University of Biskra
Mathematics Department
Module: Analysis 1

First year license
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Exercise series N°4

Exercise 1 Prove that the derivative of an even differentiable function is odd, and the derivative of an odd differentiable function is even. What about the nth derivative of an even and an odd function?

Exercise 2 Consider the function f defined by:

$$f(x) = \left(\frac{sin(2x)}{2\sqrt{1-cos(x)}}\right)^m, \quad m \in \mathbb{N}^*.$$

1. Determine the domain of the function f.
2. Discuss the parity (even or odd) of f according to the values of the parameter m.
3. Verify that f is a 2π-periodic function, then discuss the limit of f at all bounds of its domain, according to the values of the parameter m.

Exercise 3

1. Show that the curves of the following functions are symmetrical with respect to a vertical axis $x = x_0$.

$$f(x) = \sqrt{(x-1)^2+1}, \quad g(x) = x^2 + 2x + 4.$$

2. For each of the following functions, determine the point of symmetry of their graphs.

$$f(x) = \frac{2x-1}{x+1}, \quad g(x) = \frac{x^2-1}{x-2}$$

3. Show that any function having the form

$$f(x) = \frac{ax+b}{x-c} \quad \text{with } a,\ b,\ c \in \mathbb{R}.$$

admits a point of symmetry.

4. Show that any function having the form

$$f(x) = \sqrt{(x-a)^2+b}, \qquad g(x) = (x-a)^2 + b \qquad \text{with } a,\ b \in \mathbb{R}.$$

admits a vertical axe of symmetry.

Exercise 4 In each of the following cases, determine the limit, if it exists:

$$\lim_{x\to 4} \frac{x^2-7x+12}{x^2-16}, \qquad \lim_{x\to 1} \left(\frac{1}{x^2-3x+2} - \frac{1}{x-1}\right), \qquad \lim_{x\to \frac{\pi}{2}} \frac{\sqrt[3]{sin(x)}}{x-\frac{\pi}{2}}$$

$$\lim_{x\to 0} x sin(\tfrac{1}{x}), \qquad \lim_{x\to +\infty} x sin(\tfrac{1}{x}), \qquad \lim_{x\to 0} \frac{ln(1-sin(x))}{x}, \qquad \lim_{x\to 0} \frac{sin(ax)}{sin(bx)}.$$

$$\lim_{x\to \pm\infty} \frac{\sqrt{x^2-7}}{3x+5}, \qquad \lim_{x\to \pm\infty} \sqrt{x^2+6x+1} - x, \qquad \lim_{x\to 1} \frac{\sqrt{x^2-1}+\sqrt{x}-1}{\sqrt{x-1}}, \qquad \lim_{x\to 1} \frac{\sqrt{x}-1}{\sqrt[4]{x}-1}, \qquad \lim_{x\to 1} \frac{\sqrt[3]{x}-1}{\sqrt[4]{x}-1}$$

$$\lim_{x\to 0} (1+ax)^{1/x}, \qquad \lim_{x\to \pm\infty} \left(\frac{x^2+x}{x^2+x+2}\right)^{x^2+x}, \qquad \lim_{x\to \pm\infty} P_n(x)e^{-x}, \qquad \lim_{x\to \pm\infty} \frac{ln(P_n(x))}{x}$$

Note: $a, b \in \mathbb{R}^*$, $n \in \mathbb{N}^*$ and $P_n(x)$ is a positive polynomial of degree n

Exercise 5

- Find all the possible values of the constants $a,\ b$ and $c \in \mathbb{R}$ such that the following functions are continuous on their domains.

$$f(x) = \begin{cases} x^2+2x, & \text{if } x \geq 1; \\ -x+c, & \text{if } x < 1. \end{cases} \qquad g(x) = \begin{cases} x^2, & \text{if } x \leq 0; \\ a\ e^x + b, & \text{if } 0 < x < \pi; \\ 1-cos(x), & \text{if } x \geq \pi; \end{cases}$$

$$h(x) = \begin{cases} 1, & \text{if } x \leq 0; \\ ae^{-x} + be^x + cx(e^x - e^{-x}), & \text{if } 0 < x < 1; \\ e^{2-x}, & \text{if } x \geq 1; \end{cases}$$

- Study the continuity of the following function on $\mathbb{R}$, $f(x) = E(x)$. What can we conclude?

Exercise 6 For each of the following functions determine their domains and subsequently check if they have a removable discontinuity.

$$f_1(x) = e^{\frac{-1}{x^2}}, \quad f_2(x) = e^{\frac{-1}{x}}, \quad f_3(x) = \frac{1+x}{1+x^3}, \quad f_4(x) = sin(x+1)ln(|x+1|),$$

$$f_5(x) = \left(\frac{sin(2x)}{2\sqrt{1-cos(x)}}\right)^{2m}, \quad m \in \mathbb{N}^* \quad f_6(x) = cos(x)cos(1/x).$$

Exercise 7

$I)$ Let f and g two increasing continuous functions on an interval I. Show that:

$$\text{if } (f(I) \subset g(I)) \text{ or } (g(I) \subset f(I)) \text{ then } \exists c \in I \text{ such as } f(c) = g(c)$$

$II)$ Show that the following equation has at least one solution on $]-\infty; 2[$.

$$sin(x) = \frac{2x+1}{x-2}.$$

$III)$ We consider the equation (1), of unknown $x > 0$.

$$ln(x) = ax. \tag{1}$$

1. Prove that if $a \leq 0$, the equation (1) admits a unique solution and that this solution belongs to $]0, 1]$
2. Show that if $a \in]0, 1/e[$, the equation (1) admits exactly two solutions.
3. Show that if $a = 1/e$, the equation admits a unique solution whose value will be specified. Prove that if $a > 1/e$, equation (1) has no solution.

Exercise 8

- Let $f : \mathbb{R} \to \mathbb{R}$. Suppose $c \in \mathbb{R}$ and that $f'(c)$ exists. Prove that f is continuous at c.
- Prove that:

1) $(e^x)' = e^x$ 2) $\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x)-f(x)g'(x)}{[g(x)]^2}$ 3) $arcsin(x)' = \frac{1}{\sqrt{(1-x^2)}}$

4) $arctan(x)' = \frac{1}{1+x^2}$ 5) $\left(f^{-1}(x)\right)' = \frac{1}{f' \circ f^{-1}(x)}$ 6) $f \circ g(x) = g'(x)\ f' \circ g(x)$

Exercise 9

- Return to the examples of the Exercise 5, and determine the domain of differentiability of the considered functions according to the parameters a, b, and c.

- Determine the two real numbers a and b, so that the function f, defined on $\mathbb{R}$ by:

$$f(x) = \begin{cases} \sqrt{x}, & \textit{if } 0 \leq x \leq 1; \\ ax^2 + bx + c, & x{>}1, \end{cases}$$

is differentiable on $\mathbb{R}_+.^*$.

- Study the differentiability of the following functions:

$$f_1(x) = \begin{cases} x^2 cos(1/x), & \text{if } x \neq 0; \\ 0, & \text{else.} \end{cases} \qquad f_2(x) = \begin{cases} sin(x)sin(1/x), & \text{if } x \neq 0; \\ 0, & \text{else.} \end{cases}$$

$$f_3(x) = \begin{cases} \frac{|x|\sqrt{x^2-2x+1}}{x-1}, & \text{if } x \neq 1; \\ 1, & \text{else.} \end{cases}$$

- Study the differentiability of the following functions at x_0:

$$f_1(x) = \sqrt{x}, \quad x_0 = 0, \qquad f_2(x) = (1-x)\sqrt{1-x^2}, \quad x_0 = -1, \qquad f_3(x) = (1-x)\sqrt{1-x^2}, \quad x_0 = 1.$$

What can we conclude?

Exercise 10 Calculate the derivatives of the following functions.

1) $e^{sin(x^3)}$ 2) $ln\left(x^2 + e^{-x^2}\right)$ 3) $ln\left(\frac{x+1}{x-1}\right)$ 4) $sin(2x^2 + cos(x))$

5) $arcsin(x^2 + x)$ 6) $arctan(x^2 + x)$ 7) $\sqrt[3]{x^2 + x}$ 8) $a^{\left(\frac{x-1}{x+1}\right)}$, $a \in \mathbb{R}_+^*$

9) $e^{e^{x^2+1/x}}$ 10) $log_a(arcsin(x))$, $a \in \mathbb{R}_+^*$ 11) $\sqrt{|x^2 - 4x + 3|}$ 12) $\frac{1-tan^2(x)}{(1+tan(x))^2}$

Exercise 11

1. In the application of mean value theorem's to the function

$$f(x) = \alpha x^2 + \beta x + \gamma, \qquad \alpha,\ \beta,\ \gamma \in \mathbb{R}^*$$

on the interval $[a;b]$ specify the number $c \in]a;b[$. Give a geometric interpretation.

2. Let x and y two reals with $0 < x < y$, show that

$$x < \frac{y-x}{ln(y) - ln(x)} < y.$$

Exercise 12 Let f and $g \rightarrow [a;b]$ be two continuous functions on $[a;b]$ $(a < b)$ and differentiable on $]a;b[$. We suppose that $g'(x) \neq 0$ for all $x \in]a;b[$.

1. Show that $g(x) \neq g(a)$, for all $x \in]a;b[$.

2. Let us set $\alpha = \frac{f(b)-f(a)}{g(b)-g(a)}$ and consider the function $h(x) = f(x) - \alpha g(x)$ for $x \in [a;b]$. Show that h satisfies the hypotheses of Rolle's theorem and deduce that there exists a real number $c \in]a;b[$ such that
$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}.$$

3. We assume that $\lim_{x \to b} \frac{f'(x)}{g'(x)} = l$, where l is a finite real number. Show that
$$\lim_{x \to b-} \frac{f(x)-f(b)}{g(x)-g(b)} = l.$$

4. Application. Calculate the following limit:
$$\lim_{x \to 1^-} \frac{arccos(x)}{\sqrt{1-x^2}}.$$

Exercise 13 Using the derivative notions, determine the following limits:

$$1)\ \lim_{x \to 0} \frac{e^{3x-2}-e^2}{x} \qquad 2)\ \lim_{x \to 1} \frac{ln(2-x)}{x-1} \qquad 3)\ \lim_{x \to \pi} \frac{sin(x)}{x^2-\pi^2}$$

$$4)\ \lim_{x \to \frac{\pi}{2}} \frac{e^{cos(x)}}{x-\frac{\pi}{2}} \qquad 5)\ \lim_{x \to 0} \frac{ln(1-sin(x))}{x} \qquad 6)\ \lim_{x \to +\infty} \left(ln(x+1) - ln(x)\right).$$

Exercise 14 Give the domain of differentiability of the following functions then calculate the nth-order derivative, by justifying its existence.

$$f(x) = 2x^k,\ k \in \mathbb{N}^*,\quad f(x) = 1/x,\quad f(x) = 1/x^2,\quad f(x) = sin(2x),\quad f(x) = sin(x)cos(x),$$

$$f(x) = \frac{1}{1-x^2},\quad f(x) = x^2e^x.$$