University of Biskra
Mathematics Department
Module: Analysis 1

First year license
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Worksheet No. 2

Exercise 1 Let $z \in \mathbb{C}$, $x,\ y \in \mathbb{R}$, $r \in \mathbb{R}_+^*$, $\theta \in [0; 2\pi[$ and $i^2 = -1$.

1. Rewrite each z into polar form ($re^{i\theta}$).

$$a)\ z = 6, \quad b)\ z = 6i, \quad c)\ z = 2+2i, \quad d)\ z = -2+2i, \quad e)\ z = 3+\sqrt{3}i.$$

2. Rewrite z from polar into $x+iy$ form.

$$a)\ z = 3e^{\frac{5\pi}{4}i}, \quad b)\ z = 5e^{\frac{7\pi}{4}i}, \quad c)\ z = re^{\frac{\pi}{12}i}, \quad d)\ z = \sqrt{16e^{\frac{2\pi}{3}i}}.$$

3. Compute the following, simplifying the results into $x+iy$ form.

$$a)\ z = (2+2i)^8, \quad b)\ \sqrt{3+\sqrt{3}i}.$$

4. Let $z = \sqrt{\frac{(1+i)}{\sqrt{2}}}$,
   (a) Compute z, and simplifying the results into $x+iy$ form.
   (b) Deduce the values of $cos(\frac{\pi}{8})$ and $sin(\frac{\pi}{8})$

Exercise 2 Let $z,\ w \in \mathbb{C}$ and $i^2 = -1$.

1. Let $z = 1+i$ and $w = z^n$ with $n \in \mathbb{Z}$.
   (a) Determine the values of n for which w is a pure imaginary number ($Re(w) = 0$).
   (b) Determine the values of n for which w is a real number ($Im(w) = 0$).
2. Let $w = \frac{z-i}{z+1}$ with $z \neq -1$. Determine the set of points M with affix z of which
   (a) w is a pure imaginary number ($Re(w) = 0$).
   (b) w is a real number ($Im(w) = 0$).

Exercise 3 Let $z,\ z_0 = x_0 + y_0\, i \in \mathbb{C}$, $r \in \mathbb{R}_+^*$ and $i^2 = -1$. Solve the following inequations.

1. $|z - z_0| \leq r$.
2. $|2z + i| \leq |\overline{z} + 1|$.
3. $\left|\frac{z-3}{z-5}\right| \leq r$, with $z \neq 5$(Left to the student).
4. $|2z + z_0| \leq |\overline{z} + z_1|$, with $z_0,\ z_1 \in \mathbb{C}$ (Left to the student).

Exercise 4 Let $z \in \mathbb{C}$ and $i^2 = -1$. Solve the following equations

$$a)\ 5z + 2i = (i+1)z - 3, \quad b)\ \frac{z-i}{z+1} = 4i, \quad c)\ 2z + i\overline{z} = 3, \quad d)\ z^2 + z\overline{z} = 0.$$

$$e)\ z^2 + 2z + 2 = 0, \quad f)\ -2z^2 + 6z - 5 = 0, \quad g)\ 2z^2 - z(1+5i) - 2(1-i) = 0.$$

Exercise 5 We consider the following polynomial $P(z) = z^3 + 9iz^2 + 2(6i - 11)z - 3(4i + 12)$, with $Z \in \mathbb{C}$.

1. Demonstrate that the equation $P(z) = 0$ admits a real solution z_1.
2. Determine a polynomial $Q(z)$ such that $P(z) = (z - z_1)Q(z)$.
3. Solve the equation $P(z) = 0$ in $\mathbb{C}$.
4. Demonstrate that the points of the complex plane corresponding to the solutions of the equation $P(z) = 0$ are aligned.

Exercise 6 Let Z_n be a complex number defined by:

$$Z_n = \begin{cases} 8, & \text{if } n = 0; \\ \frac{1+i\sqrt{3}}{4} Z_{n-1}, & \text{else.} \end{cases}$$

and $(M_n)_{n\in\mathbb{N}}$ are the points of affix Z_n on the complex plane $\mathbf{P}$.

1. Calculate z based on n.
2. For any natural number n, calculate the ratio
$$\frac{Z_n - Z_{n-1}}{Z_n}.$$
3. We note $|Z_n| = r_n$, gives the limit of r_n when n tends towards infinity. What geometric interpretation can we give?

Exercise 7

1. Show that
$$\forall\ u,\ v \in \mathbb{C}: \quad |u + v|^2 + |u - v|^2 = 2\left(|u|^2 + |v|^2\right)$$
2. Show that the following equivalence is false
$$\text{for } u \in \mathbb{C} \text{ and } v \in \mathbb{C} : u = v \Leftrightarrow |u| = |w|.$$

Exercise 8 *(Left to the student).* We consider the following polynomial $P(z) = z^3 + 2(\sqrt{2} - 1)z^2 - 4(\sqrt{2} - 1)z - 8$, with $z \in \mathbb{C}$.

1. Compute $P(2)$. Determine a factorization of $P(z)$ by $(z - 2)$.
2. Solve the equation $P(z) = 0$ in $\mathbb{C}$.

Exercise 9 *(Left to the student)* We consider the function f of the plane which at any point M associates the affix point:

$$w = \frac{z + i}{z - 2i}, \quad \text{with } z \neq 2i.$$

1. For $z \neq 2i$, we set $z = 2i + re^{i\theta}$, with and $r > 0$ and $\theta \in [0; 2\pi[$. Write $w - 1$ using r and θ.
2. A is the affix point $2i$,
   (a) Determine the set E_1 of points M for which $|w - 1| = 3$.
   (b) Determine the set E_2 of points M for which $arg(w - 1) = \frac{\pi}{4}$.
   (c) Represent the sets E_1 and E_2

Correction of worksheet No. 2

Course note: Let's consider the complex number $z = x + iy$.

- $\overline{z} = x - iy$ (See figure 1, left side).
- $\forall (z_1, z_2) \in \mathbb{C}^2, \overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$
- The modulus of z is defined by: $|z| = \sqrt{x^2 + y^2}$.
- For any complex number we have : $|z|^2 = z\,\overline{z}$.
- For any integer n we have $|z^n| = |z|^n$.
- The polar form of a complex number $z = x + iy$ is given by

$$z = r(\cos(\theta) + i\sin(\theta)), \text{ with } r = |z| \geq 0 \text{ and } \theta = arg(z) \in \mathbb{R} \text{ (see figure 1, right side).}$$

- The exponential form of a complex number $z = x + iy$ is given by

$$z = re^{i\theta}, \text{ with } r = |z| \geq 0 \text{ and } \theta = arg(z) \in \mathbb{R}.$$

- If $z = re^{i\theta}$ then

$$z = r\left(\cos(\theta) + i\sin(\theta)\right).$$

- For any natural number n we have

$$z^n = r^n e^{in\theta} = r^n \left(\cos(n\theta) + i\sin(n\theta)\right).$$

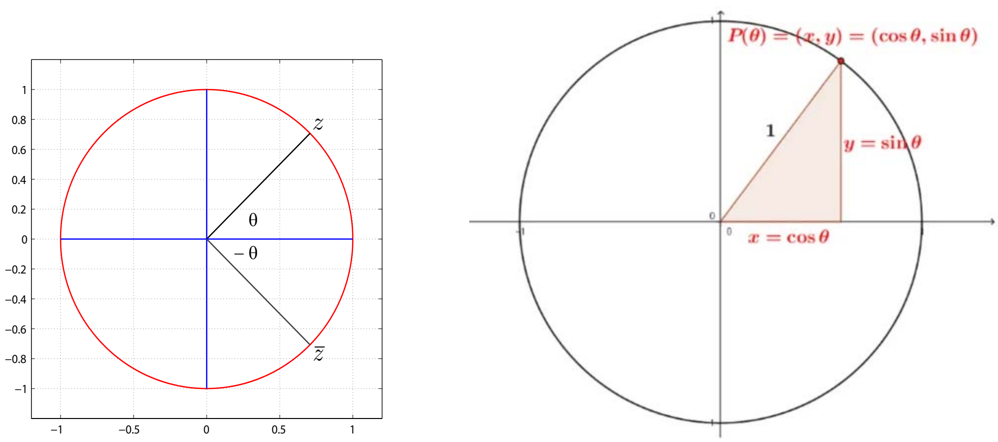
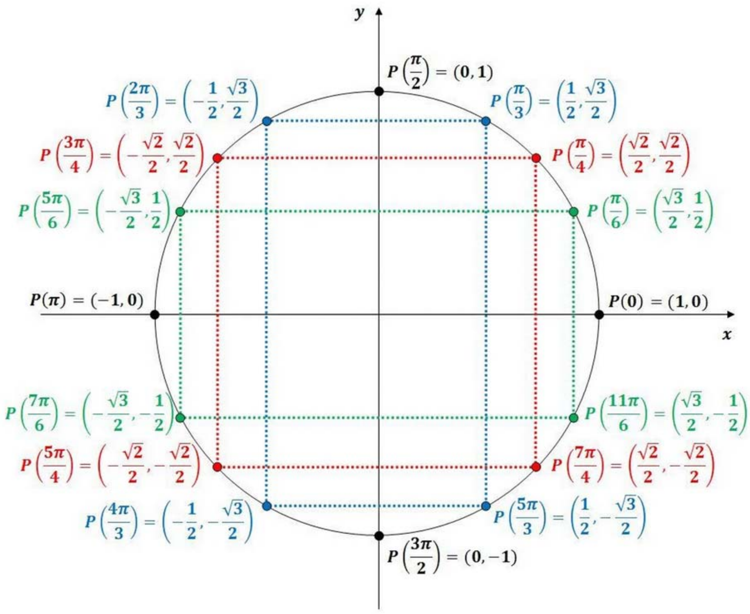


Figure 1: Graphical illustrations of the conjugate of a complex number and its trigonometric form.



Solution of the Exercise 1

1. Rewrite each z into polar form ($re^{i\theta}$).

   (a) $z = 6 = 6 + 0i \Rightarrow |z| = \sqrt{6^2 + 0^2} = 6$. So,

   $$z = 6(1 + 0i) = 6(cos(\theta) + i\sin(\theta)) \Rightarrow \begin{cases} \cos(\theta) = 1 \\ \sin(\theta) = 0 \end{cases} \Rightarrow \theta = 2\pi k,\ k \in Z.$$

   As $\theta \in [0, 2\pi[$ then $\theta = 0 \Rightarrow z = 6(cos(0) + i\sin(0))$.

   (b) $z = 6i = 0 + 6i \Rightarrow |z| = \sqrt{0^2 + 6^2} = 6$. So,

   $$z = 6(0 + i) = 6(cos(\theta) + i\sin(\theta)) \Rightarrow \begin{cases} \cos(\theta) = 0 \\ \sin(\theta) = 1 \end{cases} \Rightarrow \theta = \frac{\pi}{2} \Rightarrow z = 6(cos\left(\frac{\pi}{2}) + i\sin(\frac{\pi}{2})\right).$$

   (c) $z = 2 + 2i \Rightarrow |z| = \sqrt{2^2 + 2^2} = 2\sqrt{2}$. So,

   $$z = 2\sqrt{2}\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) = 2\sqrt{2}(cos(\theta) + i\sin(\theta)) \Rightarrow \begin{cases} \cos(\theta) = \frac{\sqrt{2}}{2} \\ \sin(\theta) = \frac{\sqrt{2}}{2} \end{cases} \Rightarrow \theta = \frac{\pi}{4}$$

   Hence,

   $$z = 2\sqrt{2}(cos\left(\frac{\pi}{4}) + i\sin(\frac{\pi}{4})\right).$$

   (d) $z = -2 + 2i \Rightarrow |z| = \sqrt{(-2)^2 + 2^2} = 2\sqrt{2}$. So,

   $$z = 2\sqrt{2}\left(\frac{-\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) = 2\sqrt{2}(cos(\theta) + i\sin(\theta)) \Rightarrow \begin{cases} \cos(\theta) = \frac{-\sqrt{2}}{2} \\ \sin(\theta) = \frac{\sqrt{2}}{2} \end{cases} \Rightarrow \theta = \frac{3\pi}{4}.$$

   Hence,

   $$z = 2\sqrt{2}\left(cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right).$$

(e) $z = 3+\sqrt{3}i \Rightarrow |z| = \sqrt{3^2+(\sqrt{3})^2} = 2\sqrt{3}$. So,

$$z = 2\sqrt{3}\left(\frac{\sqrt{3}}{2}+i\frac{1}{2}\right) = 2\sqrt{3}(cos(\theta)+i\sin(\theta)) \Rightarrow \begin{cases} \cos(\theta) = \frac{\sqrt{3}}{2} \\ \sin(\theta) = \frac{\sqrt{1}}{2} \end{cases} \Rightarrow \theta = \frac{\pi}{6}.$$

Hence,

$$z = 2\sqrt{3}\left(cos(\frac{\pi}{6})+i\sin(\frac{\pi}{6})\right).$$

2. Rewrite z from polar into $x+y\,i$ form.

(a) $z = 3e^{\frac{5\pi}{4}i}$

$$\begin{aligned} z &= 3\left(\cos\left(\frac{5\pi}{4}\right)+i\sin\left(\frac{5\pi}{4}\right)\right) \\ &= 3\left(\cos\left(\pi+\frac{\pi}{4}\right)+i\sin\left(\pi+\frac{\pi}{4}\right)\right) \\ &= 3\left(-\cos\left(\frac{\pi}{4}\right)+i\sin\left(\frac{\pi}{4}\right)\right) \\ &= \frac{-3\sqrt{2}}{2}+i\frac{3\sqrt{2}}{2}. \end{aligned}$$

(b) $z = 5e^{\frac{7\pi}{4}i}$

$$\begin{aligned} z &= 5\left(\cos\left(\frac{7\pi}{4}\right)+i\sin\left(\frac{7\pi}{4}\right)\right) \\ &= 5\left(\cos\left(2\pi-\frac{\pi}{4}\right)+i\sin\left(2\pi-\frac{\pi}{4}\right)\right) \\ &= 5\left(\cos\left(-\frac{\pi}{4}\right)+i\sin\left(-\frac{\pi}{4}\right)\right) \\ &= \frac{5\sqrt{2}}{2}-i\frac{5\sqrt{2}}{2}. \end{aligned}$$

(c) $z = re^{\frac{\pi}{12}i}$

$$\begin{aligned} z &= re^{\frac{\pi}{12}i} \\ &= 5\left(\cos\left(2\pi-\frac{\pi}{4}\right)+i\sin\left(2\pi-\frac{\pi}{4}\right)\right) \end{aligned}$$

$$\begin{cases} \cos\left(\frac{\pi}{12}\right) &= ? \\ \sin\left(\frac{\pi}{12}\right) &= ? \end{cases}$$

We have

$$\begin{cases} \cos^2(\theta)+\sin^2(\theta) = 1 \\ \cos(2\theta) = \cos^2(\theta)-\sin^2(\theta) \end{cases} \Longrightarrow \begin{cases} \cos^2(\theta) = 1-\sin^2(\theta) \\ \cos(2\theta) = 1-2\sin^2(\theta) \end{cases}$$

$$\Longrightarrow \begin{cases} \cos(\theta) = \sqrt{1-\sin^2(\theta)} \\ \sin(\theta) = \sqrt{\frac{1-\cos(2\theta))}{2}}. \end{cases}$$

Hence, if we put $\theta = \frac{\pi}{12}$ then we will have the following

$$\begin{cases} \cos(\frac{\pi}{12}) = \sqrt{1-\sin^2(\frac{\pi}{12})} \\ \sin(\frac{\pi}{12}) = \sqrt{\frac{1-\cos(\frac{\pi}{6}))}{2}} \end{cases} \implies \begin{cases} \cos\left(\frac{\pi}{12}\right) = \frac{\sqrt{2+\sqrt{3}}}{2} \\ \sin\left(\frac{\pi}{12}\right) = \frac{\sqrt{2-\sqrt{3}}}{2} \end{cases}$$

So,

$$z = r\left(\frac{\sqrt{2+\sqrt{3}}}{2}\right) + i\, r\left(\frac{\sqrt{2-\sqrt{3}}}{2}\right).$$

(d) $z = \sqrt{16e^{\frac{2\pi}{3}i}}$

$$\begin{aligned} z &= \sqrt{16e^{\frac{2\pi}{3}i}} = 4e^{\frac{\pi}{3}i} \\ &= 4\left(\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right) \\ &= 2 + i2\sqrt{3}. \end{aligned}$$

3. Compute the following, then simplifying the results into $x + y\ i$ form.

(a) Case of $z = (2+2i)^8$
Let $w = 2+2i$. Note that $|w| = \sqrt{2^2+2^2} = 2\sqrt{2}$ and $arg(w) = \frac{\pi}{4}$, so

$$\begin{aligned} z &= w^8 = (2+2i)^8 \\ &= \left[2\sqrt{2}\left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right)\right]^8 \\ &= \left(2\sqrt{2}\right)^8\left(\cos\left(\frac{8\pi}{4}\right) + i\sin\left(\frac{8\pi}{4}\right)\right) \\ &= \left(2\sqrt{2}\right)^8 = 4096. \end{aligned}$$

(b) Case of $(3+\sqrt{3}i)^{\frac{3}{2}}$
Let $w = 3+\sqrt{3}$.
Note that $|w| = \sqrt{3^2+\sqrt{3}^2} = 2\sqrt{3}$ and $arg(w) = \frac{\pi}{6}$, so

$$\begin{aligned} z &= (3+\sqrt{3}i)^{\frac{3}{2}} = w^{\frac{3}{2}} \\ &= \left[2\sqrt{3}\left(\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right)\right]^{\frac{3}{2}} \\ &= \left(2\sqrt{3}\right)^{\frac{3}{2}}\left(\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right) \\ &= \left(2\sqrt{3}\right)^{\frac{3}{2}}\left(-\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right) \\ &= \left(2\sqrt{3}\right)^{\frac{3}{2}}\left(\frac{-\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right). \end{aligned}$$

4. Let $z = \sqrt{(1+i)}$

(a) Compute z, and simplifying the results into $x + y\ i$ form.
Let's $z = x + iy$ and $w = 1 + i$.

$$w = z^2 \Longrightarrow \left\{ \begin{array}{ccc} z^2 & = & w \\ |z|^2 & = & |w| \end{array} \right. \Longrightarrow \left\{ \begin{array}{ccc} x^2 + y^2 & = & \sqrt{2} \\ x^2 - y^2 & = & 1 \\ 2xy & = & 1 \end{array} \right.$$

adding the first and the second above equation we obtain $x = \sqrt{\frac{\sqrt{2}+1}{2}}$ and $y = \sqrt{\frac{\sqrt{2}-1}{2}}$ i.e.

$$z = \left(\sqrt{\frac{\sqrt{2}+1}{2}} \right) + i \left(\sqrt{\frac{\sqrt{2}-1}{2}} \right).$$

(b) Deduce the values of $cos(\frac{\pi}{8})$ and $sin(\frac{\pi}{8})$ from the first question we have

$$w = \sqrt{2} \left(\frac{\sqrt{2}}{2} + i \frac{\sqrt{2}}{2} \right) = \sqrt{2} \left(\cos\left(\frac{\pi}{4}\right) + i \sin\left(\frac{\pi}{4}\right) \right)$$

so,

$$\sqrt{w} = w^{\frac{1}{2}} = \left(2^{\frac{1}{4}} \cos\left(\frac{\pi}{8}\right) \right) + i \left(2^{\frac{1}{4}} \sin\left(\frac{\pi}{8}\right) \right) = z.$$

By identification we find:

$$\left\{ \begin{array}{ccc} 2^{\frac{1}{4}} \cos\left(\frac{\pi}{8}\right) & = & \sqrt{\frac{\sqrt{2}+1}{2}} \\ 2^{\frac{1}{4}} \sin\left(\frac{\pi}{8}\right) & = & \sqrt{\frac{\sqrt{2}-1}{2}} \end{array} \right. \Longrightarrow \left\{ \begin{array}{ccc} \cos\left(\frac{\pi}{8}\right) & = & \sqrt{\frac{2+\sqrt{2}}{4}} \\ \sin\left(\frac{\pi}{8}\right) & = & \sqrt{\frac{2-\sqrt{2}}{4}} \end{array} \right.$$

Solution of the Exercise 2

1. Note that

$$z = 1 + i = \sqrt{2} \left(\cos\left(\frac{\pi}{4}\right) + i\ \sin\left(\frac{\pi}{4}\right) \right) \Longrightarrow\ w = z^n = 2^{\frac{n}{2}} \left(\cos\left(\frac{n\pi}{4}\right) + i\ \sin\left(\frac{n\pi}{4}\right) \right). \tag{1}$$

(a) Case w is a pure imaginary number

$$Re(w) = 0 \Longrightarrow \cos\left(\frac{n\pi}{4}\right) \Longrightarrow \frac{n\pi}{4} = \frac{\pi}{2} + k\pi \Longrightarrow n = 2 + 4k, \quad \text{with } k \in \mathbb{Z}.$$

(b) Case w is a pure real number

$$Im(w) = 0 \Longrightarrow \sin\left(\frac{n\pi}{4}\right) \Longrightarrow \frac{n\pi}{4} = k\pi \Longrightarrow n = 4k, \quad \text{with } k \in \mathbb{Z}.$$

2. Let $w = \frac{z-i}{z+1}$ with $z = x + iy$, $v = (x+1) - iy$ and $z \neq -1$.

$$\begin{array}{rcl} w & = & \dfrac{z-i}{z+1} = \dfrac{x + (y-1)i}{(x+1) + iy} \\ & = & \dfrac{x + (y-1)i}{(x+1) + iy} \times \dfrac{(x+1) - iy}{(x+1) - iy} \\ & = & \left(\dfrac{x(x+1) + y(y-1)}{|v|^2} \right) + i \left(\dfrac{(x+1)(y-1) - xy}{|v|^2} \right). \end{array}$$

(a) Case w is a pure imaginary number.

$$\begin{aligned} Re(w) = 0 &\implies x(x+1) + y(y-1) = 0 \\ &\implies x^2 + 2\frac{1}{2}x + \frac{1}{4} + y^2 - 2\frac{1}{2}y + \frac{1}{4} = \frac{1}{2} \\ &\implies \left(x - \frac{-1}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{2}. \end{aligned}$$

From this last equation, we conclude that the values of z, such that w is a pure imaginary number, are the points belonging to the circle with center $(\frac{-1}{2}, \frac{1}{2})$ of radius $\frac{\sqrt{2}}{2}$ without the point $(-1, 0)$ (because $z \neq -1$).

(b) w is a pure real number.

$$\begin{aligned} Im(w) = 0 &\implies (x+1)(y-1) - xy = 0 \\ &\implies xy - x + y - 1 - xy = 0 \\ &\implies y = x + 1. \end{aligned}$$

From this last equation, we conclude that the values of z, so that w is a real number, are the points belonging to the line $y = x + 1$ without the point $(-1, 0)$.

Solution of the Exercise 3

Let $z,\ z_0 = x_0 + i\ y_0 \in \mathbb{C}$, $r \in \mathbb{R}_+^*$ and $i^2 = -1$.

1. Case $|z - z_0| \leq r$.

$$|z - z_0| \leq r \implies \left(\sqrt{(x - x_0)^2 + (y - y_0)^2}\right)^2 \leq r^2 \implies (x - x_0)^2 + (y - y_0)^2 \leq r^2.$$

According to this last result, we conclude that the solutions of the inequality are the set of points belonging to the disk delimited by the circle with the center (x_0, y_0) and radius r.

Remark 1 *In the previous inequality, if we replace the operator " less or equal than ($\leq$)" by "strictly less than" ($<$), then the solutions of the inequality are the set of points that belonging only to the interior of disk delimited by the circle with center (x_0, y_0) of radius r.*

2. Case $|2z + i| \leq |\overline{z} + 1|$.

$$\begin{aligned} |2z + i| \leq |\overline{z} + 1| &\implies |2x + i(2y+1)| \leq |(x+1) - iy| \\ &\implies |2x + i(2y+1)|^2 \leq |(x+1) - iy|^2 \\ &\implies 4x^2 + (2y+1)^2 \leq (x+1)^2 + y^2 \\ &\implies 4x^2 + 4y^2 + 4y + 1 \leq x^2 + 2x + 1 + y^2 \\ &\implies \left(x^2 - 2\frac{1}{3}x + \frac{1}{9}\right) + \left(y^2 + 2\frac{2}{3}x + \frac{4}{9}\right) \leq \frac{1}{9} + \frac{4}{9} \\ &\implies \left(x - \frac{1}{3}\right)^2 \left(y + \frac{2}{3}\right)^2 \leq \left(\frac{\sqrt{5}}{3}\right)^2. \end{aligned}$$

According to this last result, we conclude that the solutions of the inequality are the set of points belonging to the disk delimited by the center circle $(\frac{1}{3}, \frac{-2}{3})$ of radius $\frac{\sqrt{5}}{3}$.

Solution of the Exercise 4

Let $z \in \mathbb{C}$ and $i^2 = -1$. Solve the following equations.

1. Case of the equation $5z + 2i = (i+1)z - 3$,

$$\begin{aligned} 5z + 2i = (i+1)z - 3 &\Longrightarrow 5z - (i+1)z = -3 - 2i \\ &\Longrightarrow z = \frac{-3-2i}{4-i} \\ &\Longrightarrow z = \frac{(-3-2i)(4+i)}{(4-i)(4+i)}. \end{aligned}$$

   So,

$$z = \frac{-10}{17} + i\frac{-12}{17}.$$

2. Case of the equation $\frac{z-i}{z+1} = i$.

$$\frac{z-i}{z+1} = i \Longrightarrow z - i = iz + i \Longrightarrow (1-i)z = 2i \Longrightarrow z = \frac{2i}{1-i} \Longrightarrow z = \frac{2i(1+i)}{(1-i)(1+i)}.$$

   So,

$$z = -1 + i.$$

3. Case of the equation $2z + i\overline{z} = 3$.

   Let's assume that $z = x + iy$ with $x,\ y \in \mathbb{R}$.

$$2z + i\overline{z} = 3 \Longrightarrow 2x + i(2y) + i(x - iy) = 3 \Longrightarrow (2x + y) + i(x + 2y) = 3 + i\ 0.$$

   So, to find the solution to the above equation, we must solve the following system of linear equations:

$$\begin{cases} 2x & + & y & = & 3 \\ x & + & 2y & = & 0 \end{cases}$$

   After the resolution of this system, we get $x = 2$ and $y = -1$, hence the solution if the original equation is :

$$z = 2 - i.$$

4. Case of the equation $z^2 + z\overline{z} = 0$.

   Let's assume that $z = x + iy$ with $x,\ y \in \mathbb{R}$.

$$z^2 + z\overline{z} = 0 \Longrightarrow z(z + \overline{z}) = 0 \Longrightarrow \begin{cases} z & = & 0 \\ z + \overline{z} & = & 0 \end{cases} \Longrightarrow \begin{cases} (x, y) & = & (0, 0) \\ x & = & 0, \text{ and y} \in \mathbb{R} \end{cases}$$

   hence, the solution of the original equation is the set of the pure imaginary numbers ($z \in i \times\ \mathbb{R}$).

5. Case of the equation $z^2 + 2z + 2 = 0$.

$$\Delta = 2^2 - 4 * 2 = -4 \Longrightarrow \sqrt{\Delta} = \pm 2i \Longrightarrow \begin{cases} z_1 & = & \frac{-2-2i}{2} & = & -1 - i \\ z_2 & = & \frac{-2+2i}{2} & = & -1 + i. \end{cases}$$

6. Case equation $-2z^2 + 6z - 5 = 0$,

$$\Delta = 6^2 - 4 * (-2) * (-5) = -4 \Longrightarrow \sqrt{\Delta} = \pm 2i \Longrightarrow \begin{cases} z_1 & = & \frac{-6-2i}{-4} & = & \frac{3}{2} - \frac{1}{2}\ i \\ z_2 & = & \frac{-6+2i}{-4} & = & \frac{3}{2} + \frac{1}{2}\ i. \end{cases}$$

7. Case equation $2z^2 - (1+5i)z - 2(1-i) = 0$

$$\Delta = (1+5i)^2 - 4*2*(-2)(1-i) = -8-6i \Longrightarrow \sqrt{\Delta} = ?$$

To obtain the two solutions of the equation, we must first find the root of the determinant Δ. Suppose that $\sqrt{\Delta} = w = x + iy$ with x and y are a real numbers. From this assumption, it is clear that the following holds.

$$\left\{\begin{array}{lcl} w^2 & = & \Delta \\ |w|^2 & = & |\Delta| \end{array}\right. \Longrightarrow \left\{\begin{array}{lcl} x^2+y^2 & = & 10 \\ x^2-y^2 & = & -8 \\ 2xy & = & -6 \end{array}\right.$$

Using the first and the second above equations we obtain $x = \pm 1$ and $y = \pm 3$. From the third equation, we note that x and y have different sign, so

$$\sqrt{\Delta} = \pm(1-3i)\,.$$

Then the solutions of the given second order equation are:

$$\left\{\begin{array}{lclcl} z_1 & = & \frac{(1+5i)-(1-3i)}{4} & = & 2i \\ z_2 & = & \frac{(1+5i)+(1-3i)}{4} & = & \frac{1}{2}+\frac{1}{2}i \end{array}\right.$$

Solution of the Exercise 5

1. The equation $P(z)$ admit a real solution $\Longrightarrow \exists x \in R$ such as $P(x) = 0$

$$\begin{array}{rcl} P(x)=0 & \Longrightarrow & x^3+9ix^2+2(6i-11)x-3(4i+12)=0 \\ & \Longrightarrow & \left(x^3-22x-36\right)+i\left(9x^2+12x-12\right)=0 \\ & \Longrightarrow & \left\{\begin{array}{lcl} x^3-22x-36 & = & 0 \\ 9x^2+12x-12 & = & 0. \end{array}\right. \end{array}$$

Solving the second equation of this system above, we get $x = -2$ or $x = \frac{2}{3}$ $(\Delta = \frac{64}{9})$. After replacing x with these two values ??in the first equation, we see that only $x = -2$ satisfies both equations of the system. Therefore, we conclude that the real root of the polynomial $P(z)$ is $z = -2$.

2. As $z = -2$ is a root of $P(z)$ then the latter can be rewritten as follows:

$$P(z) = (z+2)Q(z),$$

with $Q(z)$ is a second order polynomial, i.e. $Q(z) = az^z + bz + c$. The complex coefficients a, b, and c are to be determined.

To determine these coefficients, we can proceed in two ways, namely: by identifying $P(z)$ with $(z+2)(az^z+bz+c)$ or simply using the Euclidean division of $P(z)$ on $(z+2)$. Using the Euclidean division we get:

$$P(z) = (z+2)(z^2+(9i-2)z-6(3+i)).$$

$$
\begin{array}{rllll|l}
 & z^3 & +9iz^2 & +2(6i-11)z & -3(4i+12) & z+2 \\ \cline{6-6}
- & & & & & z^2-(2-9i)z-(18+6i) \\
 & z^3 & +2z^2 & & & \\ \cline{2-5}
 & & (9i-2)z^2 & +2(6i-11)z & -3(4i+12) & \\
 & - & & & & \\
 & & (9i-2)z^2 & +2(9i-2)z & & \\ \cline{3-5}
 & & & -2(3i+9)z & -3(4i+12) & \\
 & & - & & & \\
 & & & -2(3i+9)z & -3(4i+12) & \\ \cline{4-5}
 & & & & 0 &
\end{array}
$$

3. Solve the equation $P(z) = 0$

$$P(z) = 0 \Longrightarrow (z+2)(z^2 + (9i-2)z - 6(3+i)) = 0 \Longrightarrow \begin{cases} z+2 & = 0 \\ z^2 + (9i-2)z - 6(3+i) & = 0 \end{cases}$$

Let's solve the equation $z^2 + (9i-2)z - 6(3+i) = 0$.We have

$$\Delta = (9i-2)^2 + 4 * 6(3+i) = -5 - 12i.$$

Let's $w = x + iy$, with x and y are a real numbers, the root of Δ so

$$\begin{cases} w^2 & = \Delta \\ |w|^2 & = |\Delta| \end{cases} \Longrightarrow \begin{cases} x^2 + y^2 & = 13 \\ x^2 - y^2 & = -5 \\ 2xy & = -12 \end{cases} \Longrightarrow \begin{cases} x & = 2 \\ y & = -3 \end{cases} \Longrightarrow \sqrt{\Delta} = \pm(2-3i).$$

Finally, the solutions of the equation $P(z)$ are given as follows:

$$\begin{cases} z_1 & = -2 \\ z_2 & = \frac{-(9i-2)-(2-3i)}{2} = -3i \\ z_3 & = \frac{-(9i-2)+(2-3i)}{2} = 2-6i \end{cases}$$

4. The affixes of the solution are aligned?

   Let's A, B and C denotes the affixes of z_1, z_2 and z_3 respectively.

   To show that the three points are aligned, it is enough to determine the equation of the line passing through two points, for example, A and B, and to verify that the remaining point C belongs to this line.

   By a simple calculation, we can show that the line passing through the two points A and B is written as follows:

$$y = \frac{-3}{2}x - 3.$$

   We see that the coordinates (2,-6) associated with point C verify the equation of the constructed line. Consequently, the three points are therefore aligned.

Solution of the Exercise 6

1. Calculate z based on n.

   We see that for all $n \in \mathbb{N}^*$, $\frac{Z_{n+1}}{Z_n} = \frac{1+i\sqrt{3}}{4} = \textit{constant}$, So (Z_n) is a complex geometric sequence that can be defined by its first term $Z_0 = 8$ and the common ratio $q = \frac{1+i\sqrt{3}}{4}$. Hence the expression of (Z_n) can be rewritten as follows:

$$Z_n = 8\left(\frac{1+i\sqrt{3}}{4}\right)^n, \quad n \in \mathbb{N}.$$

2. For any natural number n, calculate the ratio

$$\begin{aligned}
\frac{Z_n - Z_{n-1}}{Z_n} &= \frac{8\left(\frac{1+i\sqrt{3}}{4}\right)^n - 8\left(\frac{1+i\sqrt{3}}{4}\right)^{n-1}}{8\left(\frac{1+i\sqrt{3}}{4}\right)^n} \\
&= \frac{\left(\frac{1+i\sqrt{3}}{4}\right) - 1}{\left(\frac{1+i\sqrt{3}}{4}\right)} = \frac{-3+i\sqrt{3}}{1+i\sqrt{3}} \\
&= \frac{3+i\sqrt{3}}{1+i\sqrt{3}} \times \frac{1-i\sqrt{3}}{1-i\sqrt{3}} \\
&= i\sqrt{3}.
\end{aligned}$$

3. We note $|Z_n| = r_n$, gives the limit of r_n when n tends towards infinity. What geometric interpretation can we give?

   Let's first find the expression of r_n.

$$r_n = |Z_n| = \left|8\left(\frac{1+i\sqrt{3}}{4}\right)^n\right| = 8\left|\frac{1+i\sqrt{3}}{4}\right|^n = 8\left(\sqrt{\left(\frac{1}{4}\right)^2 + \left(\frac{\sqrt{3}}{4}\right)}\right)^n = \left(\frac{1}{2}\right)^{n-3}.$$

   Hence,

$$\lim_{n \longrightarrow \infty} r_n = \lim_{n \longrightarrow \infty} \left(\frac{1}{2}\right)^{n-3} = 0.$$

   This means that the point M_n of affix Z_n, as n tend to infinity tend to the origin $(0,0)$ of the complex plane P.

Solution of the Exercise 7

1. Show that : $\forall\, u,\ v \in \mathbb{C}: \quad |u+v|^2 + |u-v|^2 = 2\left(|u|^2 + |v|^2\right).$

$$\begin{aligned}
|u+v|^2 + |u-v|^2 &= (u+v) * (\overline{u+v}) + (u-v) * (\overline{u-v}) \\
&= (u+v) * (\overline{u} + \overline{v}) + (u-v) * (\overline{u} - \overline{v}) \\
&= u\overline{u} + u\overline{v} + v\overline{u} + v\overline{v} + u\overline{u} - u\overline{v} - v\overline{u} + v\overline{v} \\
&= 2u\overline{u} + 2v\overline{v} \\
&= 2\left(|u|^2 + |v|^2\right).
\end{aligned}$$

2. Show that the following equivalence is false

$$\text{for } u \in \mathbb{C} \text{ and } v \in \mathbb{C} : u = v \Leftrightarrow |u| = |w|.$$

   (a) "$\Longrightarrow$"Let's consider $u,\ v \in \mathbb{C}$ with $u = x_1 + iy_1$ and $v = x_2 + iy_2$

$$u = v \Longrightarrow x_1 = x_2 \text{ and } y_1 = y_2 \Longrightarrow |u| = \sqrt{x_1^2 + y_1^2} = \sqrt{x_2^2 + y_2^2} = |v|.$$

   (b) "$\Longleftarrow$" This implication is not always true. Indeed, this can be shown by a simple counter-example: if we take $u = -v0$ (with $u \neq 0$) then $|u| = |v|$ although $u \neq v$.

   As final result, the equivalence is false.

Solution of the Exercise 8

We consider the following polynomial $P(z) = z^3 + 2(\sqrt{2} - 1)z^2 - 4(\sqrt{2} - 1)z - 8$, with $z \in \mathbb{C}$.

1. Compute $P(2)$. Determine a factorization of $P(z)$ by $(z - 2)$.

   By substitution $z = 2$ in the expression of the polynomial we get $P(2) = 0$ i.e. $z = 2$ is a root of P. So, P can be rewritten as follows: $P(z) = (z - 2)Q(z)$ with $Q(z)$ is a second order polynomial.

2. Using the Euclidean division of $P(z)$ on $(z - 2)$ we obtain $Q(z) = z^2 + 2\sqrt{2}z + 4$. Hence

$$P(z) = (z - 2)\left(z^2 + 2\sqrt{2}z + 4\right).$$

3. Solve the equation $P(z) = 0$ in $\mathbb{C}$.

$$P(z) = 0 \Longrightarrow (z - 2)\left(z^2 + 2\sqrt{2}z + 4\right) \Longrightarrow \begin{cases} z - 2 & = & 0 \\ z^2 + 2\sqrt{2}z + 4 & = & 0 \end{cases}$$

   We have the determinant of equation $z^2 + 2\sqrt{2}z + 4 = 0$ is:

$$\Delta = (2\sqrt{2})^2 - 4 * 4 = -8 \Longrightarrow \sqrt{\Delta} = \pm i\ 2\sqrt{2}.$$

   So, the solutions of the equation $P(z)$ are given as follows:

$$\begin{cases} z_1 & = & 2 \\ z_2 & = & \frac{2\sqrt{2} - i\ 2\sqrt{2}}{2} = \sqrt{2} - i\ \sqrt{2} \\ z_3 & = & \frac{2\sqrt{2} + i\ 2\sqrt{2}}{2} = \sqrt{2} + i\ \sqrt{2} \end{cases}$$