University of Biskra
Mathematics Department
Module: Analysis 1

First year license
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Exercise series N°1

Exercise 1 *:* Consider the following assertions:

$A_1: \exists x \in \mathbb{R}, \forall y \in \mathbb{R}$: $x + y > 0$.

$A_2: \forall x \in \mathbb{R}, \exists y \in \mathbb{R}$: $x + y > 0$.

$A_3: \forall x \in \mathbb{R}, \forall y \in \mathbb{R}$: $x + y > 0$.

$A_4: \exists x \in \mathbb{R}, \forall y \in \mathbb{R}$: $y^2 > x$.

1. Are assertions A_1, A_2, A_3 and A_4 true or false?
2. Give their negation.

Exercise 2 *:*

- If a and b are two positive or zero real numbers, show that:

$$\sqrt{a} + \sqrt{b} \leq 2\sqrt{a+b}.$$

- Prove by induction the following equalities:

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2} \quad and \quad \sum_{k=0}^{n-1} 2^k = 2^n - 1, \quad with \ \ n \in \mathbb{N}^*$$

- Show that $\sqrt{2}$ is not a rational number.

Exercise 3 Let x and $y \in \mathbb{R}$.

1. Show that the following relationships are always true:
   (a) If $|x| < y$ *then* $-y < x < y$
   (b) $|x + y| \leq |x| + |y|$.
   (c) $||x| - |y|| \leq |x - y|$.
2. Solve the following inequalities:
   (a) $|x - 2| > 5$.
   (b) $|x + 2| > |x|$.
   (c) $|2x - 1| < |x - 1|$.

Exercise 4 Determine (if they exist): the all upper and lower bounds, supremum, infimum, maximum, and minimum, of the following sets:

$$E_1 = \left\{1, \frac{1}{3}, \frac{1}{5}, ..., \frac{1}{2n+1}, ...; \ \ n \in \mathbb{N}\right\}, \quad E_2 =]0, 5], \quad E_3 = \left\{4 - \frac{1}{n}; n \in \mathbb{N}^*\right\},$$

$$E_4 = \left\{\frac{1}{2} + \frac{n}{2n+1}, \ \frac{1}{2} - \frac{n}{2n+1}; \ \ n \in \mathbb{N}^*\right\}$$

Exercise 5 Show that the following relationships are true.

- $x - 1 < E(x) \leq x$,
- $E(x) + E(y) \leq E(x + y)$,
- $E(x) - E(y) \geq E(x - y)$,
- $E\left(\frac{E(nx)}{n}\right) = E(x)$,

with $x,\ y \in \mathbb{R}$, $n \in \mathbb{N}^*$ and $E(.)$ is the integral part function.

Solution

Solution of the Exercise 1 *:*

A_1 : is false, because we can find an y in $\mathbb{R}$ such that for any x in $\mathbb{R}$ we have $x + y$ less or equal to zero $(x + y \leq 0.)$
For example, if we take $y = 0$, then for all x negative $(x \leq 0)$ we have $x + y = x \leq 0$
The negation: $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}\colon\ x + y \leq 0.$

A_2 : is true, the fact that for any x we can find an $y \in \mathbb{R}$ for which the inequality $x + y > 0$ is verified.
For exemple, if we take $y = -x + 1$ then $x + y = 1 > 0$.
The negation: $\exists x \in \mathbb{R}, \quad \forall y \in \mathbb{R} : \quad x + y \leq 0.$

A_3 : is false, because if we choose, for example, $y \leq 0$ and $x \leq 0$ then $x + y < 0$.
The negation: $\exists x \in \mathbb{R}, \quad \exists y \in \mathbb{R} : \quad x + y \leq 0.$

A_4 : is true, and it is the fact that for all $y \in \mathbb{R}$, it is enough to take an x in the interval $]-\infty, y^2[$ for the inequality $y^2 > x$ to be verified.
The negation: $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}\colon\ y^2 \leq x.$

Solution of the Exercise 2 *:*

- For two positive or zero real numbers a and b, we have:

$$\begin{cases} a \leq a + b \\ b \leq a + b \end{cases} \Longrightarrow \begin{cases} \sqrt{a} \leq \sqrt{a+b}.....(*) \\ \sqrt{b} \leq \sqrt{a+b}.....(**) \end{cases} \text{ (the fact that the root function is an increasing function)}$$

by adding the two sides of the inequalities (*) and (**), we will have:

$$\sqrt{a} + \sqrt{b} \leq 2\sqrt{a+b}.$$

- Recall that the proof by induction is based on the following three steps:

Step 1: Verify that the desired result holds for $n = n_0$

Step 2: Assume that the desired result holds for n.

Step 3: Use the assumption from step 2 to show that the result holds for $(n + 1)$.

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}, \quad with \quad n \in \mathbb{N}^* \tag{1}$$

$$\sum_{k=0}^{n-1} 2^k = 2^n - 1, \quad with \quad n \in \mathbb{N}^* \tag{2}$$

for $n = 1$:

$$\begin{cases} \sum_{k=1}^{n} k = \sum_{k=1}^{1} k = 1 \\ \frac{n(n+1)}{2} = \frac{1(1+1)}{2} = \frac{2}{2} = 1 \end{cases} \tag{3}$$

for n**:** We assume that the following equality is true for n.

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2} \tag{4}$$

for $n+1$**:** On the one hand, using the assumption (4), we have:

$$\sum_{k=1}^{n+1} k = \sum_{k=1}^{n} k + (n+1) = \frac{n(n+1)}{2} + (n+1) = \frac{(n+1)(n+2)}{2}$$

On the other hand we have:

$$\sum_{k=1}^{n+1} k = \frac{(n+1)((n+1)+1)}{2} = \frac{(n+1)(n+2)}{2}$$

Consequently, the equality (1) holds for $n+1$. From the above three steps we conclude that (1) holds for all $n \in \mathbb{N}^*$.

for $n=1$**:**

$$\begin{cases} \sum_{k=0}^{n-1} 2^k & = & \sum_{k=0}^{0} 2^k & = & 2^0 = 1 \\ 2^n - 1 & = & 2^1 - 1 & = & 2^1 - 1 = 1 \end{cases} \tag{5}$$

So the equality holds for $n=1$.

for n**:** We assume that the following equality is true for n.

$$\sum_{k=0}^{n-1} 2^k = 2^n - 1 \tag{6}$$

for $n+1$**:** On the one hand, using the assumption (6), we have:

$$\sum_{k=0}^{n+1-1} 2^k = \sum_{k=0}^{n} 2^k = \sum_{k=1}^{n-1} 2^k + 2^n = 2^n - 1 + 2^n = 2 \times 2^n - 1 = 2^{n+1} - 1$$

On the other hand we have:

$$\sum_{k=0}^{n} 2^k = 2^{n+1} - 1$$

Consequently, the equality (2) holdsF for $n+1$. From the above three steps we conclude that (2) holds for all $n \in \mathbb{N}^*$.

- Proof By Contradiction that $\sqrt{2}$ is irrational
  Recall that for $n \in \mathbb{N}$, we have:

$$n \text{ is an odd natural number } \Leftrightarrow n^2 \text{ is an odd natural number.}$$

$$n \text{ is an even natural number } \Leftrightarrow n^2 \text{ is an even natural number.}$$

Note: The demonstration of the two equivalences above is an additional exercise to be left for the student. Assume that $\sqrt{2}$ is rational.
Then, let $\sqrt{2} = \frac{p}{q}$, where $p \in \mathbb{Z}$ and $q \in \mathbb{Z}^*$, and p and q are relatively prime i.e $gcd(p,q) = 1$.

$$\sqrt{2} = \frac{p}{q} \Rightarrow 2 = \frac{p^2}{q^2} \Rightarrow p^2 = 2q^2 \Rightarrow p^2 \text{ is even } \Rightarrow p \textbf{ is even}, \text{ say } p = 2m$$
$$\Rightarrow 4m^2 = 2q^2 \Rightarrow 2m = q^2 \Rightarrow q \textbf{ is even}.$$

Thus, both p and q are even and have 2 as a common factor. But we assumed that p and q are relatively prime. This is a contradiction. Thus, $\sqrt{2}$ cannot be written as $\frac{p}{q}$ for $p \in \mathbb{Z}$ and $q \in \mathbb{Z}^*$ Thus $\sqrt{2}$ is irrational.

Solution of the Exercise 3 Let x and $y \in \mathbb{R}$.

1. From the definition of the absolute value we have:

$$\begin{cases} x < y, & \text{if } x \geq 0; \\ -x < y, & \text{if } x < 0. \end{cases} \Longrightarrow \begin{cases} x < y, & \text{if } x \geq 0; \\ x > -y, & \text{if } x < 0. \end{cases} \Longrightarrow -y < x < y. \tag{7}$$

2. We have

$$\begin{cases} -|x| \leq x \leq |x| \\ -|y| \leq y \leq |y| \end{cases} \Longrightarrow -|x| - |y| \leq x + y \leq |x| + |y| \Longrightarrow -(|x| + |y|) \leq x + y \leq (|x| + |y|) \tag{8}$$

As $|x| + |y| \geq 0$, then from (7) and (8) we can conclude that :

$$|x + y| \leq |x| + |y|. \tag{9}$$

3. $||x| - |y|| \leq |x - y|$?
We have

$$\begin{cases} |x| \leq |(x - y) + y| \\ |y| \leq |(y - x) + x| \end{cases} \text{using the inequality (9)} \Rightarrow \begin{cases} |x| \leq |(x - y) + y| \leq |(x - y)| + |y| \\ |y| \leq |(y - x) + x| \leq |(y - x)| + |x| \end{cases}$$

$$\Rightarrow \begin{cases} |x| \leq |(x - y)| + |y| \\ |y| \leq |(y - x)| + |x| \end{cases} \Rightarrow \begin{cases} |x| \leq |(x - y)| + |y| \\ |y| \leq |(y - x)| + |x| \end{cases} \Rightarrow \begin{cases} |x| - |y| \leq |x - y| \\ |y| - |x| \leq |x - y| \end{cases} \Rightarrow \begin{cases} |x| - |y| \leq |x - y| \\ |x| - |y| \geq -|x - y| \end{cases}$$

Finally,

$$-|x - y| \leq |x| - |y| \leq |x - y|.$$

Thus, from the result proven at the beginning of the exercise, we conclude that

$$||x| - |y|| \leq |x - y|.$$

Resolution of inequalities:

1. $|x - 2| > 5$. we have the inequality $|x - 2| > 5$, then using the absolute value definition, we can be rewritten the inequality as follows:

$$\begin{cases} (x - 2) > 5, & \text{if } x - 2 \geq 0; \\ -(x - 2) > 5, & \text{if } x - 2 < 0. \end{cases} \Rightarrow \begin{cases} (x - 2) > 5, & \text{if } x \geq 2; \\ -(x - 2) > 5, & \text{if } x < 2. \end{cases} \Rightarrow \begin{cases} x > 7, & \text{if } x \geq 2; \\ x < -3, & \text{if } x < 2. \end{cases} \tag{10}$$

Thus, the solutions of the inequality $|x - 2| > 5$ are:

$$x \in]-\infty, -3[\cup]7, +\infty[.$$

2. $|x + 2| > |x|$.

x	-2	0	
$\lvert x\rvert$	$-x$	$-x$	x
$\lvert x+2\rvert$	$-x-2$	$x+2$	$x+2$
	A	B	C

We notice that three situations are possible:

Case A:

$$for\ x \in]-\infty, -2[, \quad -x - 2 > -x \Rightarrow x + 2 < x \Rightarrow 2 < 0$$

Thus the set of solution in this case is empty i.e. $E_A = \{\} = \varnothing$

Case B:

$$for\ x \in [-2, 0],\ x + 2 > -x \Rightarrow x > -1$$

Thus, the set of solution in this case $x \in [-2, 0]$ and $x > -1$ i.e. $E_B =]-1, 0]$

Case C:

$$for\ x \in]0, +\infty[,\ x + 2 > x \Rightarrow 2 > 0. \text{ This latest inequality is always true}, x \in \mathbb{R}$$

Thus, the set of solution in this case $x \in]0, +\infty[$ and $x \in \mathbb{R}$ i.e. $E_C =]0, +\infty[$

From the three cases above, we conclude that the set of solutions to the inequality $|x + 2| > |x|$ is:

$$E = E_A \cup E_B \cup E_C = \varnothing \cup]-1, 0] \cup]0, +\infty[=]-1, +\infty[.$$

3. $|2x - 1| < |x - 1|$. Note that:

x	1/2	1	
$\|2x-1\|$	$-2x+1$	$2x-1$	$2x-1$
$\|x-1\|$	$-x+1$	$-x+1$	$x-1$
	A	B	C

With the same reasoning as in Example 2, we can show the following:

$$E_A = \left]0, \frac{1}{2}\right[, \quad E_B = \left[\frac{1}{2}, \frac{2}{3}\right[, \quad and \quad E_C = \varnothing$$

$$\Rightarrow E = \left]0, \frac{2}{3}\right[.$$

Solution of the Exercise 4

1. max, min, sup, inf, lb, ub of E_1 we have

$$n \in \mathbb{N} \Leftrightarrow 0 \leq n < \infty \Leftrightarrow 1 \leq n + 1 < \infty \Leftrightarrow 0 < \frac{1}{n+1} \leq 1 \Leftrightarrow E_1 =]0, 1]. \tag{11}$$

From (11), we conclude that

lb: $lb =]-\infty; 0]$.

inf: $inf = max(]-\infty; 0]) = 0$.

min: the minimum of E_1 does not exist, because E_1 is an open interval on the left side.

ub: $ub = [1; +\infty[$

sup: $sup = min([1; +\infty[) = 1$.

min: max=1 (because $1 \in E_1$).

2. max, min, sup, inf, lb, ub of E_2

lb: $lb =]-\infty; 0]$.

inf: $inf = max(]-\infty; 0]) = 0$.

min: the minimum of E_2 does not exist, because E_2 is an open interval on the left side.

ub: $ub = [5; +\infty[$

sup: $sup = min([5; +\infty[) = 5$.

min: max=5 (because $5 \in E_2$).

3. max, min, sup, inf, lb, ub of E_3

$$n \in \mathbb{N}^* \Leftrightarrow 1 \leq n < \infty \Leftrightarrow 0 < \frac{1}{n} \leq 1 \Leftrightarrow -1 \leq \frac{-1}{n} < 0 \Leftrightarrow 3 \leq 4 - \frac{1}{n} < 4 \Leftrightarrow E_1 = [3, 4]. \tag{12}$$

From (12), we conclude that

lb: $lb =]-\infty; 3]$.

inf: $inf = max(]-\infty; 3]) = 3$.

min: min=3;

ub: $ub = [4; +\infty[$

sup: $sup = min([4; +\infty[) = 4$.

min: the maximum of E_3 does not exist, because E_3 is an open interval on the right side.

4. max, min, sup, inf, lb, ub of E_3 Let's define the following subsets:

$$u_n = \frac{1}{2} + \frac{n}{2n+1}, \quad n \in \mathbb{N}^*$$

$$v_n = \frac{1}{2} - \frac{n}{2n+1}; \quad n \in \mathbb{N}^*$$

It is easy to show that u_n is an increasing sequence while v_n is a decreasing sequence. Indeed,

$$\begin{aligned} u_{n+1} - u_n &= \left(\frac{1}{2} + \frac{n+1}{2n+3}\right) - \left(\frac{1}{2} + \frac{n}{2n+1}\right) \\ &= \frac{(2n^2 + n + 2n + 1) - (2n^2 + 3n)}{(2n+3)(2n+1)} \\ &= \frac{1}{(2n+3)(2n+1) \geq 0} > 0 \\ &\Leftrightarrow u_n \text{ is an increasing sequence.} \end{aligned}$$

$$\begin{aligned} v_{n+1} - v_n &= \left(\frac{1}{2} - \frac{n+1}{2n+3}\right) - \left(\frac{1}{2} - \frac{n}{2n+1}\right) \\ &= \frac{-(2n^2 + n + 2n + 1) + (2n^2 + 3n)}{(2n+3)(2n+1)} \\ &= \frac{-1}{(2n+3)(2n+1)} < 0 \\ &\Leftrightarrow v_n \text{ is a decreasing sequence.} \end{aligned}$$

so,

$$\left\{ \begin{array}{l} u_1 \leq u_n < \lim\limits_{n\to\infty} u_n, \\ \lim\limits_{n\to+\infty} v_n < v_n \leq v_1, \end{array} \right. \left\{ \begin{array}{l} \frac{5}{6} \leq u_n < 1, \\ 0 < v_n \leq \frac{1}{6}, \end{array} \right. \tag{13}$$

At this level, to answer the main question of the exercise we can proceed in two ways:
First way:

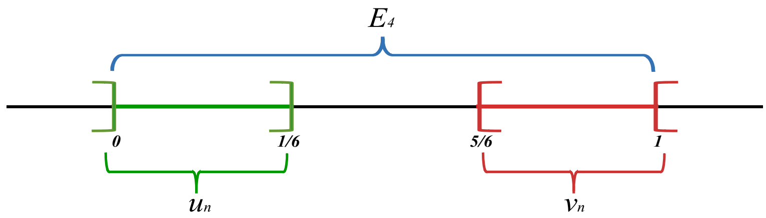
lb: we have $lb_u =]-\infty; \frac{5}{6}]$ and $lb_v =]-\infty; 0] \;\Rightarrow\; lb_{E_4} = lb_u \cap lb_u =]-\infty; 0]$.

inf: we have $inf_u = \frac{5}{6}]$ and $inf_v = 0 \;\Rightarrow\; inf_{E_4} = min(inf_u, inf_v) = 0$.

min: we have $min_u = \frac{5}{6}]$ and min_v does not exist $\Rightarrow lb_{E_4}$ does not exist.;

ub: we have $ub_u = [1; +\infty[$ and $ub_v = [\frac{1}{6}; +\infty[\;\Rightarrow\; lb_{E_4} = ub_u \cap ub_u = [1; +\infty[$.

sup: we have $sup_u = 1$ and $sup_v = \frac{1}{6} \;\Rightarrow\; sup_{E_4} = max(sup_u, sup_v) = 1$.



min: we have max_u does not exist and $max_v = \frac{1}{6} \Rightarrow lb_{E_4}$ does not exist.;

Second way: From (13), we note that

$$u_n \in \left[\frac{5}{6};1\right[\text{ and } v_n \in \left]0;\frac{1}{6}\right] \Rightarrow E_5 = \left]0;\frac{1}{6}\right] \cup \left[\frac{5}{6};1\right[,$$

thus,

lb: $lb =]-\infty;0]$.

inf: $inf = max(]-\infty;0]) = 0$.

min: minimum does not exist;

ub: $ub = [1;+\infty[$

sup: $sup = min([1;+\infty[) = 1$.

min: the maximum does not exist.

Solution of the Exercise 5

- $x - 1 < E(x) \leq x$?

  According to the definition of the integer part of a real number, we have

$$\begin{aligned} E(x) \leq x < E(x)+1 &\Leftrightarrow 0 \leq x - E(x) < 1 \\ &\Leftrightarrow 0 \leq x - E(x) < 1 \\ &\Leftrightarrow -x \leq -E(x) < -x+1 \\ &\Leftrightarrow x \geq E(x) > x-1. \end{aligned}$$

- $E(x) + E(y) \leq E(x+y)$?

  Let x and y two real numbers. We have

$$\begin{cases} x = E(x) + R_x, & \text{with } R_x \in [0,1[; \\ y = E(y) + R_y, & \text{with } R_y \in [0,1[; \end{cases}$$

  On the one hand, as $R_x + R_y < 2$ then

$$R_x + R_y = \begin{cases} 0, & \text{if } R_x + R_y \in [0;1[; \\ 1, & \text{if } R_x + R_y \in [1;2[; \end{cases}$$

  On the other hand,

$$\begin{aligned} E(x+y) &= E(E(x) + R_x + E(y) + R_y) \\ &= E\left((E(x) + E(y)) + (R_y + R_x)\right) \\ &= E(x) + E(y) + E(R_x + R_y) \end{aligned}$$

Consequently,

$$\begin{aligned} & \begin{cases} E(x+y)=E(x)+E(y), & \text{if } R_x+R_y\in[0;1[; \\ E(x+y)=E(x)+E(y)+1, & \text{if } R_x+R_y\in[1;2[; \end{cases} \\ \Rightarrow \quad & \begin{cases} E(x+y)=E(x)+E(y), & \text{if } R_x+R_y\in[0;1[; \\ E(x+y)>E(x)+E(y), & \text{if } R_x+R_y\in[1;2[; \end{cases} \\ \Rightarrow \quad & E(x+y)\geq E(x)+E(y). \end{aligned}$$

- $E(x)-E(y)\geq E(x-y)$?
  Let $x,y\in\mathbb{R}$.

$$E(x)=E((x-y)+y)\geq E(x-y)+E(y)\Rightarrow E(x)-E(y)\geq E(x-y).$$

- $E\left(\frac{E(nx)}{n}\right)=E(x)$?
  According to the definition of the integer part of a real number, we have

$$\begin{aligned} E(x)\leq x<E(x)+1 \quad & \Leftrightarrow \quad nE(x)\leq nx<nE(x)+n \\ & \Leftrightarrow \quad E\left(nE(x)\right)\leq E(nx)<E\left(nE(x)+n\right),\ (E(.) \text{ is an increasing function}) \\ & \Leftrightarrow \quad nE(x)\leq E(nx)<nE(x)+n \text{ (integer part of an integer number)} \\ & \Leftrightarrow \quad E(x)\leq \frac{E(nx)}{n}<E(x)+1 \text{ (definition of E(.))} \\ & \Leftrightarrow \quad E\left(\frac{E(nx)}{n}\right)=E(x). \end{aligned}$$