University of Biskra
Mathematics Department
Module: Analysis 1

First year license
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Exercise series N°3

Exercise 1

$$U_n = \frac{1}{1+n} + \frac{1}{2+n}, \quad U_n = n^2\left(1 - \frac{1}{1+n}\right), \quad U_n = n\left(n - (-1)^n\right), \quad U_n = \sqrt[n]{a}, \quad \text{with } a > 1.$$

1. Study the monotony of previous sequences
2. Show that if U_n is an increasing (respectively, a decreasing) sequence then $V_n = \frac{1}{n}\sum_{i=1}^{n} U_i$ is also an increasing (respectively, a decreasing) sequence.

Exercise 2 Show that:

I) Let $(Un)_{n\in\mathbb{N}}$ be a sequence of $\mathbb{R}$. What do you think of the following propositions:

1. If U_n converges to a real l then U_{2n} and U_{2n+1} converge to l.
2. If U_{2n} and U_{2n+1} are convergent, the same is true of U_n.
3. If U_{4n} and U_{4n+2} are convergent, towards the same limit, it is the same for U_n.
4. If U_{2n} and U_{2n+1} are convergent, towards the same limit, it is the same for U_n.

II) Prove that:

1. if the sequence $\{U_n\}_{n\in\mathbb{N}}$ converges to l_1 and $\{V_n\}_{n\in\mathbb{N}}$ converges to l_2, then the sequence $\{U_n + V_n\}_{n\in\mathbb{N}}$ converges to $l_1 + l_2$.
2. convergent sequences are Cauchy sequences.

Exercise 3 Let consider the following real sequences:

$$U_n = \frac{1}{n+1}, \quad V_n = \sqrt[n]{a} \text{ with } a > 1 \quad W_n = \frac{(-1)^n + bn}{n+1} \text{ with } b \in \mathbb{R}, \quad T_n = c^n \text{ with } c \in]-1, 1[.$$

1. Prove, using the definition of the limit of a real sequence, that:
$$\lim_{n\to\infty} U_n = 0, \quad \lim_{n\to\infty} V_n = 1, \quad \lim_{n\to\infty} W_n = b, \quad \lim_{n\to\infty} T_n = 0.$$
2. For each sequence determine the smallest value of N (see the below note), when $\epsilon = 0.001$, and $a = b = 2$ and $c = 1/2$.
3. Prove, using the definition of the limit of a real sequence, that the sequences K_n and S_n are divergent, with
$$K_n = \frac{-n^2 + n + 1}{n+1} \quad \text{and} \quad S_n = ln(ln(ln(n))).$$

Note: $\lim_{n\to\infty} U_n = l \Leftrightarrow \forall\epsilon > 0, \exists N \in \mathbb{N} : |U_n - l| < \epsilon, \quad for\ n \geq N.$

Exercise 4 In each of the following cases, determine the limit, if it exists.

$U_n = \frac{n+(-1)^n}{n-(-1)^n}$

$U_n = \frac{a^n - b^n}{a^n + b^n}$, with $a, b > 0$

$U_n = \frac{n^2 - 2^n}{3^n}$

$U_n = \sum_{k=1}^{n} \frac{1}{\sqrt{k}} - \frac{1}{\sqrt{k+1}}$

$U_n = \sqrt{n+a} - \sqrt{n+b}$ with $a, b \geq 0$ and $a \neq b$.

$U_n = 1 - \frac{1}{a} + \frac{1}{a^2} - \frac{1}{a^3} + ... + \frac{(-1)^n}{a^n}$, with $a > 0$.

$U_n = (1 + \frac{a}{n})^n$ with $a \in \mathbb{R}^*$

$U_n = \frac{2}{n^2}\sum_{k=1}^{n} E(kx)$ with $x \geq 0$

Exercise 5 Let $a > 0$. We define the sequence $\{U_n\}_{n\geq 0}$ by U_0 strictly positive real numbers and by the relation:

$$U_{n+1} = \frac{1}{2}\left(U_n + \frac{a}{U_n}\right).$$

1. Show that for all $n \geq 1$ we have $U_n \geq \sqrt{a}$ and then, that $\{U_n\}_{n\geq 1}$ is a decreasing sequence.
2. Deduce that the sequence U_n converges to $\sqrt{a}$.

Exercise 6

1. Let $0 < a \leq b$. Prove the following inequalities:

$$\sqrt{ab} \leq \frac{a+b}{2}, \quad a \leq \frac{a+b}{2} \leq b, \quad a \leq \sqrt{ab} \leq b.$$

2. Let U_0 and V_0 be strictly positive real numbers with $U_0 < V_0$. We define two sequences U_n and V_n as follow:

$$U_{n+1} = \sqrt{U_n V_n} \qquad \text{and} \qquad V_{n+1} = \frac{U_n + V_n}{2}.$$

   (a) Show that $U_n < V_n$ for all $n \in \mathbb{N}$.
   (b) Show that V_n is a decreasing sequence.
   (c) Show that U_n is increasing then deduce that the sequences U_n and V_n are convergent and have the same limit.

Exercise 7 We consider the two sequences:

$$U_n = 1 + \frac{1}{2!} + \frac{1}{3!} + ... + \frac{1}{n!} \quad \text{and} \quad V_n = U_n + \frac{1}{n!}$$

Show that U_n and V_n converge towards the same limit.

Exercise 8 *(Leave the exercise to the students.)*

I) If the approximate values of a real number x with precision $10^{-2}, 10^{-3}, ..., 10^{-n}...$ are given by: If the approximate values of a real number x to $2, 3, ..., n...$ decimal point are given by: $1.16; 1.166; ....; 1.\underbrace{1666...6}_{n}; ...$ then give the exact value of x.

II) Consider the following sequences, defined for $n \in \mathbb{N}^*$:

$$U_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + ... + \frac{1}{n} \qquad \text{and} \qquad V_n = ln(n+1) - ln(n).$$

1. Calculate the limit of $S_n = \sum_{i=1}^{n} V_i$.
2. Show that, for all $n \in \mathbb{N}^*$ we have $V_n \leq \frac{1}{n}$.
3. What can we conclude about the nature of U_n?

Definition 1 *Let $(U_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ be two sequences such that*

- *U_n is decreasing,*
- *V_n is increasing,*
- $\lim_{n\to\infty} (U_n - V_n) = 0.$

Sequences satisfying the above properties are called Adjacent.

If $(U_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ are adjacent then, they are convergent and have the same limit.