University of Biskra
Mathematics Department
Module: Analysis 1

First year license
2023/2024

Exercise series N°2

Exercise 1 Let $z \in \mathbb{C}$, $x,\ y \in \mathbb{R}$, $r \in \mathbb{R}_+^*$, $\theta \in [0; 2\pi[$ and $i^2 = -1$.

1. Rewrite each z into polar form ($re^{i\theta}$).

$$a)\ z = 6, \quad b)\ z = 6i, \quad c)\ z = 2 + 2i, \quad d)\ z = -2 + 2i, \quad e)\ z = 3 + \sqrt{3}i.$$

2. Rewrite z from polar into $x + iy$ form.

$$a)\ z = 3e^{\frac{5\pi}{4}i}, \quad b)\ z = 5e^{\frac{7\pi}{4}i}, \quad c)\ z = re^{\frac{\pi}{12}i}, \quad d)\ z = \sqrt{16e^{\frac{2\pi}{3}i}}.$$

3. Compute the following, simplifying the results into $x + iy$ form.

$$a)\ z = (2 + 2i)^8, \quad b)\ \sqrt{3 + \sqrt{3}i}.$$

4. Let $z = \sqrt{\frac{(1+i)}{\sqrt{2}}}$,
   (a) Compute z, and simplifying the results into $x + iy$ form.
   (b) Deduce the values of $cos(\frac{\pi}{8})$ and $sin(\frac{\pi}{8})$

Exercise 2 Let $z,\ w \in \mathbb{C}$ and $i^2 = -1$.

1. Let $z = 1 + i$ and $w = z^n$ with $n \in \mathbb{Z}$.
   (a) Determine the values of n for which w is a pure imaginary number ($Re(w) = 0$).
   (b) Determine the values of n for which w is a real number ($Im(w) = 0$).
2. Let $w = \frac{z-i}{z+1}$ with $z \neq -1$. Determine the set of points M with affix z of which
   (a) w is a pure imaginary number ($Re(w) = 0$).
   (b) w is a real number ($Im(w) = 0$).

Exercise 3 Let $z,\ z_0 = x_0 + y_0\ i \in \mathbb{C}$, $r \in \mathbb{R}_+^*$ and $i^2 = -1$. Solve the following inequations.

1. $|z - z_0| \leq r$.
2. $|2z + i| \leq |\overline{z} + 1|$.
3. $\left|\frac{z-3}{z-5}\right| \leq r$, with $z \neq 5$(Left to the student).
4. $|2z + z_0| \leq |\overline{z} + z_1|$, with $z_0,\ z_1 \in \mathbb{C}$ (Left to the student).

Exercise 4 Let $z \in \mathbb{C}$ and $i^2 = -1$. Solve the following equations

$$a)\ 5z + 2i = (i + 1)z - 3, \quad b)\ \frac{z - i}{z + 1} = 4i, \quad c)\ 2z + i\overline{z} = 3, \quad d)\ z^2 + z\overline{z} = 0.$$

$$e)\ z^2 + 2z + 2 = 0, \quad f)\ -2z^2 + 6z - 5 = 0, \quad g)\ 2z^2 - z(1 + 5i) - 2(1 - i) = 0.$$

Exercise 5 We consider the following polynomial $P(z) = z^3 + 9iz^2 + 2(6i - 11)z - 3(4i + 12)$, with $Z \in \mathbb{C}$.

1. Demonstrate that the equation $P(z) = 0$ admits a real solution z_1.
2. Determine a polynomial $Q(z)$ such that $P(z) = (z - z_1)Q(z)$.
3. Solve the equation $P(z) = 0$ in $\mathbb{C}$.
4. Demonstrate that the points of the complex plane corresponding to the solutions of the equation $P(z) = 0$ are aligned.

Exercise 6 Let Z_n be a complex number defined by:

$$Z_n = \begin{cases} 8, & \text{if } n = 0; \\ \frac{1+i\sqrt{3}}{4} Z_{n-1}, & \text{else.} \end{cases}$$

and $(M_n)_{n\in\mathbb{N}}$ are the points of affix Z_n on the complex plane $\mathbf{P}$.

1. Calculate z based on n.
2. For any natural number n, calculate the ratio
$$\frac{Z_n - Z_{n-1}}{Z_n}.$$
3. We note $|Z_n| = r_n$, gives the limit of r_n when n tends towards infinity. What geometric interpretation can we give?

Exercise 7

1. Show that
$$\forall\ u,\ v \in \mathbb{C}: \quad |u+v|^2 + |u-v|^2 = 2\left(|u|^2 + |v|^2\right)$$
2. Show that the following equivalence is false
$$\text{for } u \in \mathbb{C} \text{ and } v \in \mathbb{C} : u = v \Leftrightarrow |u| = |w|.$$

Exercise 8 *(Left to the student).* We consider the following polynomial $P(z) = z^3 + 2(\sqrt{2} - 1)z^2 - 4(\sqrt{2} - 1)z - 8$, with $z \in \mathbb{C}$.

1. Compute $P(2)$. Determine a factorization of $P(z)$ by $(z - 2)$.
2. Solve the equation $P(z) = 0$ in $\mathbb{C}$.

Exercise 9 *(Left to the student)* We consider the function f of the plane which at any point M associates the affix point:

$$w = \frac{z+i}{z-2i}, \quad \text{with } z \neq 2i.$$

1. For $z \neq 2i$, we set $z = 2i + re^{i\theta}$, with and $r > 0$ and $\theta \in [0; 2\pi[$. Write $w - 1$ using r and θ.
2. A is the affix point $2i$,
   (a) Determine the set E_1 of points M for which $|w - 1| = 3$.
   (b) Determine the set E_2 of points M for which $arg(w - 1) = \frac{\pi}{4}$.
   (c) Represent the sets E_1 and E_2