x) > 0\} = \{x \in \mathbb{R} : \; \cos(x) < 1\}.$$

as $-1 \leq \cos(x) \leq 1$ and $\cos(x) = 1 \Longrightarrow x = 2k\pi, \; k \in \mathbb{Z}$, then

$$D_f = \mathbb{R}/\{2k\pi, \; k \in \mathbb{Z}\} = \bigcup_{k \in \mathbb{Z}}]2k\pi, 2(k+1)\pi[.$$

2. Discussion of the parity (even or odd) of f according to the values of the parameter m.

$$\begin{aligned} f(-x) &= \left(\frac{\sin(-2x)}{2\sqrt{1-\cos(-x)}}\right)^m = (-1)^m \left(\frac{\sin(2x)}{2\sqrt{1-\cos(x)}}\right)^m \\ &= \begin{cases} -f(x), & \text{if } m \text{ is an odd number} \\ f(x), & \text{if } m \text{ is an even number} \end{cases} \\ &\Longrightarrow \begin{cases} f \text{ is an odd function,} & \text{if } m \text{ is an odd number;} \\ f \text{ is an even function,} & \text{if } m \text{ is an even number.} \end{cases} \end{aligned}$$

3. To show that f is 2π-periodic function means that we must check if $f(x+2\pi) = f(x)$.

   We have

$$f(x+2\pi) = \left(\frac{\sin(2(x+2\pi))}{2\sqrt{1-\cos(x+2\pi)}}\right)^m = \left(\frac{\sin(2x+4\pi)}{2\sqrt{1-\cos(x+2\pi)}}\right)^m = \left(\frac{\sin(2x)}{2\sqrt{1-\cos(x)}}\right)^m = f(x).$$

   So, f is a 2π-periodic function.

4. From response of the question 3, we conclude that the limit of f.

   - to the left of all the upper bounds of the subintervals $]2k\pi, 2(k+1)\pi[,\ k \in \mathbb{Z}$ that constituting the domain of f are the same. Consequently, the limit equal to

$$\lim_{x \to (2(k+1)\pi)^-} f(x) = \lim_{x \to 0^-} f(x).$$

   - to the left of all the upper bounds of the subintervals $]2k\pi, 2(k+1)\pi[,\ k \in \mathbb{Z}$ that constituting the domain of f are the same. Consequently, the limit equal to

$$\lim_{x \to (2k\pi)^+} f(x) = \lim_{x \to 0^+} f(x)$$

$$\begin{aligned} \lim_{x \to 0^-} f(x) &= \lim_{x \to 0^-} \left(\frac{\sin(2x)}{2\sqrt{1-\cos(x)}}\right)^m = \lim_{x \to 0^-} \left(\frac{\cos(x)\sin(x)\sqrt{1+\cos(x)}}{\sqrt{1-\cos^2(x)}}\right)^m \\ &= \lim_{x \to 0^-} \left(\frac{\cos(x)\sin(x)\sqrt{1+\cos(x)}}{|\sin(x)|}\right)^m = \lim_{x \to 0^-} (-1)^m \left(\cos(x)\sqrt{1+\cos(x)}\right)^m \\ &= \begin{cases} (\sqrt{2})^m & \text{if } m \text{ is an even number} \\ \text{does not exists} & \text{if } m \text{ is an odd number.} \end{cases} \end{aligned}$$

$$\begin{aligned} \lim_{x \to 0^+} f(x) &= \lim_{x \to 0^+} \left(\frac{\sin(2x)}{2\sqrt{1-\cos(x)}}\right)^m = \lim_{x \to 0^+} \left(\frac{\cos(x)\sin(x)\sqrt{1+\cos(x)}}{|\sin(x)|}\right)^m \\ &= \lim_{x \to 0^+} \left(\cos(x)\sqrt{1+\cos(x)}\right)^m = (\sqrt{2})^m \end{aligned}$$

   We conclude that, for any $k \in \mathbb{Z}$ we will have:

$$\begin{aligned} \lim_{x \to (2k\pi)^-} f(x) &= \begin{cases} (\sqrt{2})^m & \text{if } m \text{ is an even number} \\ \text{does not exists} & \text{if } m \text{ is an odd number.} \end{cases} \\ \lim_{x \to (2k\pi)^+} f(x) &= (\sqrt{2})^m. \end{aligned}$$

Solution of the Exercise 3

1. Show that the curves of the following functions are symmetrical with respect to a vertical axis $x = x_0$.

   (a) case of function f:
   We have $D_f = \mathbb{R}$. So for all x in D_f, $x - x_0$ and $x + x_0$ are in D_f. Thus, to show that the curve of the function f is symmetrical with respect to a vertical axis $x = x_0$, it remains to verify the existence of a real x_0 such that $f(x_0 - x) = f(x_0 + x)$. In other words, we must check the existence of the solution to the above equation with respect to x_0.

$$
\begin{aligned}
f(x_0 - x) = f(x_0 + x) &\Longrightarrow \sqrt{(x_0 - x - 1)^2 + 1} = \sqrt{(x_0 + x - 1)^2 + 1} \\
&\Longrightarrow (x_0 - x - 1)^2 = (x_0 + x - 1)^2 \\
&\Longrightarrow \begin{cases} x_0 - x - 1 = x_0 + x - 1 \\ x_0 - x = -x_0 - x + 1 \end{cases} \\
&\Longrightarrow \begin{cases} x = -x \text{ (impossible)} \\ x_0 = 1. \end{cases}
\end{aligned}
$$

   (b) Case of the function g:
   We have $D_g = \mathbb{R}$. So for all x in D_g, $x - x_0$ and $x + x_0$ are in D_g. Thus, to show that the curve of the function g is symmetrical with respect to a vertical axis $x = x_0$, it remains to verify the existence of a real x_0 such that $g(x_0 - x) = g(x_0 + x)$. In other words, we must check the existence of the solution to the above equation with respect to x_0.

$$
\begin{aligned}
g(x_0 - x) = g(x_0 + x) &\Longrightarrow (x_0 - x)^2 + 2(x_0 - x) + 4 = (x + x_0)^2 + 2(x + x_0) + 4 \\
&\Longrightarrow x^2 - 2xx_0 + x_0^2 + 2x_0 - 2x + 4 = x^2 + 2xx_0 + x_0^2 + 2x + 2x_0 + 4 \\
&\Longrightarrow -2xx_0 - 2x = 2xx_0 + 2x \\
&\Longrightarrow -4x(x_0 + 1) = 0 \\
&\Longrightarrow x_0 = -1.
\end{aligned}
$$

2. For each of the following functions, determine the point of symmetry of their graphs.

$$
f(x) = \frac{2x - 1}{x + 1}, \quad g(x) = \frac{x^2 - 1}{x - 2}
$$

   (a) To prove that the function f admits the point with coordinates (a, b) as a point of symmetry, we must check the following for all x of D_f:
      i. $a - x$ and $a + x$ belong to D_f,
      ii. $f(a - x) + f(a - x) = 2b$.

   Let's check the first condition.
   As the $D_f = \mathbb{R}/\{-1\}$ then $a - x$ and $a + x$ are in D_f only if $a = -1$.
   Now, checking the second condition (finding the value of b).

$$
\begin{aligned}
f(a - x) + f(a - x) = 2b &\Longrightarrow f(-1 - x) + f(-1 - x) = 2b \\
&\Longrightarrow \frac{2(-1 - x) - 1}{(-1 - x) + 1} + \frac{2(-1 + x) - 1}{(-1 + x) + 1} = 2b \\
&\Longrightarrow 4 = 2b \\
&\Longrightarrow b = 2.
\end{aligned}
$$

   Thus, the desired point is the one whose coordinates are $(-1, 2)$.

(b) case if function g.
As the $D_g = \mathbb{R}/\{2\}$ then $a - x$ and $a + x$ are in D_g only if $a = 2$.

$$\begin{aligned} g(a-x) + g(a-x) = 2b &\implies f(2-x) + f(2-x) = 2b \\ &\implies \frac{(2-x)^2 - 1}{(2-x) - 2} + \frac{(2+x)^2 - 1}{(2+x) + -2} = 2b \\ &\implies 4 = 2b \\ &\implies b = 2. \end{aligned}$$

Thus, the desired point is the one whose coordinates are $(2, 2)$.

3. Show that any function having the form

$$f(x) = \frac{ax + b}{x - c} \quad \text{with } a,\ b,\ c \in \mathbb{R}.$$

admits a point of symmetry.

Suppose that the coordinates of the point whose existence we want to prove are (α, β).
As the $D_f = \mathbb{R}/\{c\}$, then $\alpha - x$ and $\alpha + x$ are in D_f only if $\alpha = c$.

$$\begin{aligned} f(\alpha - x) + f(\alpha - x) = 2b &\implies f(c-x) + f(c-x) = 2b \\ &\implies \frac{a(c-x) + b}{(c-x) - c} + \frac{a(c+x) + b}{(c+x) - c} = 2b \\ &\implies \frac{2ax}{x} = 2b \\ &\implies \beta = a. \end{aligned}$$

Thus, the desired point is the one whose coordinates are (c, a).

4. Show that any function having the form

$$f(x) = \sqrt{(x-a)^2 + b}, \qquad g(x) = (x-a)^2 + b \qquad \text{with } a,\ b \in \mathbb{R}.$$

admits a vertical axe of symmetry

(a) Case of function f:
Suppose that the equation of the vertical line whose existence we want to prove is $x = \alpha$.
Note that

$$D_f = \begin{cases} \mathbb{R}, & \text{if } b \geq 0 \\ ]-\infty, -\sqrt{-b} + a[\cup], \sqrt{-b} + a, +\infty[, & \text{if } b < 0 \end{cases}$$

On the one hand, we have

$$\begin{aligned} f(\alpha - x) = f(\alpha + x) &\implies \sqrt{(\alpha - x + a)^2 + b} = \sqrt{(\alpha + x - a)^2 + b} \\ &\implies (\alpha - x + a)^2 = (\alpha + x - a)^2 \\ &\implies \begin{cases} \alpha - x + a &= \alpha + x - a \\ \alpha - x + a &= -\alpha - x + a \end{cases} \\ &\implies \begin{cases} x &= -x \text{ (rejected)} \\ \alpha &= a \end{cases} \end{aligned}$$

and on the other hand, for any x in D_f, $a-x$ and $a+x$ are also in D_g. So, the equation of the vertical line sought is indeed that $x=a$.

(b) Case of function g:
Suppose that the equation of the vertical line whose existence we want to prove is $x=\alpha$.
Note that $D_f=\mathbb{R}$.
On the one hand, we have

$$\begin{aligned} f(\alpha-x)=f(\alpha+x) &\Longrightarrow (\alpha-x+a)^2+b=(\alpha+x-a)^2+b \\ &\Longrightarrow \begin{cases} \alpha-x+a &= \alpha+x-a \\ \alpha-x+a &= -\alpha-x+a \end{cases} \\ &\Longrightarrow \begin{cases} x &= -x \text{ (rejected)} \\ \alpha &= a \end{cases} \end{aligned}$$

and on the other hand, for any x in D_f, $a-x$ and $a+x$ are also in D_g. Then, the equation of the vertical line sought is indeed that $x=a$.

Solution of the Exercise 4

In each of the following cases, determine the limit, if it exists:

- $\lim\limits_{x\to 4}\frac{x^2-7x+12}{x^2-16}=\lim\limits_{x\to 4}\frac{(x-4)(x-3)}{(x-4)(x+4)}=\lim\limits_{x\to 4}\frac{x-3}{x+4}=\frac{1}{8}$.
- $\lim\limits_{x\to 1}\left(\frac{1}{x^2-3x+2}-\frac{1}{x-1}\right)=\lim\limits_{x\to 1}\left(\frac{1}{(x-1)(x-2)}-\frac{1}{x-1}\right)=\lim\limits_{x\to 1}\frac{3-x}{(x-1)(x-2)}=\begin{cases}\frac{2}{0^+}=+\infty, & \text{if } x\longrightarrow 1^+\\ \frac{2}{0^-}=-\infty, & \text{if } x\longrightarrow 1^-\end{cases}$
- $\lim\limits_{x\to\frac{\pi}{2}}\frac{\sqrt[3]{\sin(x)}}{x-\frac{\pi}{2}}$
- $\lim\limits_{x\to 0}x\sin(\frac{1}{x})=?$

Case of $x\longrightarrow 0^-$	Case of $x\longrightarrow 0^+$
$-1\leq\sin\left(\frac{1}{x}\right)\leq 1$	$-1\leq\sin\left(\frac{1}{x}\right)\leq 1$
$-x\geq x\sin\left(\frac{1}{x}\right)\geq x$	$-x\leq x\sin\left(\frac{1}{x}\right)\leq x$
$\lim\limits_{x\to 0^-}-x\geq\lim\limits_{x\to 0^-}x\sin\left(\frac{1}{x}\right)\geq\lim\limits_{x\to 0^-}x$	$\lim\limits_{x\to 0^-}-x\leq\lim\limits_{x\to 0^-}x\sin\left(\frac{1}{x}\right)\leq\lim\limits_{x\to 0^-}x$
$0\geq\lim\limits_{x\to 0^-}x\sin\left(\frac{1}{x}\right)\geq 0$	$0\leq\lim\limits_{x\to 0^-}x\sin\left(\frac{1}{x}\right)\leq 0$

Form the above results we conclude that: $\lim\limits_{x\to 0}x\sin\left(\frac{1}{x}\right)=0$.

- $\lim\limits_{x\to+\infty}x\sin(\frac{1}{x})=\lim\limits_{x\to+\infty}\frac{\sin(\frac{1}{x})}{\frac{1}{x}}=\lim\limits_{y\to 0^+}\frac{\sin(y)}{y}=1$.
  with the same manner we can show
  $\lim\limits_{x\to-\infty}x\sin(\frac{1}{x})=\lim\limits_{x\to-\infty}\frac{\sin(\frac{1}{x})}{\frac{1}{x}}=\lim\limits_{y\to 0^-}\frac{\sin(y)}{y}=1$.
- $\lim\limits_{x\to 0}\frac{ln(1-\sin(x))}{x}=\lim\limits_{x\to 0}\left(\frac{-\sin(x)}{x}\right)\left(\frac{ln(1-\sin(x))}{-\sin(x)}\right)=-1*1=-1$.
- $\lim\limits_{x\to 0}\frac{\sin(ax)}{\sin(bx)}=\frac{a}{b}\lim\limits_{x\to 0}\frac{\frac{\sin(ax)}{ax}}{\frac{\sin(ax)}{bx}}=\frac{a}{b}\frac{\lim\limits_{x\to 0}\frac{\sin(ax)}{ax}}{\lim\limits_{x\to 0}\frac{\sin(ax)}{bx}}=\frac{a}{b}$

- $\lim\limits_{x\to+\infty} \frac{\sqrt{x^2-7}}{3x+5} = \lim\limits_{x\to+\infty} \frac{x\sqrt{1-\frac{7}{x^2}}}{x\left(3+\frac{5}{x}\right)} = \lim\limits_{x\to+\infty} \frac{\sqrt{1-\frac{7}{x^2}}}{3+\frac{5}{x}} = \frac{1}{3}.$

- $\lim\limits_{x\to-\infty} \frac{\sqrt{x^2-7}}{3x+5} = \lim\limits_{-x\to+\infty} \frac{-x\sqrt{1-\frac{7}{x^2}}}{x\left(3+\frac{5}{x}\right)} = \lim\limits_{x\to+\infty} \frac{-\sqrt{1-\frac{7}{x^2}}}{3+\frac{5}{x}} = \frac{-1}{3}.$

- $\lim\limits_{x\to+\infty} \sqrt{x^2+6x+1} - x = \lim\limits_{x\to+\infty} \frac{x^2+6x+1-x^2}{\sqrt{x^2+6x+1}+x} = \lim\limits_{x\to+\infty} \frac{6x(1+\frac{1}{6x})}{x\left(\sqrt{1+\frac{6}{x}+\frac{1}{x^2}}+\frac{1}{x}\right)} = 6$

- $\lim\limits_{x\to-\infty} \sqrt{x^2+6x+1} - x = +\infty + \infty = +\infty.$

- $\lim\limits_{x\to1} \frac{\sqrt{x^2-1}+\sqrt{x}-1}{\sqrt{x-1}} = \lim\limits_{x\to1} \frac{\sqrt{x-1}\sqrt{x+1}}{\sqrt{x-1}} + \frac{\sqrt{x}-1}{\sqrt{\sqrt{x}-1}\sqrt{\sqrt{x}+1}} = \lim\limits_{x\to1} \sqrt{x+1} + \frac{\sqrt{\sqrt{x}-1}}{\sqrt{\sqrt{x}+1}} = \sqrt{2}.$

- $\lim\limits_{x\to1} \frac{\sqrt{x}-1}{\sqrt[4]{x}-1} = \lim\limits_{x\to1} \frac{x^{\frac{1}{2}}-1}{x^{\frac{1}{4}}-1} = \lim\limits_{x\to1} \frac{\left(x^{\frac{1}{4}}\right)^2-1^2}{x^{\frac{1}{4}}-1} = \lim\limits_{x\to1} \frac{\left(x^{\frac{1}{4}}-1\right)\left(x^{\frac{1}{4}}+1\right)}{x^{\frac{1}{4}}-1} = \lim\limits_{x\to1} \left(x^{\frac{1}{4}}+1\right) = 2.$

- $\lim\limits_{x\to1} \frac{\sqrt[3]{x}-1}{\sqrt[4]{x}-1} = \lim\limits_{x\to1} \frac{x^{\frac{1}{3}}-1}{x^{\frac{1}{4}}-1} = ?.$

  Since the least common multiplier of 3 and 4 is 12 ($12 = LCM(3,4)$), we put $y^12 = x$. Note that as x tends toward 1, y also tends toward 1, and $(y^3-1) = (y-1)(y^2+y+1)$ and $(y^2)^2 - 1 = (y^2-1)(y^2+1) = (y-1)(y+1)(y^2+1)$. Therefore,

  $\lim\limits_{x\to1} \frac{x^{\frac{1}{3}}-1}{x^{\frac{1}{4}}-1} = \lim\limits_{y\to1} \frac{y^4-1}{y^3-1} = \lim\limits_{y\to1} \frac{(y-1)(y+1)(y^2+1)}{(y-1)(y^2+y+1)} = \lim\limits_{y\to1} \frac{(y+1)(y^2+1)}{(y^2+y+1)} = \frac{4}{3}.$

- $\lim\limits_{x\to0} (1+ax)^{1/x}$

  We have,

  $\lim\limits_{x\to0} \frac{\ln(1+ax)}{x} = \lim\limits_{x\to0} a\frac{\ln(1+ax)}{ax} = \lim\limits_{y\to0} a\frac{\ln(1+y)}{y} = a.$

  $\lim\limits_{x\to0} (1+ax)^{1/x} = \lim\limits_{x\to0} e^{\ln(1+ax)^{1/x}} = \lim\limits_{x\to0} e^{\frac{\ln(1+ax)}{x}} = e^{\lim\limits_{x\to0}\frac{\ln(1+ax)}{x}} = e^a.$

- $$\begin{aligned}\lim_{x\to\pm\infty} \left(\frac{x^2+x}{x^2+x+2}\right)^{x^2+x} &= \lim_{x\to\pm\infty} \left(\frac{x^2+x+2-2}{x^2+x+2}\right)^{x^2+x} = \lim_{x\to\pm\infty} \left(1-\frac{2}{x^2+x+2}\right)^{x^2+x+2}\left(1-\frac{2}{x^2+x+2}\right)^{-2} \\ &= \lim_{x\to\pm\infty} \left(1-\frac{2}{x^2+x+2}\right)^{x^2+x+2} \lim_{x\to\pm\infty} \left(1-\frac{2}{x^2+x+2}\right)^{-2} = ?\end{aligned}$$

  – The second limit is easy to be calculated $\lim\limits_{x\to\pm\infty} \left(1-\frac{2}{x^2+x+2}\right)^{-2} = 1.$

  – For the first limit if we put $y = \frac{1}{x^2+x+2}$ then we get:

  $$\lim_{x\to\pm\infty} \left(1-\frac{2}{x^2+x+2}\right)^{x^2+x+2} = \lim_{y\to0} (1-2y)^{\frac{1}{y}} = e^{-2}.$$

  Therefore,

  $$\lim_{x\to\pm\infty} \left(\frac{x^2+x}{x^2+x+2}\right)^{x^2+x} = e^{-2}.$$

- $\lim\limits_{x\to\pm\infty} P_n(x)e^{-x} \equiv \lim\limits_{x\to\pm\infty} e^{-x} = \begin{cases} 0, & \text{when } x\to+\infty \\ +\infty, & \text{when } x\to-\infty \end{cases}$

- $\lim\limits_{x\to\pm\infty}\frac{ln(P_n(x))}{x}\equiv\lim\limits_{x\to\pm\infty}\frac{1}{x}=0.$

Note: $a,b\in\mathbb{R}^*$, $n\in\mathbb{N}^*$ and $P_n(x)$ is a positive polynomial of degree n.

Solution of the Exercise 5

- Find all the possible values of the real constants a, b and c such that the following functions are continuous in their domains.

1. Case of the function f:

$$f(x)=\begin{cases}x^2+2x, & \text{if } x\geq 1;\\ -x+c, & \text{if } x<1.\end{cases}\iff f(x)=\begin{cases}x^2+2x, & \text{if } x>1;\\ 3, & \text{if } x=1;\\ -x+c, & \text{if } x<1.\end{cases}$$

The function f is composed of two continuous functions on $\mathbb{R}$, so if f has a continuity problem it will surely be at $x=1$ the point of the decomposition of f. Thus, for the function f to be continuous on R it is necessary that:

$$\lim_{x\to 1^-\infty}f(x)=\lim_{x\to 1^+}f(x)=f(1)$$

we have $f(1)=3$, $\lim\limits_{x\to 1^+}f(x)=\lim\limits_{x\to 1^+}x^2+2x=3$ and $\lim\limits_{x\to 1^-}f(x)=\lim\limits_{x\to 1^-}-x+c=c-1.$
Consequently, f is continuous on $\mathbb{R}$ if and only if $c-1=3\Longrightarrow c=4$

$$f(x)=\begin{cases}x^2+2x, & \text{if } x\geq 1;\\ -x+4, & \text{if } x<1.\end{cases}$$

2. Case of function g:

$$g(x)=\begin{cases}x^2, & \text{if } x\leq 0;\\ a\ e^x+b, & \text{if } 0<x<\pi;\\ 1-\cos(x), & \text{if } x\geq\pi;\end{cases}\iff g(x)=\begin{cases}x^2, & \text{if } x<0;\\ 0, & \text{if } x=0;\\ a\ e^x+b, & \text{if } 0<x<\pi;\\ 2, & \text{if } x\leq 0;\\ 1-\cos(x), & \text{if } x>\pi;\end{cases}$$

To ensure the continuity of the g on $\mathbb{R}$ we must check the existence of the constants a and b that satisfy the following:

$$\begin{cases}\lim\limits_{x\to 0^-}g(x)=\lim\limits_{x\to 0^+}g(x)=g(0)\\ \lim\limits_{x\to\pi^-}g(x)=\lim\limits_{x\to\pi^+}g(x)=g(\pi)\end{cases}\Longrightarrow\begin{cases}\lim\limits_{x\to 0^+}ae^x+b=0\\ \lim\limits_{x\to\pi^-}ae^x+b=2\end{cases}\Longrightarrow\begin{cases}a+b=0\\ ae^\pi+b=2\end{cases}\Longrightarrow\begin{cases}a=\frac{2}{e^\pi-1}\\ b=\frac{-2}{e^\pi-1}\end{cases}$$

We conclude that on order that the function g be a continuous on $\mathbb{R}$ it's must have the form:

$$g(x)=\begin{cases}x^2, & \text{if } x\leq 0;\\ \frac{2}{e^\pi-1}\ (e^x-1)\,, & \text{if } 0<x<\pi;\\ 1-\cos(x), & \text{if } x\geq\pi;\end{cases}$$

3. Case of the function h:

$$h(x)=\begin{cases}1, & \text{if } x<0;\\ 1, & \text{if } x=0;\\ ae^{-x}+be^x+cx(e^x-e^{-x}), & \text{if } 0<x<1;\\ e^1, & \text{if } x=1;\\ e^{2-x}, & \text{if } x>1;\end{cases}$$

- Study the continuity of the following function on $\mathbb{R}$, $f(x) = E(x)$. What can we conclude?

  Note that the integer party function can be written as follows:

$$f(x) = E(x) = \begin{cases} k-1, & \text{if } k-1 < x < k; \\ k, & \text{if } x = k; \\ k, & \text{if } k < x < k+1; \end{cases} \qquad \text{with } k \in \mathbb{Z}.$$

From the expression of the function $E(x)$, it is clear that to check its continuity on $\mathbb{R}$, it is sufficient to check its continuity at $x_0 = k$ with $k \in \mathbb{Z}$.

For any integer number k, we have,

1. $f(k) = E(k) = k$.
2. $\lim\limits_{x \to k^+} f(x) = k$;
3. $\lim\limits_{x \to k^-} f(x) = k - 1$;

Hence, this above results indicate that the function $E(x)$ is continuous on the right of the integer numbers but not continuous at their left side.

We conclude that the $E(x)$ is continuous only on $\mathbb{R}/\mathbb{Z}$.

Solution of the Exercise 6

For each of the following functions determine their domains and subsequently check if they have a removable discontinuity.

1. Case of the function f_1:

$$f_1(x) = e^{\frac{-1}{x^2}}$$

   (a) $D_{f_1} = \{x \in \mathbb{R} : \ x \neq 0\} = \mathbb{R}^*$.

   (b) We note that $\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0} e^{\frac{-1}{x^2}} = 0$;

   (c) So, we can remove the discontinuity of the function f_1 at $x_0 = 0$, and the function will be rewritten as follows:

$$F_1(x) = \begin{cases} e^{\frac{-1}{x^2}}, & \text{if } x \in D_{f_1}; \\ 0, & \text{if } x = 0; \end{cases}$$

2. Case of the function f_2:

   (a) $D_{f_2} = \{x \in \mathbb{R} : \ x \neq 0\} = \mathbb{R}^*$.

   (b) We note that $\lim\limits_{x \to 0^-} f(x) \neq \lim\limits_{x \to 0^+} f(x)$. Indeed, where $\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^-} e^{\frac{-1}{x}} = +\infty$ while $\lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^+} e^{\frac{-1}{x}} = 0$;

   (c) So, the function f_2 hasn't a removable discontinuity.

3. Case of the function f_3:

$$f_3(x) = \frac{1+x}{1+x^3}$$

   (a) $D_{f_3} = \left\{x \in \mathbb{R} : \ 1 + x^3 \neq 0\right\} = \mathbb{R}/\{-1\}$.

(b) We note that $\lim\limits_{x\to -1^-} f(x) = \lim\limits_{x\to -1^+} f(x) = \lim\limits_{x\to -1} f(x) = \lim\limits_{x\to -1} \frac{1+x}{1+x^3} = \lim\limits_{x\to -1} \frac{1}{x^2-x+1} = \frac{1}{3}$.

(c) So, we can remove the discontinuity of the function f_3 at $x_0 = -1$, and the new function will be defined as follows:

$$F_3(x) = \begin{cases} \frac{1+x}{1+x^3}, & \text{if } x \in D_{f_3}; \\ \frac{1}{3}, & \text{if } x = -1; \end{cases}$$

4. Case of the function f_4:

$$f_4(x) = \sin(x+1)\ln(|x+1|),$$

(a) $D_{f_4} = \{x \in \mathbb{R} : \ |1+x| \neq 0\} = \mathbb{R}/\{-1\}$.

(b) We note that we can rewrite the limit at the point $x_0 = -1$ as follows:

$$\lim_{x\to -1} f(x) = \lim_{x\to -1} \sin(x+1)\ln(|x+1|) = \lim_{y\to 0} \sin(y)\ln(|y|)$$

$$\lim_{y\to 0} \sin(y)\ln(|y|) = \lim_{y\to 0} \left(\frac{\sin(y)}{y}\right)(y\ln(|y|)) = \lim_{y\to 0} \left(\frac{\sin(y)}{y}\right) \lim_{y\to 0} (y\ln(|y|)) = 1 * 0 = 0.$$

.

(c) So, we can remove the discontinuity of the function f_4 at $x_0 = -1$, and the new function will be defined as follows:

$$F_4(x) = \begin{cases} \frac{1+x}{1+x^3}, & \text{if } x \in D_{f_4}; \\ 0, & \text{if } x = -1; \end{cases}$$

5. Case of the function f_5:

$$f_5(x) = \left(\frac{\sin(2x)}{2\sqrt{1-\cos(x)}}\right)^{2m}, \quad m \in \mathbb{N}^*$$

From the results obtained in exercise 2, in the case where the power is an even number, we deduce that f_5 admits a removable discontinuity and the new function will be defined on $\mathbb{R}$ and its have the form :

$$F_5(x) = \begin{cases} \left(\frac{\sin(2x)}{2\sqrt{1-\cos(x)}}\right)^{2m}, & \text{if } x \in D_{f_4}; \\ 2^m, & \text{if } x = 2k\pi \text{ with } k \in \mathbb{Z}; \end{cases}$$

6. Case of the function f_6:

$$f_6(x) = \cos(x)\cos(1/x).$$

(a) $D_{f_6} = \mathbb{R}^*$.

(b) As $\lim\limits_{x\to 0} \cos(x) = 1$ and $\lim\limits_{x\to 0} \cos\left(\frac{1}{x}\right)$ doesn't exist we deduce that $\lim\limits_{x\to 0} \cos(x)\cos\left(\frac{1}{x}\right)$ doesn't exist also. Consequently, the function f_6 hasn't a removable discontinuity point.

Solution of the Exercise 7

$I)$ Let f and g two increasing continuous functions on an interval I. Show that:

$$\text{if } (f(I) \subset g(I)) \text{ or } (g(I) \subset f(I)) \text{ then } \exists c \in I \text{ such as } f(c) = g(c)$$

Let's consider $h(x) = g(x) - f(x)$, and that $I = [a, b]$ and $f(I) = [m_1; M_1]$ and $g(I) = [m_2; M_2]$

Note that:

- as f and g are continuous functions on I then h is also continuous on I.
- as f and g are increasing functions on I then $m_1 = f(a)$, $m_2 = g(a)$, $M_1 = f(b)$, and $M_2 = g(b)$.

1st case : $f(I) \subset g(I)$.
The statement $f(I) \subset g(I)$ means that

$$m_2 \leq m_1 \leq M_1 \leq M_2.$$

Hence,

$$\begin{cases} h(a) = f(a) - g(a) = m_1 - m_2 \geq 0, \\ h(b) = f(b) - g(b) = M_1 - M_2 \leq 0. \end{cases} \Longrightarrow h(a) * h(b) < 0.$$

Thus, based on the intermediate values theorem we deduce that

$$\exists c \in [a,b]: \quad h(c) = 0 \Longrightarrow \exists c \in [a,b]: \quad f(c) - g(c) = 0 \Longrightarrow \exists c \in [a,b]: \quad f(c) = g(c).$$

2ed case : we assume that $g(I) \subset f(I)$.
The statement $fg(I) \subset f(I)$ means that

$$m_1 \leq m_2 \leq M_2 \leq M_1.$$

Hence,

$$\begin{cases} h(a) = f(a) - g(a) = m_1 - m_2 \leq 0, \\ h(b) = f(b) - g(b) = M_1 - M_2 \geq 0. \end{cases} \Longrightarrow h(a) * h(b) < 0.$$

Thus, based on the intermediate values theorem we deduce that

$$\exists c \in [a,b]: \quad h(c) = 0 \Longrightarrow \exists c \in [a,b]: \quad f(c) - g(c) = 0 \Longrightarrow \exists c \in [a,b]: \quad f(c) = g(c).$$

$II)$ Let f and g two continuous functions on an interval I. Show that:

$$\text{if } (f(I) \subset g(I)) \text{ or } (g(I) \subset f(I)) \text{ then } \exists c \in I \text{ such as } f(c) = g(c). \tag{1}$$

Assume that

$$\begin{array}{ll} \min\limits_{x \in I} f(x) = a_1 \text{ and } f(a_1) = m_1 & \max\limits_{x \in I} f(x) = b_1 \text{ and } f(b_1) = M_1 \\ \min\limits_{x \in I} g(x) = a_2 \text{ and } g(a_2) = m_2 & \max\limits_{x \in I} f(x) = b_2 \text{ and } g(b_2) = M_2 \end{array}$$

Let's consider the function f defined by

$$h(x) = g(x) - f(x).$$

Note that as f and g are continuous functions on I then h is also continuous on I.

1st case : $f(I) \subset g(I)$.
The statement $f(I) \subset g(I)$ means that

$$\forall x \in I: \; m_2 \leq f(x) \leq M_2. \Longrightarrow m_2 \leq f(a_2) \leq M_2 \text{ and } m_2 \leq f(b_2) \leq M_2.$$

Hence,

$$\begin{cases} h(a_2) = f(a_2) - g(a_2) = f(a_2) - m_2 \geq 0, \\ h(b_2) = f(b_2) - g(b_2) = f(b_2) - M_2 \leq 0. \end{cases} \Longrightarrow h(a_2) * h(b_2) < 0.$$

Thus, based on the intermediate values theorem, we deduce that

$$\begin{aligned}\exists c \in [a_2, b_2]: \quad h(c) = 0 \quad &\Longrightarrow \quad \exists c \in [a, b]: \quad h(c) = 0\\ &\Longrightarrow \quad \exists c \in [a, b]: \quad f(c) - g(c) = 0\\ &\Longrightarrow \quad \exists c \in [a, b]: \quad f(c) = g(c).\end{aligned}$$

2ed case : we assume that $g(I) \subset f(I)$.

This statement $g(I) \subset f(I)$ means that:

$$\forall x \in I: \ m_1 \leq g(x) \leq M_1. \Longrightarrow m_1 \leq g(a_1) \leq M_1 \text{ and } m_1 \leq f(b_1) \leq M_1.$$

Hence,

$$\begin{cases} h(a_1) = f(a_1) - g(a_1) = m_1 - g(a_1) \geq 0, \\ h(b_1) = f(b_1) - g(b_1) = M_1 - g(b_1) \leq 0. \end{cases} \Longrightarrow h(a_1) * h(b_1) < 0.$$

Thus, based on the intermediate values theorem, we deduce that

$$\begin{aligned}\exists c \in [a_1, b_1]: \quad h(c) = 0 \quad &\Longrightarrow \quad \exists c \in [a, b]: \quad h(c) = 0\\ &\Longrightarrow \quad \exists c \in [a, b]: \quad f(c) - g(c) = 0\\ &\Longrightarrow \quad \exists c \in [a, b]: \quad f(c) = g(c).\end{aligned}$$

III) Show that the following equation has at least one solution on $]-\infty; 2[$.

$$\sin(x) = \frac{2x+1}{x-2}.$$

Let's consider the function $f(x) = \sin(x) - \frac{2x+1}{x-2}$.

We have on one hand, the function $\sin(x)$ is a continuous function on $\mathbb{R}$ and $\frac{2x+1}{x-2}$ is a continuous function on $\mathbb{R}/\{2\}$ so the function$f$ is continuous on $\mathbb{R}/\{2\}$ consequently continuous on $]-\infty; 2[$.

On the other hand, $\lim\limits_{x\to-\infty} f(x) * \lim\limits_{x\to 2^-} f(x) < 0$.

Based on the intermediate values theorem we conclude that $\exists\ c \in]-\infty; 2[$ such that $f(c) = 0 \Longrightarrow \sin(c) - \frac{2c+1}{c-2} \Longrightarrow \sin(c) = \frac{2c+1}{c-2}$.

Calculation of the limits.

$$\begin{aligned}\sin(x) \leq 1 \quad &\Longrightarrow \quad \sin(x) - \frac{2x+1}{x-2} \leq 1 - \frac{2x+1}{x-2}\\ &\Longrightarrow \quad \lim_{x\to-\infty} f(x) \leq \lim_{x\to-\infty}\left(1 - \frac{2x+1}{x-2}\right)\\ &\Longrightarrow \quad \lim_{x\to-\infty} f(x) \leq -1 < 0.\end{aligned}$$

$$\lim_{x\to 2^-} f(x) \quad = \quad \lim_{x\to 2^-}\left(\sin(x) - \frac{2x+1}{x-2}\right) = +\infty > 0$$

Remark 1 *To show that the following equation has at least one solution on* $]-\infty; 2[$.

$$\sin(x) = \frac{2x+1}{x-2},$$

we can use the results (1). Indeed, if we take $f = \sin(x)$, $g(x) = \frac{2x+1}{x-2}$, *and* $I =]-\infty; 2[$ *then the result is immediate, because* $f(I) \subset g(I)$.

III) We consider the following equation, of unknown $x > 0$ and the real parameter a.

$$\ln(x) = ax. \tag{2}$$

Let $h(x)$ be a function defined by: $h(x) = \ln(x) - ax$ and $h'(x) = \frac{1}{x} - a$.

	x	0	$1/a$	∞
if $a \leq 0$	h'	$+$	0	$+$
if $a > 0$	h'	$+$	0	$-$

From the variation table of h, we see that when $a > 0$, the function h only changes direction of variation once and that it reaches its maximum at the point $1/a$. Hence, intuitively:

1. $h(x)$ admits exactly two solutions if the maximum value of the function is strictly positive,
2. $h(x)$ admits exactly one solution if the maximum value of the function is equal to zero
3. $h(x)$ don't admit solutions if its maximum value is negative.

1. To show that the equation admits a unique solution on $]0, 1]$, we check the condition of the intermediate values on the function h and that this latest is monotonous on the same interval.
   (a) as the functions $\ln(x)$ and ax are a continuous function on $]0, 1]$ then function h is also a continuous function on this interval.
   (b) For $a \leq 0$, and $x \in]0, 1]$, we have $h'(x) = \frac{1}{x} - a > 0$ this mean that h is monotonous on $]0, 1]$.
   (c) we have $h(1) = -a > 0$ and $\lim_{x\to 0^+} h(x) = -\infty < 0$

   From these three above results we conclude that:
   there exists a unique $c \in]0, 1]$ such that $h(c) = 0 \Longrightarrow$ there exists a unique $c \in]0, 1]$ such that $\ln(c) = ac$.
2. For $a \in]0, 1/e[$, we have $\lim_{x\to 0^+} h(x) = -\infty < 0$ and $\lim_{x\to+\infty} h(x) = -\infty$.
   In addition,

   $$0 < a < 1/e \Longrightarrow 1/a > e \Longrightarrow \ln(1/a) > 1 \Longrightarrow \ln(1/a) - 1 > 0 \Longrightarrow f(1/a) > 0$$

   then,

   $$\begin{cases} f(1/a) \times \lim_{x\to 0^+} h(x) < 0 \\ f(1/a) \times \lim_{x\to+\infty} h(x) < 0 \end{cases} \Longrightarrow \begin{cases} \exists c_1 \in]0, 1/a[& \text{such that} & h(c_1) = 0 \\ \exists c_2 \in]1/a, +\infty[& \text{such that} & h(c_2) = 0 \end{cases}$$

   We conclude that there exists exactly two real numbers c_1 and c_2 as solution of the given equation.
3. Show that if $a = 1/e$, the equation admits a unique solution whose value will be specified. we have $h(1/e) = 0$ and $h(x)$ is negative on all $]0, 1/e[\cup]1/e, \infty[$ then $1/e$ is the unique solution of the equation.
4. we have if $a > 1/e$, the $h(x)$ is strictly negative for any $x \in D_h$. Consequently, the equation (2) doesn't have any solution on $\mathbb{R}$.

Solution of the Exercise 8

- Let $f : \mathbb{R} \to \mathbb{R}$. Suppose $c \in \mathbb{R}$ and that $f'(c)$ exists. Prove that f is continuous at c.

  Recall $f'(c)$ exist, means that $\lim_{x\to c} \frac{f(x)-f(c)}{x-c} = l$ (exist).

$$\begin{aligned}
\lim_{x\to c} f(x) &= \lim_{x\to c} f(x) - f(c) + f(c) \\
&= \lim_{x\to c} \frac{f(x)-f(c)}{x-c}(x-c) + f(c) \\
&= \lim_{x\to c} \frac{f(x)-f(c)}{x-c} \lim_{x\to c}(x-c) + f(c) \\
&= l * 0 + f(c) \\
&= f(c).
\end{aligned}$$

$$\lim_{x\to c} f(x) = f(c) \Longleftrightarrow f \text{ is continuous at } c.$$

Remark 2 *To show the proposition, we can use the link between differentiability and the continuity of a function at a given point. Indeed, it's sufficient to show that*

$$\textit{if } f \textit{ is not continuous at } c \Longrightarrow f \textit{ is not differentiable at } c.$$

Knowing that f is not continuous at c means that:

1. *$f(c)$ don't exist.*
2. $\lim_{x\to c^+} \neq f(c)$ *or* $\lim_{x\to c^-} \neq f(c)$
3. $\lim_{x\to c^+} = \infty$ *or* $\lim_{x\to c^-} = \infty$.

- Prove that:

1. $(e^x)' = e^x$

$$(e^x)' = \lim_{h\to 0} \frac{e^{x+h} - e^x}{h} = \lim_{h\to 0} \frac{e^x(e^h - 1)}{h} = e^x \lim_{h\to 0} \frac{e^h - 1}{h} = ?$$

Let $y = e^h - 1 \Longrightarrow h = \ln(1+y)$. if $h \to 0$ then $h \to 0$. So,

$$\lim_{h\to 0} \frac{e^h - 1}{h} = \lim_{y\to 0} \frac{y}{\ln(1+y)} = \lim_{y\to 0} \frac{1}{\ln(1+y)} y = 1.$$

$$(e^x)' = e^x \lim_{h\to 0} \frac{e^h - 1}{h} = e^x.$$

2. $\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$

By definition we have:

$$\begin{aligned}
\left(\frac{f(x)}{g(x)}\right)' &= \lim_{h\to 0} \frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h} = \lim_{h\to 0} \frac{\frac{g(x)f(x+h) - f(x)g(x+h)}{g(x)g(x+h)}}{h} \\
&= \lim_{h\to 0} \frac{g(x)f(x+h) - g(x)f(x)}{h} - \frac{f(x)g(x+h) - f(x)g(x)}{h} \times \frac{1}{g(x)g(x+h)} \\
&= g(x) \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} - f(x) \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} \times \lim_{h\to 0} \frac{1}{g(x)g(x+h)}
\end{aligned}$$

as

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=f'(x),\quad \lim_{h\to 0}\frac{g(x+h)-g(x)}{h}=g'(x)\ \text{ and } \lim_{h\to 0}\frac{1}{g(x)g(x+h)}=\frac{1}{[g(x)]^2},$$

then

$$\left(\frac{f(x)}{g(x)}\right)'=\frac{f'(x)g(x)-f(x)g'(x)}{[g(x)]^2}.$$

3. $\arcsin(x)'=\frac{1}{\sqrt{(1-x^2)}}$
   Recall that $f(f^{-1})(x)=f^{-1}(f)(x)=x$ and $\cos(x)=\sqrt{1-\sin^2(x)}$ $(\cos^2(x)+\sin^2(x)=1)$.

$$\arcsin(\sin(x))=x\Longrightarrow(\arcsin(\sin(x)))'=x'\Longrightarrow\cos(x)\arcsin'(\sin(x))=1\Longrightarrow\arcsin'(\sin(x))=\frac{1}{\cos(x)}$$

   We put $y=\sin(x)$ then

$$\arcsin'(\sin(x))=\frac{1}{\sqrt{1-\sin^2(x)}}\Longrightarrow\arcsin'(y)=\frac{1}{\sqrt{1-y^2}}\Longrightarrow\arcsin'(x)=\frac{1}{\sqrt{1-x^2}}.$$

4. $\arctan(x)'=\frac{1}{1+x^2}$
   To show the result, we proceed in the same way as in the case of arcsin.
   Note that:

$$\tan'(x)=\left(\frac{\sin(x)}{\cos(x)}\right)'=\frac{\sin^2(x)+\cos^2(x)}{\cos^2(x)}=1+\tan^2(x).$$

$$\begin{aligned}(\arctan(\tan(x)))'=x\quad&\Longrightarrow\quad\left(1+\tan^2(x)\right)\arctan'(\tan(x))=1\\&\Longrightarrow\quad\arctan'(\tan(x))=\frac{1}{1+\tan^2(x)}\\&\Longrightarrow\quad\arctan'(x)=\frac{1}{1+x^2}.\end{aligned}$$

5. $\left(f^{-1}(x)\right)'=\frac{1}{f'\circ f^{-1}(x)}$

$$\left(f\left(f^{-1}\right)(x)\right)'=x'\Longrightarrow\left(f^{-1}(x)\right)'\left(f'\left(f^{-1}\right)(x)\right)=1\Longrightarrow\left(f^{-1}(x)\right)'=\frac{1}{\left(f'\left(f^{-1}\right)(x)\right)}.$$

6. $(f\circ g(x))'=g'(x)\ f'\circ g(x)$

Solution of the Exercise 9

- Return to the examples of the Exercise 5, and determine the domain of differentiability of the considered functions according to the parameters a, b, and c.

  Recall that a function f is differentiable at point x_0 if and only if

  1. f is continuous at point x_0
  2. $\lim\limits_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}=l$ (exist).

For the first condition it's already checked in exercise 5, where the functions f and g to be continuous they must be :

$$f(x) = \begin{cases} x^2 + 2x, & \text{if } x \geq 1; \\ -x + 4, & \text{if } x < 1. \end{cases}$$

$$g(x) = \begin{cases} x^2, & \text{if } x \leq 0; \\ \frac{2}{e^{\pi}-1}\ (e^x - 1)\,, & \text{if } 0 < x < \pi; \\ 1 - cos(x), & \text{if } x \geq \pi; \end{cases}$$

Let's now check the second condition.

Case of f

$$\lim_{x\to 1^+} \frac{f(x) - f(1)}{x - 1} = \lim_{x\to 1^+} \frac{x^2 + 2x - 3}{x - 1} = \lim_{x\to 1^+} \frac{(x-1)(x+3)}{x - 1} = \lim_{x\to 1^+} x + 3 = 4$$

$$\lim_{x\to 1^-} \frac{f(x) - f(x_0)}{x - 1} = \lim_{x\to 1^-} \frac{-x + 4}{x - 1} = -\infty.$$

We see that the limit at the left side of $x = 1$ doesn't exist so f isn't differentiable at $x = 1$. We conclude that f is differentiable only on $\mathbb{R}/\{1\}$.

Case of g

$$\lim_{x\to 0^-} \frac{g(x) - g(0)}{x - 0} = \lim_{x\to 0^-} \frac{x^2}{x} = \lim_{x\to 0^-} x = 0$$

$$\lim_{x\to 0^+} \frac{g(x) - g(0)}{x - 0} = \lim_{x\to 0^+} \frac{\frac{2}{e^{\pi}-1}\ (e^x - 1)}{x} = \frac{2}{e^{\pi} - 1} \lim_{x\to 0^-} \frac{(e^x - 1)}{x} = \frac{2}{e^{\pi} - 1}$$

$$g(x) = \begin{cases} x^2, & \text{if } x \leq 0; \\ \frac{2}{e^{\pi}-1}\ (e^x - 1)\,, & \text{if } 0 < x < \pi; \\ 1 - cos(x), & \text{if } x \geq \pi; \end{cases}$$

- Determine the two real numbers a and b, so that the function f, defined on $\mathbb{R}$ by:

$$f(x) = \begin{cases} \sqrt{x}, & \text{if } \ 0 \leq x \leq 1; \\ ax^2 + bx + c, & \text{x>1}, \end{cases}$$

is differentiable on $\mathbb{R}_+.^*$.

- *Study the differentiability of the following functions:*

$$f_1(x) = \begin{cases} x^2 cos(1/x), & \textit{if } \ x \neq 0; \\ 0, & \textit{else.} \end{cases} \qquad f_2(x) = \begin{cases} \sin(x)\sin(1/x), & \textit{if } \ x \neq 0; \\ 0, & \textit{else.} \end{cases}$$

$$f_3(x) = \begin{cases} \frac{|x|\sqrt{x^2-2x+1}}{x-1}, & \textit{if } \ x \neq 1; \\ 1, & \textit{else.} \end{cases}$$

- Study the differentiability of the following functions at x_0:

$$f_1(x) = \sqrt{x}, \ \ x_0 = 0, \qquad f_2(x) = (1-x)\sqrt{1-x^2}, \ \ x_0 = -1, \qquad f_3(x) = (1-x)\sqrt{1-x^2}, \ \ x_0 = 1.$$

What can we conclude?

- does f_1 differentiable at $x_0 = 0$?

$$\lim_{x\to 0^+} \frac{f_1(x) - f_1(0)}{x - 0} = \lim_{x\to 0^+} \frac{\sqrt{x}}{x} = \lim_{x\to 0^+} \frac{1}{\sqrt{x}} = +\infty.$$

We conclude that the function f_1 is not differentiable at $x_0 = 0$.

- does f_2 differentiable at $x_0 = -1$?

Remark 3 *1) From the result obtained on f_1, it's clear that $\sqrt{1-x^2}$ is not differentiable at $x_0 = -1$. 2) $1 - x$ is a first order polynomial function so it differentiable on $\mathbb{R}$.*

$$\begin{aligned}\lim_{x\to -1} \frac{f_2(x) - f_2(-1)}{x + 1} &= \lim_{x\to -1} \frac{(1-x)\sqrt{1-x}\sqrt{1+x}}{x+1} \\ &= \lim_{x\to -1} \frac{(1-x)\sqrt{1-x}}{\sqrt{1+x}} = \frac{2\sqrt{2}}{0^+} = +\infty.\end{aligned}$$

We conclude that the function f_2 is not differentiable at $x_0 = -1$.

- does f_3 differentiable at $x_0 = 1$?

Remark 4 *1) From the result obtained on f_1, it's clear that $\sqrt{1-x^2}$ is not differentiable at $x_0 = 1$. 2) $1 - x$ is a first order polynomial function so it differentiable on $\mathbb{R}$.*

$$\begin{aligned}\lim_{x\to 1} \frac{f_2(x) - f_2(1)}{x - 1} &= \lim_{x\to 1} \frac{(1-x)\sqrt{1-x}\sqrt{1+x}}{x-1} \\ &= \lim_{x\to 1} \sqrt{1-x^2} = 0.\end{aligned}$$

We conclude that the function f_3 is differentiable at $x_0 = 1$.

From these last two examples, we see that if we have two real functions f and g. Knowing the non-differentiability of the function f at the point x_0 does not allow us to deduce the differentiability of their product $f * g$ at x_0.

Solution of the Exercise 10

Calculate the derivatives of the following functions.

1. $\left(e^{\sin(x^3)}\right)' = \left(\sin(x^3)\right)' e^{\sin(x^3)} = (x^3)' \sin'(x^3) e^{\sin(x^3)} = 3x^2 \cos(x^3) e^{\sin(x^3)}$.
2. $\left(\ln\left(x^2 + e^{-x^2}\right)\right)' = \frac{\left(x^2+e^{-x^2}\right)'}{x^2+e^{-x^2}} = \frac{(x^2)'+\left(e^{-x^2}\right)'}{x^2+e^{-x^2}} = \frac{2x-2xe^{-x^2}}{x^2+e^{-x^2}}$.
3. $\left(\ln\left(\frac{x+1}{x-1}\right)\right)' = \frac{\left(\frac{x+1}{x-1}\right)'}{\frac{x+1}{x-1}} = \left(\frac{1*(x-1)-(x+1)*1}{(x-1)^2}\right)\left(\frac{x-1}{x+1}\right) = \frac{2}{1-x^2}$.
4. $\left(\sin(2x^2 + cos(x))\right)' = \left(2x^2 + cos(x)\right)' = \sin'(2x^2 + cos(x)) = 4x - \sin(x) + \cos(2x^2 + cos(x))$.
5. $\left(\arcsin(x^2 + x)\right)' = (x^2 + x)' \arcsin'(x^2 + x) = (2x + 1)\frac{1}{\sqrt{1-(x^2+x)^2}}$.
6. $\left(\arctan(x^2 + x)\right)' = (x^2 + x)' \arctan'(x^2 + x) = (2x + 1)\frac{1}{1+(x^2+x)^2}$.
7. $\left(\sqrt[3]{x^2 + x}\right)' = \left[\left(x^2 + x\right)^{\frac{1}{3}}\right]' = \frac{1}{3}(x^2 + x)' \left(x^2 + x\right)^{\frac{1}{3}-1} = \frac{1}{3}(x + 1)\frac{1}{\sqrt[3]{(x^2+x)^2}}$.

8. $\left(a^{\left(\frac{x-1}{x+1}\right)}\right)' = \left(e^{\ln(a)\ \left(\frac{x-1}{x+1}\right)}\right)' \left(\ln(a)\ \left(\frac{x-1}{x+1}\right)\right)' e^{\ln(a)\ \left(\frac{x-1}{x+1}\right)} = \frac{2\ln(a}{(x+1)^2} a^{\left(\frac{x-1}{x+1}\right)}$

9. $\left(e^{e^{x^2+1/x}}\right)' = \left(e^{x^2+1/x}\right)' e^{e^{x^2+1/x}} = (x^2+1/x)' e^{x^2+1/x} e^{e^{x^2+1/x}} = \left(2x - \frac{1}{x^2}\right) e^{x^2+1/x} e^{e^{x^2+1/x}}.$

10. $(\log_a(\arcsin(x)))' = \left(\frac{\ln(\arcsin(x))}{\ln(a)}\right)' = \frac{1}{\ln(a)} \frac{(\arcsin(x))'}{\arcsin(x)} = \frac{1}{\ln(a)\sqrt{1-x^2}\arcsin(x)}.$

11. $\left(\sqrt{|x^2-4x+3|}\right)'$. To compute the derivative in this case we must eliminate the absolute value. Note that,

x	$-\infty$	1		3		$+\infty$
x^2-4x+3	$+$		$-$		$+$	

then

$$\left(\sqrt{|x^2-4x+3|}\right)' = \begin{cases} \left(\sqrt{x^2-4x+3}\right)' = \frac{(x^2-4x+3)'}{2\sqrt{x^2-4x+3}} = \frac{2x-4}{2\sqrt{x^2-4x+3}} & \text{if } x \notin]1;3[\\ \left(\sqrt{-x^2+4x-3}\right)' = \frac{-(x^2-4x+3)'}{2\sqrt{-(x^2-4x+3)}} = \frac{4-2x}{2\sqrt{-(x^2-4x+3)}} & \text{if } x \in]1;3[\end{cases}$$

12. $\left(\frac{1-\tan^2(x)}{(1+tan(x))^2}\right)' = ?$

We have: $\frac{1-\tan^2(x)}{(1+tan(x))^2} = \frac{(1-\tan(x))(1+\tan(x))}{(1+tan(x))^2} = \frac{(1-\tan(x))}{(1+tan(x))} = \frac{\cos(x)-\sin(x)}{\cos(x)+\sin(x)}.$

then

$\left(\frac{\cos(x)-\sin(x)}{\cos(x)+\sin(x)}\right)' = \frac{-(\sin(x)+\cos(x))^2-(\cos(x)-\sin(x))^2}{(\cos(x)+\sin(x))^2} = -1-\left(\frac{\cos(x)-\sin(x)}{\cos(x)+\sin(x)}\right)^2 = -1-\left(\frac{1-\tan(x)}{1+\tan(x)}\right)^2 = \frac{-2}{1+\sin(2x)}.$

Solution of the Exercise 11

1. In the application of mean value theorem's to the function

$$f(x) = \alpha x^2 + \beta x + \gamma, \qquad \alpha,\ \beta,\ \gamma \in \mathbb{R}^*$$

on the interval $[a;b]$

(a) specify the number $c \in]a;b[$. Note that the function f is a second order polynomial, so f is continuons and differentiable on $\mathbb{R}$. The application of mean value theorem's to the function f mean that:

$$\exists c \in]a,b[,\ \text{such that } \frac{f(b)-f(a)}{b-a} = f'(c)$$
$$\Longrightarrow \frac{\alpha b^2+\beta b+\gamma-(\alpha a^2+\beta a+\gamma)}{b-a} = 2\alpha c+\beta$$
$$\Longrightarrow \frac{\alpha(b^2-a^2)+\beta(b-a)}{b-a} = 2\alpha c+\beta$$
$$\Longrightarrow \alpha(b+a)+\beta = 2\alpha c+\beta$$
$$\Longrightarrow c = \frac{b+a}{2}.$$

(b) Give a geometric interpretation. The above result means that, if we have two points $(a, f(a))$ and $(b, f(b))$ then the line passing through these points is parallel to the tangent of the polynomial at the points $x_0 = \frac{b+a}{2}$.

2. Let x and y two reals with $0 < x < y$, show that
$$x < \frac{y-x}{ln(y)-ln(x)} < y.$$
Let f defined by $f(x) = \ln(x)$; hence $f'(x) = \frac{1}{x}$. Applying the mean value theorem's on the interval $]x, y[$, we get:
$$\exists c \in]x, y[, \text{ such that } \frac{f(y)-f(x)}{y-x} = f'(c) \Longrightarrow \frac{\ln(y)-\ln(x)}{y-x} = \frac{1}{c} \Longrightarrow c = \frac{y-x}{\ln(y)-\ln(x)}.$$
As $c \in]x, y[$ then $x < c < y$ and if we substitue c by its expression we get:
$$x < \frac{y-x}{\ln(y)-\ln(x)} < y.$$

Solution of the Exercise 12 In the rest of this exercise the notation $f(x)|'_{x=x_0}$ designates the derivative of the function f at the point x_0.

1. $\lim\limits_{x\to 0} \frac{e^{3x-2}-e^{-2}}{x} = \lim\limits_{x\to 0} \frac{e^{3x-2}-e^{3*0-2}}{x-0} = e^{3x-2}\big|'_{x=0} = 3e^{3x-2}\big|_{x=0} = 3e^{-2}.$
2. $\lim\limits_{x\to 1} \frac{\ln(2-x)}{x-1} = \lim\limits_{x\to 1} \frac{\ln(2-x)-\ln(2-1)}{x-1} = \ln(2-x)|'_{x=1} = \frac{-2}{2-x}\Big|_{x=1} = -2.$
3. $\lim\limits_{x\to \pi} \frac{\sin(x)}{x^2-\pi^2} = \lim\limits_{x\to \pi} \frac{\sin(x)}{(x-\pi)(x+\pi)} = \lim\limits_{x\to \pi} \frac{\sin(x)-\sin(\pi)}{(x-\pi)} \times \lim\limits_{x\to \pi} \frac{1}{x+\pi} = \frac{1}{2\pi} \times \sin(x)|'_{x=\pi} = \frac{1}{2\pi} \times \cos(x)|_{x=\pi} = \frac{-1}{2\pi}.$
4. $\lim\limits_{x\to \frac{\pi}{2}} \frac{e^{cos(x)}}{x-\frac{\pi}{2}} = \lim\limits_{x\to \frac{\pi}{2}} \frac{e^{cos(x)}-e^{\cos(\frac{\pi}{2})}}{x-\frac{\pi}{2}} = e^{\cos(x)}\big|'_{x=\frac{\pi}{2}} = -\sin(x)\, e^{\cos(x)}\big|_{x=\frac{\pi}{2}} = -1.$
5. $\lim\limits_{x\to 0} \frac{\ln(1-\sin(x))}{x} = \lim\limits_{x\to 0} \frac{\ln(1-\sin(x))-\ln(1-\sin(0))}{x-0} = \ln(1-\sin(x))|'_{x=0} = \frac{-\cos(x)}{1-\sin(x)}\Big|_{x=0} = -1.$
6. To compute the present limit we use the mean values theorem.

   From the mean values theorem, we have,
$$\frac{\ln(x+1)-\ln(x)}{(x+1)-x} = \ln(y)|'_{y=c} = \frac{1}{c}, \quad \text{with } c \in]x, x+1[$$
   In addition, if x tend toward $+\infty$ then c also tend toward $+\infty$. So,
$$\lim_{x\to+\infty} (\ln(x+1)-\ln(x)) = \lim_{x\to+\infty} \frac{\ln(x+1)-\ln(x)}{(x+1)-x} = \lim_{c\to+\infty} \frac{1}{c} = 0.$$

Solution of the Exercise 13

Give the domain of differentiability of the following functions then calculate the nth-order derivative, by justifying its existence.

1. $f(x) = 2x^k,\ k \in \mathbb{N}^*$

n	$f^{(n)}$
1	$2kx^{k-1} = \frac{2k!}{(k-1)!}x^{k-1}$
2	$2k(k-1)x^{k-2} = \frac{2k!}{(k-2)!}x^{k-2}$
$\vdots$	$\vdots$
i	$\frac{2k!}{(k-i)!}x^{k-i}$
$\vdots$	$\vdots$
k	$2k!$
$\geq k+1$	0

2. $f(x) = 1/x$, and $f(x) = \ln(1+x)$.

n	$f^{(n)} = \left(\frac{1}{x}\right)^{(n)}$	$f^{(n)} = (\ln(1+x))^{(n)}$
1	$\frac{-1}{x^2} = \frac{(-1)^1 * 1}{x^{1+1}}$	$\frac{1}{x+1} = \frac{1}{(x+1)^1}$
2	$\frac{1*2}{x^3} = \frac{(-1)^2 * 2!}{x^{2+1}}$	$\frac{-1}{(x+1)^2} = \frac{(-1)^1 * 1!}{(x+1)^2}$
3	$\frac{-1*2*3}{x^4} == \frac{(-1)^3 * 3!}{x^{3+1}}$	$\frac{1*2}{(x+1)^3} = \frac{(-1)^2 * 2!}{(x+1)^3}$
$\vdots$	$\vdots$	$\vdots$
n	$\frac{(-1)^n n!}{x^{n+1}}$	$\frac{(-1)^{n-1}(n-1)!}{(1+x)^n}$
$\vdots$	$\vdots$	$\vdots$

3. $f(x) = \sin(x)$, and $f(x) = \sin(x)cos(x) = \frac{1}{2}\sin(2x)$.

n	$f^{(n)} = (\sin(x))^{(n)}$	$f^{(n)} = \left(\frac{1}{2}\sin(2x)\right)^{(n)}$
1	$\cos(x) = \sin\left(x + \frac{\pi}{2}\right)$	$\cos(2x) = 2^{1-1}\sin\left(2x + \frac{\pi}{2}\right)$
2	$\cos\left(x + \frac{\pi}{2}\right) = \sin\left(x + \frac{2\pi}{2}\right)$	$\cos\left(2x + \frac{\pi}{2}\right) = 2^{2-1}\sin\left(2x + \frac{2\pi}{2}\right)$
3	$\cos\left(x + \frac{2\pi}{2}\right) = \sin\left(x + \frac{3\pi}{2}\right)$	$2^2\cos\left(x + \frac{2\pi}{2}\right) = 2^{3-1}\sin\left(x + \frac{3\pi}{2}\right)$
$\vdots$	$\vdots$	$\vdots$
n	$\cos\left(x + \frac{(n-1)\pi}{2}\right) = \sin\left(x + \frac{n\pi}{2}\right)$	$2^{n-1}\cos\left(x + \frac{(n-1)\pi}{2}\right) = 2^{n-1}\sin\left(x + \frac{n\pi}{2}\right)$
$\vdots$	$\vdots$	$\vdots$

Proof: Let's check that the proposed formulas of the nth derivative of $\sin(x)$ given by

$$\sin^{(n)}(x) = \sin\left(x + \frac{n\pi}{2}\right), \tag{3}$$

is correct using the induction method.

- **for** $n = 1$**:** we have by definition $\sin'(x) = \cos(x) = \sin\left(x + \frac{\pi}{2}\right)$.
  using the formula (3) we get $\sin^{(1)}(x) = \sin\left(x + \frac{1*\pi}{2}\right) = \sin\left(x + \frac{\pi}{2}\right)$.
  We conclude that the proposition is correct for $n = 1$.
- **for** n (≥ 1)**:** Let's assume that (3) is correct for n.
- **for** $n+1$**:** Let's show that the proposition (3) is correct for $n+1$.
  we have by definition

$$\begin{aligned} \sin^{(n+1)}(x) &= \left(\sin^{(n)}(x)\right)' = \left(\sin\left(x + \frac{n\pi}{2}\right)\right)' \\ &= \cos\left(x + \frac{n\pi}{2}\right) = \cos\left(x + \frac{n\pi}{2} + \frac{\pi}{2}\right) \\ &= \cos\left(x + \frac{(n+1)\pi}{2}\right). \end{aligned}$$

Using the formula (3), we get directly

$$\sin^{(n+1)}(x) = \sin\left(x + \frac{(n+1)\pi}{2}\right).$$

We conclude that the proposition is correct for $n+1$.

From the above results, we conclude that

$$\forall n \in \mathbb{N}^*, \quad \sin^{(n)}(x) = \sin\left(x + \frac{n\pi}{2}\right).$$

To proof that the proposed formulas of the nth derivative of $f(x) = \frac{1}{2}\sin(2x)$ is correct, we proceed with same manner as the case $(\sin(x))^{(n)}$.

4. $f(x) = \frac{1}{1-x^2}$, To extract the expression for the nth derivative of f, in this situation, we can proceed in two ways :

   (a) we directly differentiate f and obtain a general expression for its derivative.

   (b) we decompose the function into the sum of two subfunctions and we look for the general expression of the nth derivative of each of the two subfunctions and the expression of that of f is only their sum.

$f(x) = \frac{1}{1-x^2} = \frac{1}{(1-x)(1+x)} = \frac{a}{1+x} + \frac{b}{1-x} = \frac{1}{2}\left(\frac{1}{1+x} + \frac{1}{1-x}\right),$

Let's consider $f_1(x) = \frac{1}{1+x}$, and $f_2(x) = \frac{1}{1-x}$,

n	$f_1^{(n)} = \left(\frac{1}{1+x}\right)^{(n)}$	$f_2^{(n)} = \left(\frac{1}{1-x}\right)^{(n)}$
1	$\frac{-1}{(x+1)^2} = \frac{(-1)^1}{(x+1)^{1+1}}$	$\frac{1}{(1-x)^2} = \frac{1!}{(1-x)^{1+1}}$
2	$\frac{1*2}{(x+1)^3} = \frac{(-1)^3*2!}{(x+1)^{2+1}}$	$\frac{2}{(1-x)^3} = \frac{2!}{(1-x)^{2+1}}$
3	$\frac{-1*2*3}{(x+1)^4} = \frac{(-1)^3*3!}{(x+1)^{3+1}}$	$\frac{2*3}{(1-x)^4} = \frac{3!}{(1-x)^{3+1}}$
$\vdots$	$\vdots$	
n	$\frac{(-1)^n n!}{(1+x)^{n+1}}$	$\frac{n!}{(1-x)^{n+1}}$
$\vdots$	$\vdots$	

Hence, for any $n \in \mathbb{N}^*$, we have

$$f^{(n)}(x) = \left(\frac{1}{1-x^2}\right)^{(n)} = \frac{(-1)^n n!}{2(1+x)^{n+1}} + \frac{n!}{2(1-x)^{n+1}}.$$