University of Biskra
Mathematics Department
Module: Analysis 1

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Solution of the Worksheet 3

Solution of the Exercise 1

1. Study the variation of the sequences.

   (a) Case of (U_n):

$$\begin{aligned} U_{n+1}-U_n &= \left(\frac{1}{n+2}-\frac{1}{n+3}\right)-\left(\frac{1}{n+1}-\frac{1}{n+2}\right) \\ &= \frac{n+1-3-n}{(n+1)(n+3)} \\ &= \frac{-2}{(n+1)(n+3)}<0 \end{aligned}$$

   As $U_{n+1}-U_n<0$ for all $n\in\mathbb{N}$ then we conclude that (U_n) is a strictly decreasing sequence.

   (b) Case of (V_n): Note that, (V_n) can be rewritten as follows: $V_n = n^2\left(1-\frac{1}{n+1}\right)=\frac{n^3}{n+1}$.

$$\begin{aligned} V_{n+1}-V_n &= \left(\frac{(n+1)^3}{n+2}\right)-\left(\frac{n^3}{n+1}\right) \\ &= \frac{(n+1)^4-(n^4+2n^3)}{(n+1)(n+2)} \\ &= \frac{2n^3+6n^2+4n+1}{(n+1)(n+2)}>0 \end{aligned}$$

   As $V_{n+1}-v_n>0$ for all $n\in\mathbb{N}$ then we conclude that (V_n) is a strictly increasing sequence.

   (c) Case of (W_n): Note that, for any $n\in\mathbb{N}^*$ we have $(-1)^{n+1}=-(-1)^n$

$$\begin{aligned} W_{n+1}-W_n &= \left((n+1)\left((n+1)-(-1)^{n+1}\right)\right)-\left(n\left(n-(-1)^n\right)\right) \\ &= (n+1)^2-(n+1)(-1)^{n+1}-n^2+n(-1)^n \\ &= 2n+2n(-1)^n+1+(-1)^n \\ &= \begin{cases} 4n+2>0, & \text{if } n \text{ is an even number;} \\ 0, & \text{if } n \text{ is an odd number.} \end{cases} \end{aligned}$$

   As $W_{n+1}-W_n\geq 0$ for all $n\in\mathbb{N}$ then we conclude that (W_n) is an increasing sequence.

   (d) Case of (T_n): In this case we note that the sequence is an power sequence so its preferably to compare the ration T_{n+1}/T_n with 1 rather that the analysis of the sign of $T_{n+1}-T_n$

$$\begin{aligned} \frac{T_{n+1}}{T_n} &= \frac{a^{\frac{1}{n+1}}}{a^{\frac{1}{n}}}=a^{\frac{1}{n+1}-\frac{1}{n}} \\ &= a^{\frac{-1}{n(n+1)}}=\left(\frac{1}{a}\right)^{\frac{1}{n(n+1)}}<1 \quad (\text{ because } \frac{1}{a}<1 \text{ and } \frac{1}{n(n+1)}>0). \end{aligned}$$

   As $\frac{T_{n+1}}{T_n}<1$ for all $n\in\mathbb{N}$ then we conclude that (T_n) is a strictly decreasing sequence.

Remark: We can also show that (T_n) is a strictly decreasing sequence by doing the following:

$$\begin{aligned}\forall n \in \mathbb{N}^*, \text{ we have } n+1>0 \quad &\Longrightarrow \quad \frac{1}{n+1} < \frac{1}{n}\\ &\Longrightarrow \quad a^{\frac{1}{n+1}} < a^{\frac{1}{n}} \text{ (As the power function is strictly increasing)}\\ &\Longrightarrow \quad T_{n+1} < T_{n+1}\end{aligned}$$

By definition, it is a strictly decreasing sequence.

2. Case of (S_n).

$$\begin{aligned}S_{n+1}-S_n &= \frac{1}{n+1}\sum_{i=1}^{n+1} U_i - \frac{1}{n}\sum_{i=1}^{n} U_i\\ &= \frac{1}{n(n+1)}\left(n\left(\sum_{i=1}^{n+1} U_i\right) - (n+1)\left(\sum_{i=1}^{n} U_i\right)\right)\\ &= \frac{1}{n(n+1)}\left(n\left(\sum_{i=1}^{n} U_i\right) + nU_{n+1} - n\left(\sum_{i=1}^{n} U_i\right) - \left(\sum_{i=1}^{n} U_i\right)\right)\\ &= \frac{1}{n(n+1)}\left(\sum_{i=1}^{n} U_{n+1} - \sum_{i=1}^{n} U_i\right)\\ &= \frac{1}{n(n+1)}\left(\sum_{i=1}^{n}(U_{n+1}-U_i)\right).\end{aligned}$$

We note that

$$\text{If } (U_n) \searrow \quad \Longrightarrow \forall i \leq n, \quad U_{n+1}-U_i \leq 0 \Longrightarrow \sum_{i=1}^{n}(U_{n+1}-U_i) \leq 0 \Longrightarrow S_{n+1}-S_n \leq 0 \Longrightarrow (S_n) \searrow;$$

$$\text{If } (U_n) \nearrow \quad \Longrightarrow \forall i \leq n, \quad U_{n+1}-U_i \geq 0 \Longrightarrow \sum_{i=1}^{n}(U_{n+1}-U_i) \geq 0 \Longrightarrow S_{n+1}-S_n \geq 0 \Longrightarrow (S_n) \nearrow.$$

We conclude that the sequence (S_n) is of the same nature as U_n.

Solution of the Exercise 2

I) Let $(Un)_{n\in\mathbb{N}}$ be a sequence of $\mathbb{R}$. What do you think of the following propositions:

1. If U_n converges to a real l then U_{2n} and U_{2n+1} converge to l.
   This statement is true, because if a sequence converges then all its subsequences converge to the same limit.
2. If U_{2n} and U_{2n+1} are convergent, the same is true of U_n.
   This statement is false, because it is possible that U_{2n} and U_{2n+1} converge to two different values. For instance, if we consider the sequence $U_n = (-1)^n$, despite that the sequences U_{2n} and U_{2n+1} are convergent (toward 1 and -1, respectively) the sequence (U_n) is a divergent sequence.

3. If U_{4n} and U_{4n+2} are convergent, towards the same limit, it is the same for U_n.
   This statement is false, because nothing is known about the remaining subsequences U_{4n+1} and U_{4n+3}. For example, if we consider the sequence $U_n = \cos(n\pi)$ then $\lim_{n\to\infty} U_{4n} = \lim_{n\to\infty} U_{4n+2} = 1$ but $\lim_{n\to\infty} U_{4n+2} = \lim_{n\to\infty} U_{4n+3} = -1$. Consequently, U_n is not convergent.
4. If U_{2n} and U_{2n+1} are convergent, towards the same limit, it is the same for U_n.
   This statement is true, because $\{U_{2n}\} \cup \{U_{2n+1}\} = \{U_n\}$.

II) Prove that:

1. if the sequence $\{U_n\}_{n\in\mathbb{N}}$ converges to l_1 and $\{V_n\}_{n\in\mathbb{N}}$ converges to l_2, then the sequence $\{U_n + V_n\}_{n\in\mathbb{N}}$ converges to $l_1 + l_2$.
   By definition we have

$$\begin{cases} (U_n) \text{ converges to } l_1 \\ (V_n) \text{ converges to } l_2 \end{cases} \Longleftrightarrow \begin{cases} \forall \epsilon > 0, \exists N_1 \in \mathbb{N} : |U_n - l_1| < \epsilon/2, \quad for\ n \geq N_1. \\ \forall \epsilon > 0, \exists N_2 \in \mathbb{N} : |V_n - l_2| < \epsilon/2, \quad for\ n \geq N_2. \end{cases}$$

   Let $N = \max\{N_1,\ N_2\}$ then for all $n \geq N$ we have:

$$\begin{array}{rcl} |(U_n + V_n) - (l_1 + l_2)| & = & |(U_n - l_1) + (V_n - l_2)| \\ & \leq & |(U_n - l_1)| + |(V_n - l_2)| \\ & < & \epsilon/2\ +\ \epsilon/2 \\ & < & \epsilon \\ & \Longleftrightarrow & (U_n + V_n) \text{ converges to } l_1 + l_2. \end{array}$$

2. convergent sequences are Cauchy sequences.
   Let (U_n) to be a convergent sequence toward l. By definition we have

$$\begin{cases} (U_n) \text{ converges to } l \\ (U_m) \text{ converges to } l \end{cases} \Longleftrightarrow \begin{cases} \forall \epsilon > 0, \exists N_1 \in \mathbb{N} : |U_n - l_1| < \epsilon/2, \quad for\ n \geq N_1. \\ \forall \epsilon > 0, \exists N_2 \in \mathbb{N} : |U_m - l_2| < \epsilon/2, \quad for\ m \geq N_2. \end{cases}$$

   Let $N = \max\{N_1,\ N_2\}$ then for all $n \geq N$, we have:

$$\begin{array}{rcl} |U_n - U_m| & = & |U_n - l + l - U_m| \\ & \leq & |(U_n - l)| + |(U_m - l)| \\ & < & \epsilon/2\ +\ \epsilon/2 \\ & < & \epsilon \\ & \Longleftrightarrow & (U_n) \text{ is a Cauchy's sequence.} \end{array}$$

Solution of the Exercise 3

Let's consider the following real sequences:

$$U_n = \frac{1}{n+1}, \qquad V_n = \sqrt[n]{a} \text{ with } a > 1 \qquad W_n = \frac{(-1)^n + bn}{n+1} \text{ with } b \in \mathbb{R}, \qquad T_n = c^n \text{ with } c \in]-1, 1[.$$

1. The use of the definition of the limit of a real sequence.

(a) Case of $\lim_{n\to\infty} U_n = 0$

$$|U_n - 0| < \epsilon \Longrightarrow \left|\frac{1}{n+1}\right| < \epsilon \Longrightarrow \frac{1}{n+1} < \epsilon \Longrightarrow n+1 > \frac{1}{\epsilon} \Longrightarrow n > \frac{1}{\epsilon} - 1 \Longrightarrow N = E\left(\frac{1}{\epsilon} - 1\right) + 1.$$

(b) Case of $\lim_{n\to\infty} V_n = 1$

$$|V_n - 1| < \epsilon \Longrightarrow |\sqrt[n]{a} - 1| < \epsilon \Longrightarrow \sqrt[n]{a} - 1 < \epsilon \Longrightarrow a^{\frac{1}{n}} < \epsilon + 1 \Longrightarrow \frac{1}{n}\ln(a) < \ln(1+\epsilon) \Longrightarrow n > \frac{\ln(a)}{\ln(1+\epsilon)}.$$

So, $N = E\left(\frac{\ln(a)}{\ln(1+\epsilon)}\right) + 1$.

(c) Case of $\lim_{n\to\infty} W_n = b$

$$|W_n - b| < \epsilon \Longrightarrow \left|\frac{(-1)^n + bn}{n+1} - b\right| < \epsilon \Longrightarrow \left|\frac{(-1)^n - b}{n+1}\right| < \epsilon. \tag{1}$$

we have

$$\begin{aligned} -1 - b \leq (-1)^n - b \leq 1 - b &\Longrightarrow |(-1)^n - b| \leq max\{|-1-b|, |1-b|\} \\ &\Longrightarrow \begin{cases} |(-1)^n - b| \leq 1 + b, & \text{if } b \geq 0; \\ |(-1)^n - b| \leq 1 - b & \text{if } b < 0 \end{cases} \\ &\Longrightarrow |(-1)^n - b| \leq 1 + |b| \end{aligned} \tag{2}$$

From the inequalities (1) and (2) we deduce that

$$\frac{1+|b|}{n+1} < \epsilon \Longrightarrow n+1 > \frac{1+|b|}{\epsilon} \Longrightarrow n > \frac{1+|b|}{\epsilon} - 1 \Longrightarrow N = E\left(\frac{1+|b|}{\epsilon} - 1\right) + 1 = E\left(\frac{1+|b|}{\epsilon}\right).$$

(d) Case of $\lim_{n\to\infty} T_n = 0$. For this sequence we distinct two cases namely: case where $c = 0$ and the case where $c \in]-1, 0[\cup]0, 1[$.

Case of $c = 0$ in this case the sequence is a constant and equal zero. So, $|T_n| < \epsilon \Longleftrightarrow 0 < \epsilon \Longrightarrow N = 1$.

Case of $c \neq 0$

$$\begin{aligned} |T_n - 0| < \epsilon &\Longrightarrow |c^n| < \epsilon \\ &\Longrightarrow |c|^n < \epsilon \\ &\Longrightarrow n\ln(|c|) < \ln(\epsilon) \\ &\Longrightarrow n > \frac{\ln(\epsilon)}{\ln(|c|)} \\ &\Longrightarrow N = E\left(\frac{\ln(\epsilon)}{\ln(|c|)}\right) + 1. \end{aligned}$$

2. For each sequence determine the smallest value of N (see the below note), when $\epsilon = 0.001$, and $a = b = 2$ and $c = 1/2$.

Case of (V_n) **for** $a = 2$ **and** $\epsilon = 0.001$**:** By replacing the fixed values of a and ϵ in the expression of N, which we obtained in the first question, we will have:

$$N = E(693.4937) + 1 = 693 + 1 = 694.$$

Case of (W_n) for $b = 2$ and $\epsilon = 0.001$: By replacing the fixed values of b and ϵ in the expression of N, which we obtained in the first question, we will have:

$$N = E(3000) = 3000.$$

Case of (T_n) for $c = 1/2$ and $\epsilon = 0.001$ By replacing the fixed values of c and ϵ in the expression of N, which we obtained in the first question, we will have:

$$N = E(9.9657) + 1 = 9 + 1 = 10.$$

3. Prove, using the definition of the limit of a real sequence, that the sequences K_n and S_n are divergent, with

$$K_n = \frac{-n^2 + n + 1}{n + 1} \quad \text{and} \quad S_n = ln(ln(ln(n))).$$

Note that $\lim_{n\to\infty} K_n = -\infty$ and $\lim_{n\to\infty} S_n = +\infty$.

(a) Case of the sequence K_n: Using the definition of the limit we will have

$$\begin{aligned} K_n < A \quad &\Longrightarrow \quad \frac{-n^2 + n + 1}{n + 1} < A \\ &\Longrightarrow \quad \frac{(-n^2 - 2n - 2) + (2n + 2) + n}{n + 1} < A \\ &\Longrightarrow \quad -(n + 1) + 2 + \frac{n}{n + 1} < A \\ &\Longrightarrow \quad -(n + 1) + 3 < A \quad \left(\text{ since } \frac{n}{n + 1} < 1\right) \\ &\Longrightarrow \quad n > 2 - A. \\ &\Longrightarrow \quad N = E(2 - A) + 1. \end{aligned}$$

(b) Case of the sequence S_n: Before proceeding with the analysis of the divergence of the sequence S_n, it is first necessary to determine the domain of this sequence. To do this, let's define the following functions:

$$S_n = \ln(f(n)), \quad f(n) = \ln(g(n)) \quad \text{and} \quad g(n) = \ln(n).$$

S_n is defined if and only if:

$$f(n) > 0 \Longrightarrow \ln(g(n)) > 0 \Longrightarrow g(n) > 1 \Longrightarrow \ln(n) > 1 \Longrightarrow n > e \Longrightarrow n_0 = E(e) + 1 = 3.$$

We conclude that S_n is defined on $N/\{0, 1, 2\}$. Using the definition of the limit we will have

$$S_n > A \Longrightarrow ln(ln(ln(n))) > A \Longrightarrow n > e^{e^{e^A}} \Longrightarrow N = \max\left\{3, e^{e^{e^A}}\right\}.$$

Solution of the Exercise 4

In each of the following cases, determine the limit, if it exists.

- $\lim_{n\to+\infty} U_n = \lim_{n\to+\infty} \frac{n+(-1)^n}{n-(-1)^n} = \lim_{n\to+\infty} \frac{n\left(1+\frac{(-1)^n}{n}\right)}{n\left(1-\frac{(-1)^n}{n}\right)} = \lim_{n\to+\infty} \frac{1+\frac{(-1)^n}{n}}{1-\frac{(-1)^n}{n}} = 1 \quad (\text{since } \lim_{n\to+\infty} \frac{(-1)^n}{n} = 0).$

- To calculate the limit, simply multiply and divide U_n by the conjugate of $\sqrt{n+a}-\sqrt{n+b}$.

$$\begin{aligned}
\lim_{n\to+\infty} U_n &= \lim_{n\to+\infty}\left(\sqrt{n+a}-\sqrt{n+b}\right)\\
&= \lim_{n\to+\infty}\frac{\left(\sqrt{n+a}-\sqrt{n+b}\right)\left(\sqrt{n+a}+\sqrt{n+b}\right)}{\sqrt{n+a}+\sqrt{n+b}}\\
&= \lim_{n\to+\infty}\frac{n+a-n-b}{\sqrt{n+a}+\sqrt{n+b}} = \lim_{n\to+\infty}\frac{a-b}{\sqrt{n+a}+\sqrt{n+b}}\\
&= 0.
\end{aligned}$$

- To determine the limit of U_n in this case, we must compare a and b, where we distinguish three possible cases:

  Case of $a=b>0$: As $a^n-b^n=a^n-a^n=0 \Longrightarrow \forall n\in N,\ U_n=0$ then:

$$\lim_{n\to+\infty} U_n = \lim_{n\to+\infty}\frac{a^n-b^n}{a^n+b^n}=0.$$

  Case of $a>b>0$: As, for $a>b>0 \Longrightarrow \frac{b}{a}<1 \Longrightarrow \lim\limits_{n\to+\infty}\left(\frac{b}{a}\right)^n=0$ then

$$\lim_{n\to+\infty} U_n = \lim_{n\to+\infty}\frac{a^n-b^n}{a^n+b^n} = \lim_{n\to+\infty}\frac{a^n\left(1-\left(\frac{b}{a}\right)^n\right)}{a^n\left(1+\left(\frac{b}{a}\right)^n\right)} = \lim_{n\to+\infty}\frac{1-\left(\frac{b}{a}\right)^n}{1+\left(\frac{b}{a}\right)^n}=1$$

  Case of $0<a<b$: As, for $0<a<b \Longrightarrow \frac{a}{b}<1 \Longrightarrow \lim\limits_{n\to+\infty}\left(\frac{b}{a}\right)^n=0$ then

$$\lim_{n\to+\infty} U_n = \lim_{n\to+\infty}\frac{a^n-b^n}{a^n+b^n} = \lim_{n\to+\infty}\frac{b^n\left(\left(\frac{a}{b}\right)^n-1\right)}{b^n\left(\left(\frac{a}{b}\right)^n+1\right)} = \lim_{n\to+\infty}\frac{\left(\frac{a}{b}\right)^n-1}{\left(\frac{a}{b}\right)^n+1}=-1$$

- $U_n = 1-\frac{1}{a}+\frac{1}{a^2}-\frac{1}{a^3}+....+\frac{(-1)^n}{a^n}$, with $a>0$.
  It is easy to notice that U_n is the sum of $(n+1)th$ successive terms of a geometric sequence defined by the first term which is worth 1 and its ration $r=\frac{-1}{a}$, therefore U_n can be rewritten in the following form

$$U_n = \frac{1-\left(\frac{-1}{a}\right)^{n+1}}{1+\frac{1}{a}};$$

  so,

$$\lim_{n\to+\infty} U_n = \lim_{n\to+\infty}\frac{1-\left(\frac{-1}{a}\right)^{n+1}}{1+\frac{1}{a}} = \begin{cases}\frac{1}{1+\frac{1}{a}}, & \text{if } a\in]1,+\infty[;\\ \nexists, & \text{if } a\in]0,1].\end{cases}$$

- $\lim\limits_{n\to+\infty} U_n = \lim\limits_{n\to+\infty}\frac{n^2-2^n}{3^n} = \lim\limits_{n\to+\infty}\frac{n^2}{3^n}-\left(\frac{2}{3}\right)^n = \lim\limits_{n\to+\infty}\frac{n^2}{3^n}-\lim\limits_{n\to+\infty}\left(\frac{2}{3}\right)^n = 0-0=0.$

- $\lim\limits_{n\to+\infty} U_n = e^a$. Indeed, we have on one hand

$$\lim_{n\to+\infty} U_n = \lim_{n\to+\infty} e^{\ln(U_n)} = e^{\lim\limits_{n\to+\infty}\ln(U_n)}.$$

  On the other hand,

$$\lim_{n\to+\infty}\ln(U_n) = \lim_{n\to+\infty}\ln\left(\left(1+\frac{a}{n}\right)^n\right) = \lim_{n\to+\infty} n\ln\left(1+\frac{a}{n}\right) =?$$

We put $y = \frac{a}{n}$. So, when n tend to infinity y tend to zero. Then

$$\lim_{n\to+\infty} \ln(U_n) = \lim_{y\to 0} a \, \frac{\ln(1+y)}{y} = a.$$

Consequently,

$$\lim_{n\to+\infty} U_n = e^{\lim\limits_{n\to+\infty} \ln(U_n)} = e^a.$$

- It is easy to notice that U_n can be simplified as following:

$U_n = \sum\limits_{k=1}^{n} \frac{1}{\sqrt{k}} - \frac{1}{\sqrt{k+1}} = \sum\limits_{k=1}^{n} \frac{1}{\sqrt{k}} - \sum\limits_{k=1}^{n} \frac{1}{\sqrt{k+1}} = 1 + \sum\limits_{k=2}^{n} \frac{1}{\sqrt{k}} - \sum\limits_{k=2}^{n} \frac{1}{\sqrt{k}} - \frac{1}{\sqrt{n+1}} = 1 - \frac{1}{\sqrt{n+1}}$. Hence,

$$\lim_{n\to+\infty} U_n = \lim_{n\to+\infty} 1 - \frac{1}{\sqrt{n+1}} = 0.$$

- we have, for any real kx

$$\begin{aligned}
kx - 1 < E(kx) \leq kx \quad &\Longrightarrow \quad \sum_{k=1}^{n} kx - 1 < \sum_{k=1}^{n} E(kx) \leq \sum_{k=1}^{n} kx \\
&\Longrightarrow \quad \frac{2}{n^2}\sum_{k=1}^{n} (kx-1) < \frac{2}{n}\sum_{k=1}^{n} E(kx) \leq \frac{2}{n^2}\sum_{k=1}^{n} kx \\
&\Longrightarrow \quad \frac{2}{n^2}\sum_{k=1}^{n} kx - \frac{2}{n} < U_n \leq \frac{2}{n^2}\sum_{k=1}^{n} kx
\end{aligned}$$

And, as $\sum\limits_{k=1}^{n} kx = x\sum\limits_{k=1}^{n} k = x\frac{n(n+1)}{2}$ then after simplifications we get

$$\begin{aligned}
x\left(\frac{n+1}{n}\right) - \frac{2}{n} < U_n \leq x\left(\frac{(n+1)}{n}\right) \quad &\Longrightarrow \quad \lim_{n\to+\infty} x\left(\frac{n+1}{n}\right) - \frac{2}{n} < \lim_{n\to+\infty} U_n \leq \lim_{n\to+\infty} x\left(\frac{(n+1)}{n}\right) \\
&\Longrightarrow \quad x < \lim_{n\to+\infty} U_n \leq x \\
&\Longrightarrow \quad \lim_{n\to+\infty} U_n = x
\end{aligned}$$

Solution of the Exercise 5

Let $a > 0$. We define the sequence $\{U_n\}_{n\geq 0}$ by U_0 strictly positive real numbers and by the relation:

$$U_{n+1} = \frac{1}{2}\left(U_n + \frac{a}{U_n}\right).$$

1. Show that for all $n \geq 1$ we have $U_n \geq \sqrt{a}$.

   To show that $U_n > \sqrt{a}$ it is enough to show that for all $n \in N^*$ we have $U_{n+1} - \sqrt{a} > 0$.

   On one hand,

$$\begin{aligned}
U_{n+1} - \sqrt{a} &= \frac{1}{2}\left(U_n + \frac{a}{U_n}\right) - \sqrt{a} \\
&= \frac{U_n^2 - 2\sqrt{a}U_n + a}{2U_n} \\
&= \frac{(U_n - \sqrt{a})^2}{2U_n}
\end{aligned}$$

On the other hand, from the expression of (U_n), as $U_0 > 0$ then we deduce that $U_n > 0,\ \forall n \in N$.

Consequently, $\frac{(U_n-\sqrt{a})^2}{2U_n} = U_{n+1} - \sqrt{a} > 0 \Longrightarrow U_{n+1} > \sqrt{a}$. that means that:

$$\forall n \in N^*, \quad U_n > \sqrt{a}.$$

2. Show that $\{U_n\}_{n\geq 1}$ is a decreasing sequence.

$$\begin{aligned} U_{n+1} - U_n &= \frac{1}{2}\left(U_n + \frac{a}{U_n}\right) - U_n \\ &= \frac{U_n^2 + a - 2U_n^2}{2U_n} \\ &= \frac{a - U_n^2}{2U_n} < 0 \ \ (as\ U_n > \sqrt{a}) \end{aligned}$$

Therefore, the sequence (U_n) is a decreasing sequence.

3. Deduce that the sequence U_n converges to $\sqrt{a}$.

From the previous questions, we have shown that U_n is lower-bounded, moreover it is a decreasing sequence then U_n is a convergent sequence. Let's note $\lim\limits_{n\to+\infty} U_n = l$.

$$\begin{aligned} \lim_{n\to+\infty} U_{n+1} = \lim_{n\to+\infty} \frac{1}{2}\left(U_n + \frac{a}{U_n}\right) &\Longrightarrow l = \frac{1}{2}\left(l + \frac{a}{l}\right) \\ &\Longrightarrow l^2 = a \\ &\Longrightarrow \begin{cases} +\sqrt{a} \\ -\sqrt{a} & \text{rejected, because } U_n > 0. \end{cases} \end{aligned}$$

We conclude that $\lim\limits_{n\to+\infty} U_n = \sqrt{a}$.

Solution of the Exercise 6

1. Let $0 < a \leq b$. Prove the following inequalities:

   (a)

$$\begin{aligned} \left(\sqrt{a} - \sqrt{b}\right)^2 \geq 0 &\Longrightarrow a - 2\sqrt{ab} + b \geq 0 \\ &\Longrightarrow \sqrt{ab} \leq \frac{a+b}{2}. \end{aligned}$$

   (b)

$$\begin{aligned} a \leq b &\Longrightarrow \begin{cases} a + a & \leq & a + b \\ b + a & \leq & b + b \end{cases} \\ &\Longrightarrow \begin{cases} 2a & \leq & a + b \\ b + a & \leq & 2b \end{cases} \\ &\Longrightarrow 2a \leq a + b \leq 2b \\ &\Longrightarrow a \leq \frac{a+b}{2} \leq b \end{aligned}$$

(c)

$$\begin{aligned}
0 < a \leq b \implies & \left\{ \begin{array}{lcl} a * a & \leq & a * b \\ b * a & \leq & b * b \end{array} \right. \\
\implies & \left\{ \begin{array}{lcl} a^2 & \leq & ab \\ ab & \leq & b^2 \end{array} \right. \\
\implies & a^2 \leq ab \leq b^2 \\
\implies & a \leq \sqrt{ab} \leq b
\end{aligned}$$

2. Let U_0 and V_0 be strictly positive real numbers with $U_0 < V_0$. We define two sequences U_n and V_n as follow:

$$U_{n+1} = \sqrt{U_n V_n} \qquad \text{and} \qquad V_{n+1} = \frac{U_n + V_n}{2}.$$

(a) Let's use the proof by induction to show that $U_n < V_n$ for all $n \in \mathbb{N}$.

i. For $n = 0$, the proposition is true because we have $U_0 < V_0$.

ii. Suppose the proposition is true for n, that is, $U_n < V_n$.

iii. Let us now show that the proposition is true for $n+1$. As $0 < U_n < V_n$ then from the first inequality of the first question, it follows that $\sqrt{U_n V_n} < \frac{U_n + V_n}{2}$, therefore $U_{n+1} < V_{n+1}$. This means that the proposition is true for $n+1$.

We conclude that $U_n < V_n$ for all $n \in \mathbb{N}$.

(b) Show that V_n is a decreasing sequence.

Since $0 < U_n < V_n$ for any n, based on the right side of the second inequality of the first equation we deduce that

$$\frac{U_n + V_n}{2} < V_n \implies V_{n+1} < V_n.$$

By definition this means that (V_n) is a strictly decreasing sequence.

(c) Show that (U_n) is an increasing sequence.

Since $0 < U_n < V_n$ for any n, based on the left side of the third inequality of the first equation we deduce that

$$U_n < \sqrt{U_n V_n} \implies U_n < U_{n+1}.$$

By definition this means that (U_n) is a strictly increasing sequence.

(d) Deduce that the sequences U_n and V_n are convergent and have the same limit.

Before checking that the two sequences have the same limit, we must first check the existence of their limits, that is to say check if the two sequences are convergent.

- The sequence (V_n) is a decreasing sequence and it's bounded below $(0 < V_n)$ then we conclude that (V_n) is convergent. Let's denote $\lim\limits_{n \longrightarrow +\infty} V_n = l_v > 0$.
- As $U_n < V_n$ for any n then $U_n < l_v$. So, the sequence (U_n) is an increasing sequence and it's bounded above $(U_n < l_v)$. As a result, (U_n) is convergent. Let's denote $\lim\limits_{n \to +\infty} U_n = l_u > 0$.

Now let's check that the two sequences have the same limit. As the two sequences are convergent then the following statements are correct.

$$\left\{ \begin{array}{lcl} l_u & = & \sqrt{l_u l_v} \\ l_v & = & \frac{l_u + l_v}{2} \end{array} \right. \implies \left\{ \begin{array}{lcl} l_u^2 & = & l_u l_v \\ 2 l_v & = & l_u + l_v \end{array} \right. \implies \left\{ \begin{array}{lcl} l_u & = & l_v (\text{ because } l_u \neq 0) \\ l_v & = & l_u \end{array} \right.$$

Consequently,

$$\lim_{n \to +\infty} U_n = \lim_{n \to +\infty} V_n$$

Solution of the Exercise 7

We consider the two sequences:

$$U_n = 1 + \frac{1}{2!} + \frac{1}{3!} + ... + \frac{1}{n!} \quad \text{and} \quad V_n = U_n + \frac{1}{n!}$$

Show that U_n and V_n converge towards the same limit.

It should be noted that in this case it is insufficient to verify that the limit of the difference (respectively ratio) of the two sequences equals zero (respectively 1). Because obtaining these last two results hardly means that the two sequences are convergent.

Example: Let a_n and b_n be two numerical sequences defined by: $a_n = n$ and $b_n = \frac{(-1)^n - n^2}{n+1}$
The sequences a_n and b_n are divergent and carrying the limit of their difference is equal to 0.
Let's check the monotonicity of the sequences.

$$U_{n+1} - u_n = \left(1 + \frac{1}{2!} + \frac{1}{3!} + ... + \frac{1}{n!} + \frac{1}{(n+1)!}\right) - \left(1 + \frac{1}{2!} + \frac{1}{3!} + ... + \frac{1}{n!}\right) = \frac{1}{(n+1)!} > 0$$

We conclude that (U_n) is an increasing sequence.

$$\begin{aligned} V_{n+1} - V_n &= \left(U_{n+1} + \frac{1}{(n+1)!}\right) - \left(U_n + \frac{1}{n!}\right) \\ &= (U_{n+1} - U_n) + \left(\frac{1}{(n+1)!} - \frac{1}{n!}\right) \\ &= \frac{1}{n!}\left(\frac{2}{n+1} - 1\right) \leq 0. \end{aligned}$$

We conclude that (V_n) is a decreasing sequence.

$$\lim_{n\to+\infty} V_n - U_n = \lim_{n\to+\infty} U_n + \frac{1}{n!} - U_n = \lim_{n\to+\infty} \frac{1}{n!} = 0.$$

Therefore, (U_n) and (V_n) are adjacent sequences. As a result, they are convergent and have the same limit.

Solution of the Exercise 8

$I)$ If the approximate values of a real number x with precision $10^{-2},\ 10^{-3}, ...,\ 10^{-n}...$ are given by:

The sequence $1.23; 1.233; ....; 1.2333...3; ...$ can be rewritten as follows:

$$\begin{aligned} U_n &= 1.2 + \sum_{k=2}^{n} 3 * 10^{-k}, \quad \text{for all } n \geq 2 \\ &= 1.2 + 3 * 10^{-2}\left(\sum_{k=0}^{n-2} 10^{-k}\right) \\ &= 1.2 + 3 * 10^{-2}\left(\frac{1 - 10^{-n+1}}{1 - 10^{-1}}\right) \end{aligned}$$

Note that the exact value of x is nothing other than the limit of U_n . So,

$$x = \lim_{n\to+\infty} U_n = \lim_{n\to+\infty} 1.2 + 3 * 10^{-2}\left(\frac{1 - 10^{-n++1}}{1 - 10^{-1}}\right) = 1.2 + \frac{3 * 10^{-2}}{9 * 10^{-1}} = 1.2 + \frac{1}{30} = \frac{37}{30}.$$

then give the exact value of x.

$II)$ Consider the following sequences, defined for $n \in \mathbb{N}^*$:

$$U_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + ... + \frac{1}{n} \qquad \text{and} \qquad V_n = ln(n+1) - ln(n).$$

1. Calculate the limit of $S_n = \sum_{i=1}^{n} V_i$.

$$x = \lim_{n \to +\infty} S_n = x = \lim_{n \to +\infty} \sum_{i=1}^{n} V_i = \lim_{n \to +\infty} \ln(n+1) - \ln(1) = +\infty.$$

2. Show that, for all $n \in \mathbb{N}^*$ we have $V_n \leq \frac{1}{n}$.
   Let $f(x) = ln(x)$ then f is continuous function on $[n, n+1]$ and differentiable in $]n, n+1[$ for any $n \in N$. Mean value theorem (see chapter 3) says there exists a real number $c \in]n, n+1[$ such that

$$f'(c) = \frac{f(n+1) - f(n)}{(n+1) - n} = \frac{\ln(n+1) - \ln(n)}{(n+1) - n} = \ln(n+1) - \ln(n).$$

   But we know that $f'(c) = (\ln(c))' = \frac{1}{c}$ and from $n < c < n+1$ we have

$$\frac{1}{n+1} < \frac{1}{c} < \frac{1}{n} \Longrightarrow \frac{1}{n+1} < f'(c) < \frac{1}{n} \Longrightarrow \frac{1}{n+1} < \ln(n+1) - \ln(n) < \frac{1}{n} \Longrightarrow V_n \leq \frac{1}{n}.$$

3. What can we conclude about the nature of U_n?
   From the second question we deduce that

$$S_n \leq 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + ... + \frac{1}{n} \Longrightarrow S_n \leq U_n \Longrightarrow \lim_{n \to +\infty} S_n \leq \lim_{n \to +\infty} U_n \Longrightarrow +\infty \leq \lim_{n \to +\infty} U_n.$$

   This means that (U_n) is a divergent sequence because $\lim_{n \to +\infty} U_n = +\infty$.