University of Biskra
Mathematics Department
Module: Analysis 1

First year license
2024/2025

Continuous assessment
Test N° 1

Exercise 1

1. Prove by induction that $n^3 + 2n$ is divisible by 3 for every $n \in \mathbb{N}^*$.
2. n is an odd natural number $\Leftrightarrow n^2$ is an odd natural number.

Indication : To prove equivalence, prove the left implication and the right implication separately.

Exercise 2

Consider the following two complex numbers.

$$u = 1 + i \quad \text{and} \quad v = 1 + i\sqrt{3}$$

1. Determine the modulus of u and v.
2. Determine an argument of u and an argument of v.
3. Deduce the module and an argument for each of the cubic roots of u.
4. Determine the modulus and an argument of $\frac{u}{v}$.
5. Deduce the values of

$$\cos\left(\frac{-5\pi}{12}\right) \quad \text{and} \quad \sin\left(\frac{-5\pi}{12}\right).$$

Good luck.

Solution of Continuous assessment Test N° 1

Solution of the Exercise 1 ./ ***(04pts)***

1. To show that the given assertion is true we must check the three following steps:

   (a) Check that the proposition is true for $n = n_0$.

   (b) Assume that the proposition is true for any n.

   (c) Check that the proposition is true for $n + 1$.

   Note that a nonnegative integer number N is divisible by 3 if and only if

   $$\exists k \in \mathbb{N}^*, \ N = 3k.$$

   So the proof consists of checking the existence of the natural number k.

   $P(n = 1)$?

   For $n = 1$ we have $n^3 + 2n = 1 \times 3 \Longrightarrow$ the proposition is true for $n = 1$. **(0.5pts)**

   $P(n)$?

   Let's assume that the proposition is true for any n i.e. **(0.5pts)**

   $$\text{for } n \in N^*, \exists k \in N^* \text{ such that } n^3 + 2n = 3k$$

   $P(n + 1)$?

   For $n + 1$ we have

   $$\begin{aligned} (n+1)^3 + 2(n+1) &= n^3 + 3n^2 + 3n + 1 + 2n + 2 \\ &= \left(n^3 + 2n\right) + 3\left(n^2 + n + 1\right) \\ &= 3k + 3\left(n^2 + n + 1\right) \quad \text{(from the assumption that } P(n) \text{ is true)} \\ &= 3\left(n^2 + n + k + 1\right) \\ &= 3k' \end{aligned}$$

   As n and k are $\in \mathbb{N}^*$ then $k' = n^2 + n + k + 1 \in \mathbb{N}^*$, consequently $P(n + 1)$ is true. **(0.5pts)**

   From the above results we conclude that the following statement is always true.

For every $n \in \mathbb{N}^*$ the quantity $n^3 + 2n$ is divisible by 3. **(0.5pts)**

2. Before proceeding to demonstrate the equivalence, let's recall that:

   - A natural number can only be an odd number or an even number.
   - We say that a natural number n is an odd number if there exists a natural number k such that $n = 2k + 1$.
   - We say that a natural number n is an even number if there exists a natural number k such that $n = 2k$.

   $\Longrightarrow$**?**

$$\begin{aligned} n\text{is an odd number} &\Longrightarrow \exists k \in \mathbb{N},\ n = 2k+1 \\ &\Longrightarrow n^2 = (2k+1)^2 \\ &\Longrightarrow n^2 = 4k^2 + 4k + 1 \\ &\Longrightarrow n^2 = 2(2k^2 + 2k) + 1 = 2k' + 1 \\ &\Longrightarrow \text{as } k' \in \mathbb{N} \text{ then } n^2 \text{ is an odd number} \end{aligned}$$

   So, the right implication is always true. **(0.5pts)**

   $\Longleftarrow$**?**
   In this case, we seek to prove the following:

$$n^2 \text{ is an odd number} \Longrightarrow n \text{ is an odd number}$$

   It is difficult to prove this implication directly, so let us proceed on to the proof by the contrapositive. That means we must prove that

$$n \text{ is not an odd number} \Longrightarrow\ n^2 \text{ is not an odd number}$$

   In other words, we must show that:

$$n \text{ is an even number} \Longrightarrow n^2 \text{ is an even number} \quad \textbf{(0.5pts)}$$

$$\begin{aligned} n\text{is an even number} &\Longrightarrow \exists k \in \mathbb{N},\ n = 2k \\ &\Longrightarrow n^2 = (2k)^2 \\ &\Longrightarrow n^2 = 4k^2 \\ &\Longrightarrow n^2 = 2(2k^2) = 2k' \\ &\Longrightarrow \text{as } k' \in N \text{ then } n^2 \text{ is an even number} \end{aligned}$$

   So, the left implication is always true. **(0.5pts)**

   From the results of the proofing of the two implications we conclude that the following equivalence is always true.

n is an odd natural number $\Leftrightarrow n^2$ is an odd natural number. **(0.5pts)**

Solution of the Exercise 2 ./ ***(04pts)***

1. The modulus of u and v:

$$|u| = \sqrt{1^2 + 1^2} = \sqrt{2}\ \textbf{(0.5pts)} \quad \text{and} \quad |v| = \sqrt{1^2 + \sqrt{3}^2} = 2.\ \textbf{(0.5pts)}$$

2. The arguments of u and v:

   We have

$$u = 1 + i = \sqrt{2}\left(\frac{\sqrt{2}}{2} + i\ \frac{\sqrt{2}}{2}\right) = r\left(\cos(\theta) + i\sin(\theta)\right).$$

   This means that:

$$\begin{cases} \cos(\theta) & = & \frac{\sqrt{2}}{2} \\ \sin(\theta) & = & \frac{\sqrt{2}}{2} \end{cases} \Longrightarrow \theta = \frac{\pi}{4}.\ \textbf{(0.5pts)}$$

   We have

$$v = 1 + i\ \sqrt{3} = 2\left(\frac{1}{2} + i\ \frac{\sqrt{3}}{2}\right) = r\left(\cos(\theta) + i\sin(\theta)\right).$$

   This means that:

$$\begin{cases} \cos(\theta) & = & \frac{1}{2} \\ \sin(\theta) & = & \frac{\sqrt{3}}{2} \end{cases} \Longrightarrow \theta = \frac{\pi}{3}.\ \textbf{(0.5pts)}$$

3. Let's z a complex number such that $z = \sqrt[3]{u}$. From the above results, we can rewrite u as follows: $u = \sqrt{2}e^{i\frac{\pi}{4}}$. Then,

$$\begin{cases} |z^3| & = & |u| \\ arg(z^3)| & = & arg(u) \end{cases} \Longrightarrow \begin{cases} |z|^3 & = & \sqrt{2} = 2^{\frac{1}{2}} \\ 3arg(z)| & = & \frac{\pi}{4} + 2k\pi,\ \text{with } k \in \mathbb{Z} \end{cases}$$
$$\Longrightarrow \begin{cases} |z| & = & 2^{\frac{1}{6}} \\ arg(z)| & = & \frac{\pi}{12} + \frac{2}{3}k\pi,\ \text{with } k \in \mathbb{Z} \end{cases} \quad \textbf{(0.5pts)}$$

   So, to obtain the three cubic roots of u, we just need to take (for example) $k \in \{0,\ 1,\ ,2\}$. For this value, we obtain the following cubic roots.

$$\sqrt[3]{u} \in \left\{2^{\frac{1}{6}}e^{i\frac{\pi}{12}},\ 2^{\frac{1}{6}}e^{i\frac{3\pi}{4}},\ 2^{\frac{1}{6}}e^{i\frac{17\pi}{12}}\right\}.\ \textbf{(0.5pts)}$$

4. The modulus and argument of $\frac{u}{v}$. We have,

$$\begin{aligned} w = \frac{u}{v} = \frac{\sqrt{2}e^{i\frac{\pi}{4}}}{2e^{i\frac{\pi}{3}}} &= \frac{\sqrt{2}}{2}e^{i\left(\frac{\pi}{4} - \frac{\pi}{3}\right)} \\ &= \frac{\sqrt{2}}{2}e^{i\frac{-\pi}{12}} = re^{i\theta} \\ &\Longrightarrow \left\{ \begin{array}{lcl} |w| & = & \frac{\sqrt{2}}{2} \ \textbf{(0.5pts)} \\ arg(w)| & = & \frac{-\pi}{12} \ \textbf{(0.5pts)} \end{array} \right. . \end{aligned}$$

5. The question has been eliminated.

Solution of the Exercise 3 *Solution of the original exercise*

In the original exercise the complex numbers u and v are given as follows:

$$u = 1 + i \quad \text{and} \quad v = -1 + i\sqrt{3}$$

1. The modulus of u and v:

$$|u| = \sqrt{1^2 + 1^2} = \sqrt{2} \quad \text{and} \quad |v| = \sqrt{(-1)1^2 + \sqrt{3}^2} = 2.$$

2. The arguments of u and v: We have

$$u = 1 + i = \sqrt{2}\left(\frac{\sqrt{2}}{2} + i\,\frac{\sqrt{2}}{2}\right) = r\left(\cos(\theta) + i\sin(\theta)\right).$$

This means that:

$$\left\{ \begin{array}{lcl} \cos(\theta) & = & \frac{\sqrt{2}}{2} \\ \sin(\theta) & = & \frac{\sqrt{2}}{2} \end{array} \right. \Longrightarrow \theta = \frac{\pi}{4}.$$

We have

$$v = -1 + i\,\sqrt{3} = 2\left(\frac{-1}{2} + i\,\frac{\sqrt{3}}{2}\right) = r\left(\cos(\theta) + i\sin(\theta)\right)$$

This means that:

$$\left\{ \begin{array}{lcl} \cos(\theta) & = & -\frac{1}{2} \\ \sin(\theta) & = & \frac{\sqrt{3}}{2} \end{array} \right. \Longrightarrow \theta = \frac{2\pi}{3}.$$

3. Let's z a complex number such that $z = \sqrt[3]{u}$.

   From the above results, we can rewrite u as follows: $u = \sqrt{2}e^{i\frac{\pi}{4}}$. Then,

$$\left\{\begin{array}{lcl} |z^3| & = & |u| \\ arg(z^3)| & = & arg(u) \end{array}\right. \Longrightarrow \left\{\begin{array}{lcl} |z|^3 & = & \sqrt{2} = 2^{\frac{1}{2}} \\ 3arg(z)| & = & \frac{\pi}{4} + 2k\pi, \text{ with } k \in \mathbb{Z} \end{array}\right.$$
$$\Longrightarrow \left\{\begin{array}{lcl} |z| & = & 2^{\frac{1}{6}} \\ arg(z)| & = & \frac{\pi}{12} + \frac{2}{3}k\pi, \text{ with } k \in \mathbb{Z} \end{array}\right.$$

   So, to obtain the three cubic roots of u, we just need to take (for example) $k \in \{0,\ 1,\ ,2\}$. And in this case, we obtain the following cubic roots.

$$\sqrt[3]{u} \in \left\{2^{\frac{1}{6}}e^{i\frac{\pi}{12}},\ 2^{\frac{1}{6}}e^{i\frac{3\pi}{4}},\ 2^{\frac{1}{6}}e^{i\frac{17\pi}{12}}\right\}.$$

4. The modulus and argument of $\frac{u}{v}$.

   We have,

$$w = \frac{u}{v} = \frac{\sqrt{2}e^{i\frac{\pi}{4}}}{2e^{i\frac{2\pi}{3}}} = \frac{\sqrt{2}}{2}e^{i\left(\frac{\pi}{4}-\ \frac{2\pi}{3}\right)} = \frac{\sqrt{2}}{2}e^{i\frac{-5\pi}{12}} = re^{i\theta} \Longrightarrow \left\{\begin{array}{lcl} |w| & = & \frac{\sqrt{2}}{2} \\ arg(w)| & = & \frac{-5\pi}{12} \end{array}\right. .$$

5. On one hand we have

$$\frac{u}{v} = \frac{\sqrt{2}}{2}e^{i\frac{-5\pi}{12}} = \frac{\sqrt{2}}{2}\left(\cos\left(\frac{-5\pi}{12}\right) + i\sin\left(\frac{-5\pi}{12}\right)\right).$$

   On the other hand we have

$$\begin{aligned} \frac{u}{v} &= \frac{1+i}{-2+i\sqrt{3}} = \frac{(1+i)}{(-1+i\sqrt{3})}\frac{(-1-i\sqrt{3})}{(-1-i\sqrt{3})} \\ &= \frac{-1+\sqrt{3}}{4} + i\frac{-1-\sqrt{3}}{4} \\ &= \frac{\sqrt{2}}{2}\left(\frac{-1+\sqrt{3}}{2\sqrt{2}} + i\frac{-1-\sqrt{3}}{2\sqrt{2}}\right). \end{aligned}$$

Consequently,

$$\left\{\begin{array}{lcl} \cos\left(\frac{-5\pi}{12}\right) & = & \frac{-1+\sqrt{3}}{2\sqrt{2}} = \frac{-\sqrt{2}+\sqrt{6}}{4} \\ \sin\left(\frac{-5\pi}{12}\right) & = & \frac{-1+\sqrt{3}}{2\sqrt{2}} = \frac{-\sqrt{2}-\sqrt{6}}{4} \end{array}\right.$$

The nth roots for a complex number

In this passage, we explain the general way of determining the nth roots of a complex number. To do this, consider the two complex numbers u and z such that $z = \sqrt[n]{u} = u^{\frac{1}{n}}$ with $n \in \mathbb{N}^*$.

First of all, let's point out that the complex number u has n roots in this case.

To avoid tedious and complex calculations, when determining the n roots of u, it is more judicious to use the exponential form of u and z rather than their Cartesian forms. And, the steps to follow are as follows:

Step 1: Write z in its exponential form $u = re^{i\theta}$

Step 2: simplify the system

$$\left\{\begin{array}{lcl} |z^n| & = & |u| \\ arg(z^n) & = & arg(z) + 2k\pi \end{array}\right. \Longrightarrow \left\{\begin{array}{lcl} |z|^n & = & r \\ n\ arg(z) & = & \theta + 2k\pi \end{array}\right.$$
$$\Longrightarrow \left\{\begin{array}{lcl} |z| & = & r^{\frac{1}{n}} \\ arg(z) & = & \frac{\theta+2k\pi}{n} \end{array}\right.$$

Step 3: determine the set of the roots.

The exponential general form of a root of u is $z_k = r^{\frac{1}{n}} e^{i\frac{\theta+2k\pi}{n}}$.

At this level, to obtain the set, S, of all the roots, it is enough to replace k by n successive integer values. For example, take $k = \overline{0 : n-1}$.

$$S = \{z_0, z_1, ..., z_{n-1}\}.$$

Example 1

Let's $u = 1 + i$. find all values of v and w such that $v = \sqrt{u}$ et $\sqrt[4]{u}$.

$$u = 1 + i = \sqrt{2}\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) = \sqrt{2}\left(\cos\left(\frac{\pi}{4}\right) + i\cos\left(\frac{\pi}{4}\right)\right) = \sqrt{2}e^{i\frac{\pi}{4}}$$

v=?

$$\left\{\begin{array}{lcl} |v^2| & = & |u| \\ arg(z^2) & = & arg(z) + 2k\pi \end{array}\right. \Longrightarrow \left\{\begin{array}{lcl} |v|^2 & = & \sqrt{2} \\ 2\ arg(v) & = & \frac{\pi}{4} + 2k\pi \end{array}\right.$$
$$\Longrightarrow \left\{\begin{array}{lcl} |v| & = & 2^{\frac{1}{4}} \\ arg(v) & = & \frac{\pi}{8} + k\pi \end{array}\right.$$

So, the set of the values of v is given by:

$$u \in \left\{2^{\frac{1}{4}}e^{i\frac{\pi}{8}+0\ \pi},\ 2^{\frac{1}{4}}e^{i\frac{\pi}{8}+1\ \pi}\right\} = \left\{2^{\frac{1}{4}}e^{i\frac{\pi}{8}},\ 2^{\frac{1}{4}}e^{i\frac{9\pi}{8}}\right\}$$

w=?

$$\begin{cases} |w^4| & = & |u| \\ arg(z^4) & = & arg(z) + 2k\pi \end{cases} \Longrightarrow \begin{cases} |v|^4 & = & \sqrt{2} \\ 4\ arg(v) & = & \frac{\pi}{4} + 2k\pi \end{cases}$$
$$\Longrightarrow \begin{cases} |v| & = & 2^{\frac{1}{8}} \\ arg(v) & = & \frac{\pi+8k\pi}{16} \end{cases}$$

In this case, we have four roots so we take four values of k If we set $k = \overline{0:3}$, then the set of the values of v is given by:

$$w \in \left\{ 2^{\frac{1}{8}} e^{i\frac{\pi}{16}},\ \ 2^{\frac{1}{8}} e^{i\frac{9\pi}{16}},\ \ 2^{\frac{1}{8}} e^{i\frac{17\pi}{16}},\ \ 2^{\frac{1}{8}} e^{i\frac{25\pi}{16}} \right\}.$$