Chapter 1

Real function

A real-valued function of a real variable relates a real value to any number within its domain. This type of numerical function makes it possible, in particular, to formulate a relationship between two physical quantities. It is characterized by its graphical representation in the coordinate plane, and can also be defined by a specific formula, differential equation, or analytical form.

1.1 Numerical function

Definition 1.1

Let E and F be two sets and f be a relation from the set E to the set F. We say that f is a function, if every element of E is associated with at most one element of F, and we write:

$$\begin{array}{rccc} f: & E & \longrightarrow & F \\ & x & \longmapsto & f(x) = y \end{array}$$

is an application

Definition 1.2

We say that f is a numerical function if and only if:

$$\begin{array}{rccc} f: & E \subset \mathbb{R} & \longrightarrow & F \subset \mathbb{R} \\ & x & \longmapsto & f(x) = y \end{array}$$

is an application

In other words, f is a numerical function if and only if for every element x in E, its image in F is at most one real number.

Example 1.1

The function: inverse of x

$$\begin{array}{rcl} f:]-\infty,0[\cup]0,+\infty[& \longrightarrow & \mathbb{R} \\ x & \longmapsto & \dfrac{1}{x}. \end{array}$$

1.1.1 Domain of definition

To determine the domain of a numerical function, we need to find the set of numbers for which the function is defined. So we can define the domain of a numerical function as follows:

Definition 1.3

Let f be a numerical function. The domain of f, denoted by $\text{Dom}(f)$, is the set of all real numbers x such that $f(x)$ is a well-defined real number and we write:

$$\text{Dom}(f) = \{x \in \mathbb{R} \mid f(x) \in \mathbb{R}\}$$

In other words, the domain of a numerical function is the set of all values for which the function is defined and has a real number output.

$$\begin{array}{rcl} f: \text{Dom}(f) \subset \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \longmapsto & f(x). \end{array}$$

Example 1.2

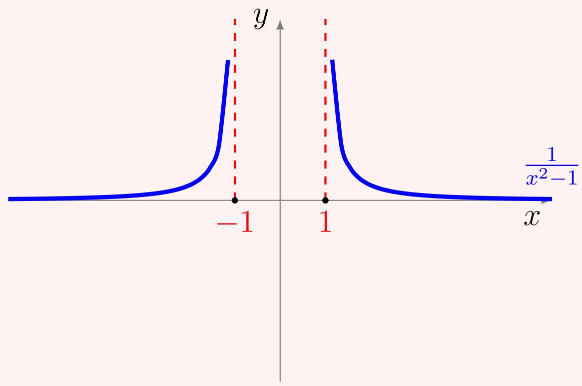
Let the function f be defined as follows

$$\begin{array}{rcl} f: \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \longmapsto & f(x) = \dfrac{1}{(x^2-1)}. \end{array}$$

The variable x is in the denominator of the function. We know that a real number cannot have a denominator equal to zero. Therefore, we cannot compute the image of the numbers 1 and -1 under the function f. Hence, f is defined for all real numbers except -1 and 1, and we write:

$$(f: \quad \text{Defined} \quad) \Longleftrightarrow x^2 - 1 \neq 0$$

$$\begin{aligned} x^2 - 1 &= 0 \Longleftrightarrow (x-1)(x+1) = 0 \\ &\Longleftrightarrow x = 1 \wedge x = -1 \\ &\Longleftrightarrow D_f = \mathbb{R}_{-\{1,-1\}} \\ &\Longleftrightarrow D_f =]-\infty, -1[\cup]-1, 1[\cup]1, +\infty[\end{aligned}$$



1.1.2 Function curve

Definition 1.4

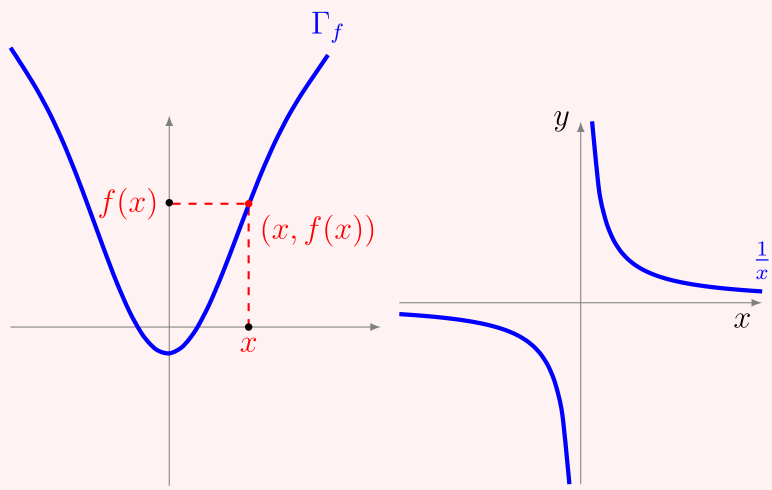
The graph of the function $f : U \to \mathbb{R}$ is the subset Γ_f of $\mathbb{R}^2$ defined as follows:

$$\Gamma_f = \{(x, f(x)) \mid x \in U\}.$$

Example 1.3

To the right the graph of the function $1/x$ and to the left of the graph of the function

$$\frac{1}{2} + \frac{x^2}{2} + \sin\left(\frac{3(x-1)}{2}\right).$$



1.2 Parity and periodicity

In this section, we will learn how to determine whether a function is even, odd, or neither, using its graph or its definition. The symmetry of the function's curve indicates whether it is odd or even.

1.2.1 Even function

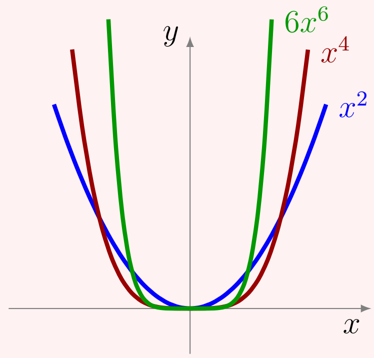
Definition 1.5

We say that f is an even function if:

$$\forall x \in D_f : f(x) = f(-x).$$

Example 1.4

Functions defined on the set $\mathbb{R}$ as $x \mapsto ax^n$ where n is even, are even functions.



If the function f is even, this means that $f(-x) = f(x)$ for all x in the domain of the function. Therefore, if we replace x with $-x$ in the point $M(x_0, f(x_0))$, the point $M'(-x_0, f(-x_0)) = (-x_0, f(x_0))$ is obtained.

Regarding the axis of symmetry, it represents the x-axis, so only the x-coordinates are exchanged. Therefore, we can see that the point $M'(-x_0, f(x_0))$ is the reflection of the point $M(x_0, f(x_0))$ with respect to the axis of symmetry. Thus, the points M and M' are symmetric with respect to the axis of symmetry.

1.2.2 Odd function

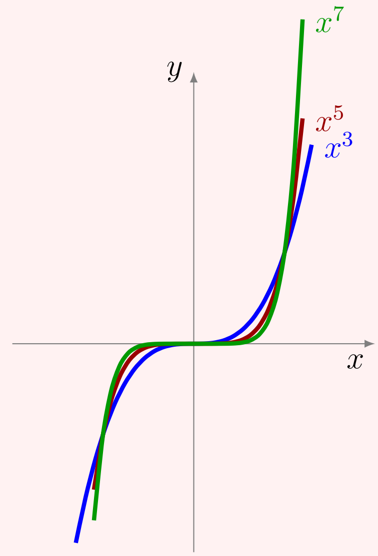
Definition 1.6

We say that f is an odd function if:

$$\forall x \in D_f : f(x) = -f(-x).$$

Example 1.5

Functions defined on the set $\mathbb{R}$ as follows $x \mapsto x^n$ where $(n \in \mathbb{N})$ is an odd functions



If the function f is odd, this means that $f(-x) = -f(x)$ for all x in the domain of the function. Therefore, if we replace x with $-x$ in the point $M(x_0, f(x_0))$, the point $M'(-x_0, f(-x_0)) = (-x_0, -f(x_0))$ is obtained.

Regarding the origin, it is the point $(0, 0)$ on the coordinate plane. Therefore, we can see that the point $M'(-x_0, f(-x_0))$ is the reflection of the point $M(x_0, f(x_0))$ with respect to the origin. Thus, the points M and M' are symmetric with respect to the origin.

1.2.3 Periodic function

Graphically, periodic functions refer to a pattern that is repeated regularly in the Cartesian plane. To fully understand the concept of periodicity, it is important to master the concepts of cycle and period.

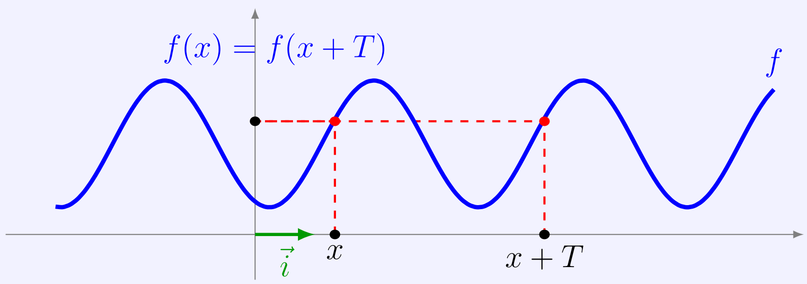
Definition 1.7

The part of the graph that corresponds to the smallest repeating pattern of a periodic function is called one cycle. The gap between two consecutive points that mark the end of the same cycle is called the period.

Definition 1.8

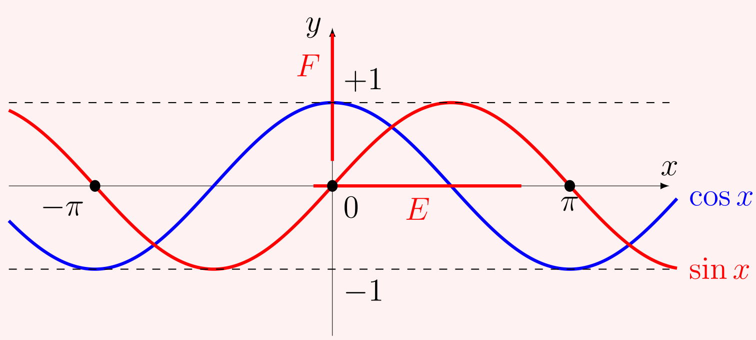
We say that f is a periodic function if there exists $k > 0$ where :

$$\forall x \in D_f : f(x+k) = f(x).$$



Example 1.6

The sine and cosine functions are periodic functions with a period of 2π, while the tangent function is a periodic function with a period of π.



1.2.4 Positive and negative functions

Let f be a numerical function defined on a set D_f, and let Δ be a subset of D_f.

Definition 1.9

The function f is said to be positive (or strictly positive) on Δ if:

$$\forall x \in \Delta : f(x) \geq 0 \quad (f(x) > 0).$$

The function f is said to be negative (or strictly negative) on Δ if:

$$\forall x \in \Delta : f(x) \leq 0 \quad (f(x) < 0).$$

Remark 1.2.1.
- *If the function f is positive, its graph lies above the x-axis, and conversely, if the function f is negative, its graph lies below the x-axis.*
- *If the function f is strictly positive or strictly negative, its graph never intersects the x-axis.*

1.2.5 Operations on functions

Let $f : U \to \mathbb{R}$ and $g : U \to \mathbb{R}$ be two defined functions on the same part U of the set $\mathbb{R}$. From this, we can define the following functions:

1) The sum of the functions f and g is the function $f + g : U \to \mathbb{R}$ defined as follows:

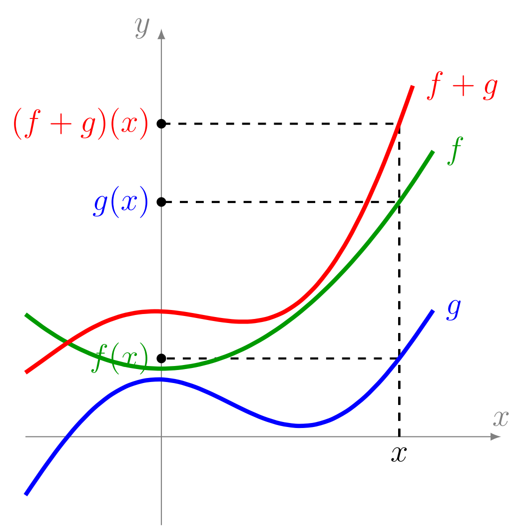
$$\forall x \in U, (f + g)(x) = f(x) + g(x).$$

2) The product of the functions f and g is the function $f \cdot g : U \to \mathbb{R}$ defined as follows:

$$\forall x \in U, (f \cdot g)(x) = f(x) \cdot g(x).$$

3) The product by scalar $\lambda \in \mathbb{R}$ and the function f is the function $\lambda \cdot f : U \to \mathbb{R}$ defined as follows:

$$\forall x \in U, (\lambda \cdot f)(x) = \lambda \cdot f(x).$$



1.2.6 Comparison of two functions

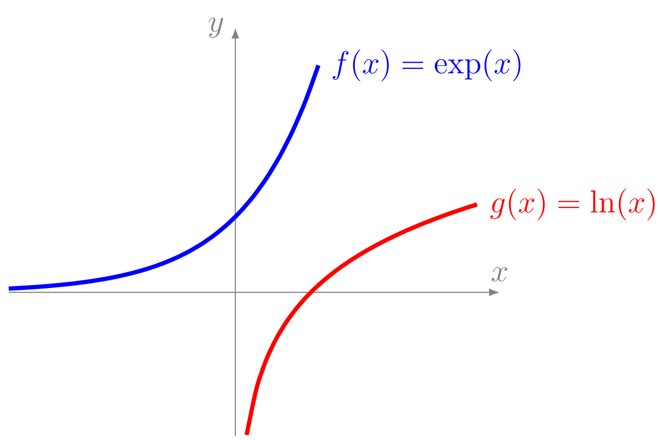
Let f and g be two defined functions on the same domain $\Delta \subset D_f \cap D_g$. We say that f is less than or equal to g, denoted as:

$$f \leq g : \quad \text{if} \quad \forall x \in \Delta, f(x) \leq g(x).$$

We say that f is greater than or equal to g, denoted as:

$$f \geq g : \quad \text{if} \quad \forall x \in \Delta, f(x) \geq g(x).$$

Remark 1.2.2. *If the function f is greater than or equal to g, then the graph of the function f lies above the graph of the function g.*



1.2.7 Function monotony

Let f be a function defined on its domain D_f, and let I be a subset of D_f.

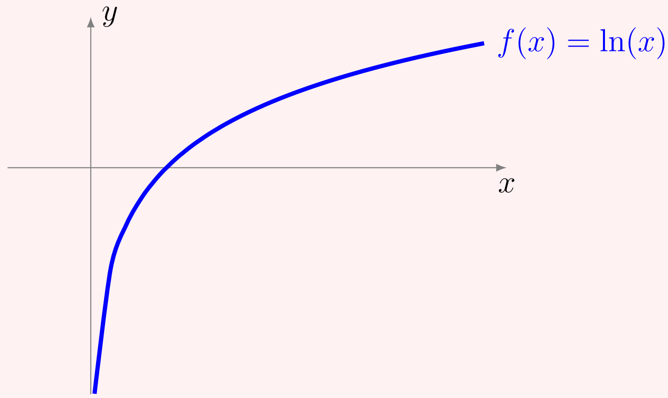
Definition 1.10

We say that f is increasing on I if and only if:

$$\forall (x, y) \in I^2 : x > y \implies f(x) \geq f(y).$$

Example 1.7

The function logarithm $x \mapsto \ln(x)$ is an increasing function on the domain $]0, +\infty[$.



Definition 1.11

We say that f is strictly increasing on I if and only if:

$$\forall (x, y) \in I^2 : x > y \implies f(x) > f(y).$$

Definition 1.12

We say that f is strictly decreasing on I if and only if:

$$\forall (x, y) \in I^2 : x > y \implies f(x) \leq f(y).$$

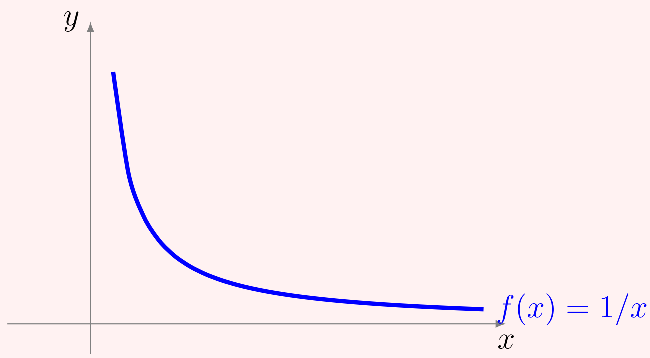
Definition 1.13

We say that f is strictly decreasing on I if and only if:

$$\forall (x, y) \in I^2 : x > y \implies f(x) < f(y).$$

Example 1.8

The inverse function $x \mapsto \frac{1}{x}$ is a strictly decreasing function on the domain $]0, +\infty[$.



1.2.8 Finite function

Before investigating whether a function is bounded or not, it must be defined on a non-empty set, and then we can start searching for the bounds of the function.

Definition 1.14

Let f be a numerical function defined on the set D_f

1) We say that f is bounded above if and only if there exists a real number M such that:

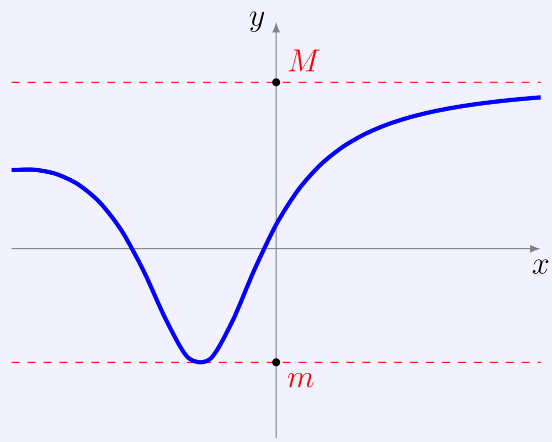
$$\forall x \in D_f : \quad f(x) \leq M.$$

2) We say that f is bounded below if and only if there exists a real number m such that:

$$\forall x \in D_f : \quad m \leq f(x).$$

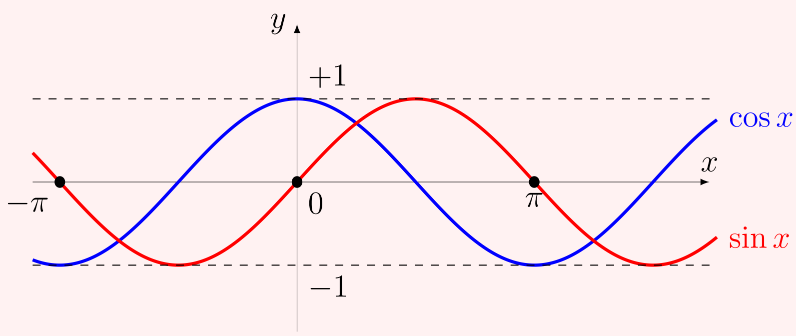
3) We say that f is bounded if and only if there exist two real numbers m and M such that:

$$\forall x \in D_f : \quad m \leq f(x) \leq M.$$



Example 1.9

The sine and cosine functions are bounded functions.



1.2.9 Max and min values of a function

Definition 1.15

Let f be a numerical function defined on the set D_f, and let $x_0 \in D_f$ and I be a subset of D_f.

1) We say that the number $f(x_0)$ is the absolute maximum value of the function f at the point x_0 if:

$$\forall x \in D_f : \quad f(x) \leq f(x_0).$$

2) We say that the number $f(x_0)$ is a relative maximum value of the function f at the

point x_0 in the domain I if $x_0 \in I$ and:

$$\forall x \in I \quad f(x) \leq f(x_0).$$

3) We say that the number $f(x_0)$ is the absolute minimum value of the function f at the point x_0 if:

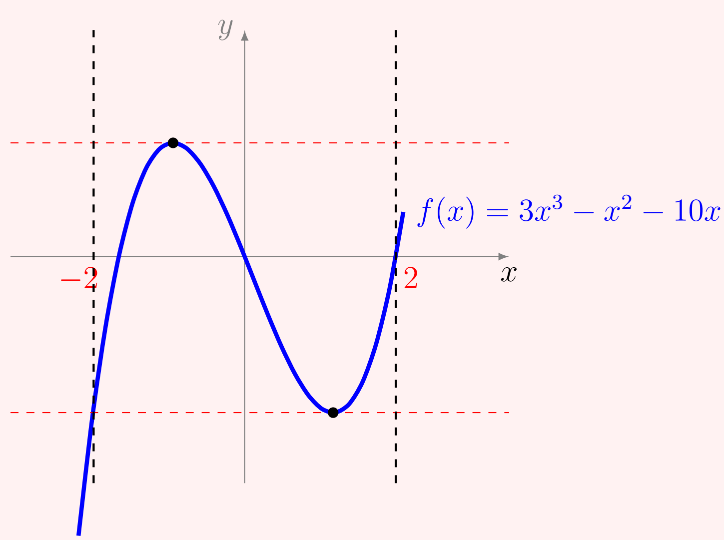
$$\forall x \in D_f \quad f(x) \geq f(x_0).$$

4) We say that the number $f(x_0)$ is a relative minimum value of the function f at the point x_0 in the domain I if $x_0 \in I$ and:

$$\forall x \in I \quad f(x) \geq f(x_0).$$

Example 1.10

The function f has an upper limit and a lower limit at the two specified points in the graph on the domain $[2, 2]$.



1.3 Limits

Limits are one of the fundamental concepts in mathematics and an important concept in analysis, upon which the concepts of continuity, differentiation, and integration rely. Undoubtedly, the reader has already studied the topic of limits, but in this chapter, we study limits in more detail.

1.3.1 Definitions

End at point

Definition 1.16

We say that a subset V of $\mathbb{R}$ is a neighborhood of the point x_0 if it contains an open set that includes the point x_0.

Let $f : I \to \mathbb{R}$ be a function defined on the domain I of $\mathbb{R}$. Let $x_0 \in \mathbb{R}$ be a point in the domain I.

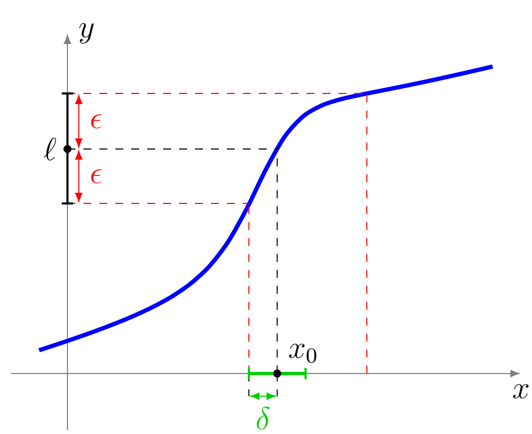
Definition 1.17

We say that the function f, defined in a neighborhood of the point x_0 (possibly undefined at the point x_0), has a limit $\ell \in \mathbb{R}$ at the point x_0 if:

$$\forall \epsilon > 0 \quad \exists \delta > 0 \quad \forall x \in I \quad |x - x_0| < \delta \implies |f(x) - \ell| < \epsilon.$$

and we say that the function $f(x)$ approaches ℓ as x approaches x_0, and we write:

$$\lim_{x \to x_0} f(x) = \ell \text{ or } \quad \lim_{x_0} f = \ell.$$



Example 1.11

Let $f(x) = 3x - 2$, the task is to find the limit at the point $x_0 = 1$. We have:

$$\lim_{x\to 1} f(x) = \lim_{x\to 1}(3x-2) = 1$$

Using the definition, we find

$$\forall \epsilon > 0, \quad \exists \delta > 0, \quad \forall x \in \mathbb{R}, \quad |x - x_0| < \delta \implies |f(x) - \ell| < \epsilon$$

$$\begin{aligned}
|x-1| < \delta \quad &\implies \quad |3x-2-1| < \epsilon \\
&\implies \quad |3x-3| < \epsilon \\
&\implies \quad |3(x-1)| < \epsilon \\
&\implies \quad 3|(x-1)| < \epsilon \\
&\implies \quad |(x-1)| < \frac{\epsilon}{3}
\end{aligned}$$

It means that taking the value $\delta = \frac{\epsilon}{3}$ is sufficient to show that for any x satisfying $|x-1| < \delta$, we have $|f(x) - 1| < \epsilon$.

$$\lim_{x\to 1} f(x) = 1.$$

Let f be a function defined on the set of points of the form $]a, x_0[\cup]x_0, b[$.

Definition 1.18

.

1) We say that the function f tends to $+\infty$ at the point x_0 if

$$\forall A > 0, \quad \exists \delta > 0, \quad \forall x \in I : \quad |x - x_0| < \delta \implies f(x) > A.$$

we write:

$$\lim_{x\to x_0} f(x) = +\infty.$$

2) We say that the function f has a limit of $-\infty$ at the point x_0 if:

$$\forall A > 0, \quad \exists \delta > 0, \quad \forall x \in I : \quad |x - x_0| < \delta \implies f(x) < -A.$$

we write:

$$\lim_{x\to x_0} f(x) = -\infty.$$

Let the function $f : I \to \mathbb{R}$ be defined on a set of the form $I =]a, +\infty[$.

Definition 1.19

1) We say that the function f converges to the limit $\ell \in \mathbb{R}$ as x approaches infinity, denoted by $+\infty$, if:

$$\forall \epsilon > 0, \quad \exists B > 0, \quad \forall x \in I : \quad x > B \implies |f(x) - \ell| < \epsilon.$$

we write:

$$\lim_{x \to +\infty} f(x) = \ell \ \text{ Or } \quad \lim_{+\infty} f = \ell.$$

2) We say that the function f converges to infinity, denoted by $+\infty$, as x approaches to $+\infty$, if:

$$\forall A > 0, \quad \exists B > 0, \quad \forall x \in I : \quad x > B \implies f(x) > A.$$

we write:

$$\lim_{x \to +\infty} f(x) = +\infty.$$

Similarly, we define the limit at negative infinity for a function f defined on a set of the form $]-\infty, a[$. We say that the function f converges to the limit $\ell \in \mathbb{R}$ as x approaches negative infinity, denoted by $-\infty$, if

$$\forall \epsilon > 0, \quad \exists B > 0, \quad \forall x \in I : \quad x < B \implies |f(x) - \ell| < \epsilon.$$

we write:

$$\lim_{x \to -\infty} f(x) = \ell \ \text{ Or } \quad \lim_{-\infty} f = \ell.$$

1.3.2 Operations on limits

Let f and g be two functions. Let x_0 be a point where $x_0 = \pm\infty$.

Proposition 1.3.1. *If we have*

$$\lim_{x_0} f = \ell \in \mathbb{R} and \ \lim_{x_0} g = \ell' \in \mathbb{R}$$

then:

- *For every* $\lambda \in \mathbb{R}$, $\lim_{x_0}(\lambda \cdot f) = \lambda \cdot \ell$.

- $\lim_{x_0}(f+g) = \ell + \ell'$

- $\lim_{x_0}(f \cdot g) = \ell \cdot \ell'$

- *If* $\ell \neq is 0$, *then* $\lim_{x_0} \frac{1}{f} = \frac{1}{\ell}$

- *If also* $\lim_{x_0} f = +\infty$ *(or* $-\infty$*), then* $\lim_{x_0} \frac{1}{f} = 0$.

1.4 Continuity

1.4.1 Continuity at a point

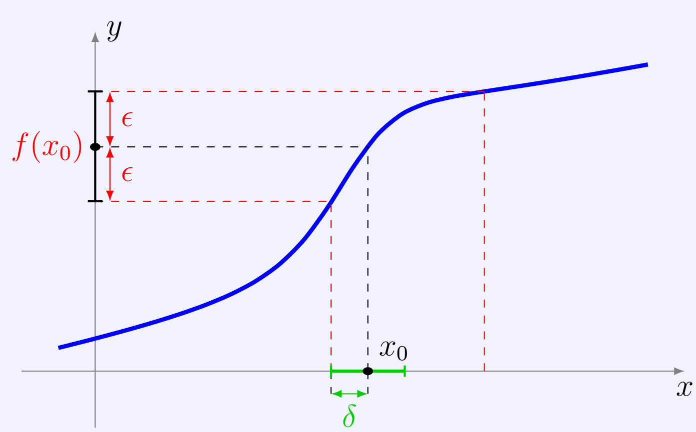
Definition 1.20

Let $f : I \to \mathbb{R}$ be a function defined on the domain I of the real numbers. Let $x_0 \in \mathbb{R}$ be a point in the domain I. We say that the function f is continuous at the point x_0 if the following holds:

$$\forall \epsilon > 0, \quad \exists \delta > 0, \quad \forall x \in I, \quad |x - x_0| < \delta \implies |f(x) - f(x_0)| < \epsilon,$$

we write:

$$\lim_{x \to x_0} f(x) = f(x_0).$$



Example 1.12

The function $f(x) = e^x$ is continuous at the point $x_0 = 0$ because

$$\lim_{x \to x_0} f(x) = \lim_{x \to 0} e^x = e^0 = 1 = f(x_0).$$

1.4.2 Continuity on domain

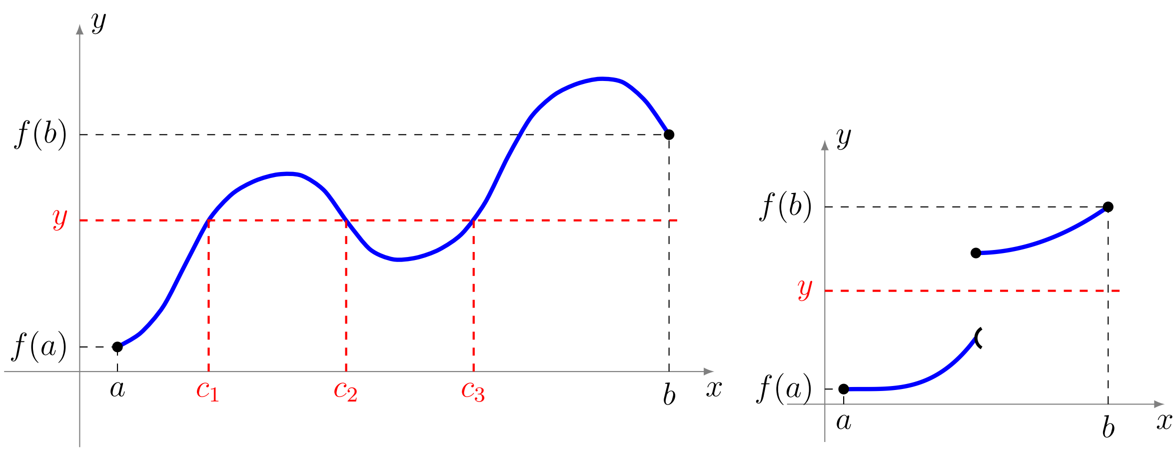
Definition 1.21

Let $f : I \to \mathbb{R}$ be a function defined on the domain I of $\mathbb{R}$.
We say that the function f is continuous on the domain I if it is continuous on all points of the domain I. We denote the set of continuous functions on the domain of I as $\mathscr{C}(I)$.

Mean Value Theorem

Theorem 1.4.1. *Let $f : [a, b] \to \mathbb{R}$ be a function that is continuous on the closed interval $[a, b]$. For any real number y that lies between $f(a)$ and $f(b)$, there exists a real number $c \in [a, b]$ such that $f(c) = y$.*

(In the left figure), the real number c is not necessarily unique. On the other hand, if the function is not continuous, then the theorem does not hold (as shown in the figure on the right).



1.4.3 Continuous extension

A continuous extension of a function allows us to extend its domain or range smoothly while preserving its continuity, enabling us to analyze its behavior in a broader context and overcome limitations imposed by its original definition.

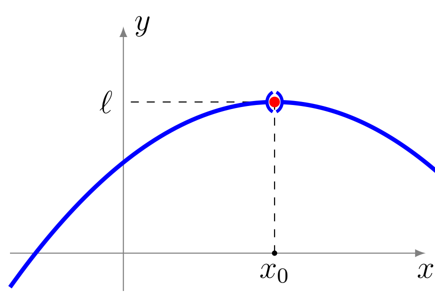
Definition 1.22

Let the domain I, x_0 be the point from I and $f : I_{\{}x_0\} \to \mathbb{R}$ be a function.

1) We say that the function f is continually extendable at the point x_0 if f accepts a finite limit at x_0, and we write:
$$\ell = \lim_{x_0} f.$$

2) We then define the function that we denote $\tilde{f} : I \to \mathbb{R}$ for each $x \in I$
$$\tilde{f}(x) = \begin{cases} f(x) & \text{if } \; x \neq x_0 \\ \ell & \text{if } \; x = x_0. \end{cases}$$

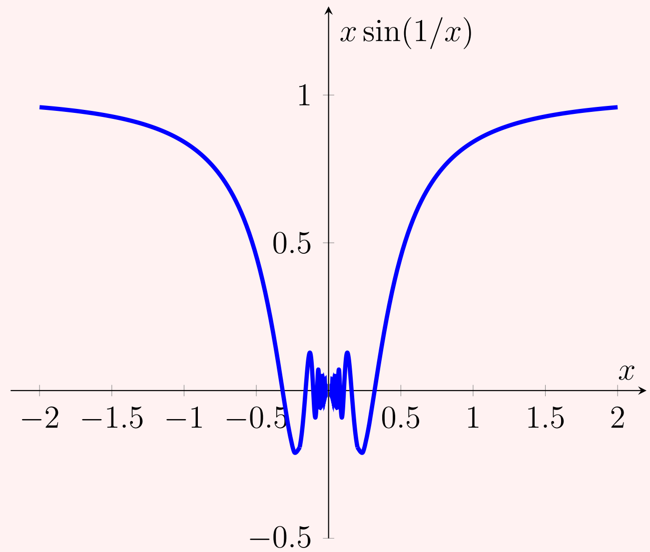
Then the function $\tilde{f}$ is continuous at point x_0, and the extension of the function f is called continuing at point x_0.



Example 1.13

Let the function defined on the set $\mathbb{R}^*$ be as follows
$$f(x) = x \sin\left(\frac{1}{x}\right).$$

Does f accept extension by continuing at 0?
We have for each $x \in \mathbb{R}^*$ that $|f(x)| \leq |x|$, we get that f goes to 0 at 0. That is, it is extendable continuously at 0 and its extension is the function $\tilde{f}$ defined on $\mathbb{R}$ as follows:
$$\tilde{f}(x) = \begin{cases} x \sin\left(\frac{1}{x}\right), & \text{if } \; x \neq 0 \\ 0, & \text{if } \; x = 0. \end{cases}$$



1.4.4 Operations on continuous functions

The primary operations on continuity are immediate consequences of analogous issues at the endpoints.

Proposition 1.4.1. *Let the two functions $f, g : I \to \mathbb{R}$ be given. Let $x_0 \in I$ be a point, hence:*

- $\lambda \cdot f$ *is continuous at* x_0 *(*$\forall \lambda \in \mathbb{R}$*).*
- $f + g$ *is continuous at* x_0*.*
- $f \cdot g$ *is continuous at* x_0*.*
- *If* $f(x_0) \neq 0$*, then* $\frac{1}{f}$ *is continuous at* x_0*.*

Proposition 1.4.2. *Let $f : I \to \mathbb{R}$ and $g : J \to \mathbb{R}$ be two functions, where $f(I) \subset J$. If f is continuous at the point $x_0 \in I$ and g is continuous at the point $f(x_0)$, then the composite function $g \circ f$ is continuous at the point x_0.*

1.5 Derivative and derivation laws

Differentiation and the rules of differentiation are fundamental concepts in calculus in mathematics. Differentiation is concerned with the instantaneous rate of change of a given function, while the rules of differentiation form a set of rules and principles that facilitate the calculation of derivatives in specific ways and provide us with information about the properties of derivative functions.

1.5.1 Derivative at a point

Let I be an open interval in $\mathbb{R}$ and $f : I \to \mathbb{R}$ be a function. Let $x_0 \in I$.

Definition 1.23

We say that the function f is differentiable at the point x_0 if the rate of increase

$$\frac{f(x) - f(x_0)}{x - x_0}$$

accepts a fixed limit as x approaches the value x_0. This fixed limit is called the derivative or the derivative value of the function f at the value x_0, denoted by $f'(x_0)$. We can write it as:

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}.$$

Definition 1.24

We say that the function f is differentiable on the interval I if it is differentiable at every point $x_0 \in I$. The function $x \mapsto f'(x)$ is called the derivative function, denoted by f' or $\frac{df}{dx}$.

Example 1.14

The function defined by $f(x) = x^2$ is differentiable at every point $x_0 \in \mathbb{R}$. We have:

$$\frac{f(x) - f(x_0)}{x - x_0} = \frac{x^2 - x_0^2}{x - x_0} = \frac{(x - x_0)(x + x_0)}{x - x_0} = x + x_0 \xrightarrow[x \to x_0]{} 2x_0.$$

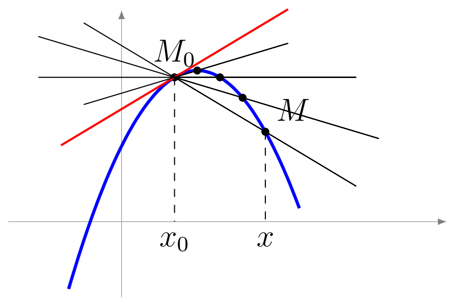
Indeed, we have shown that the derivative of the function f at x_0 is $2x_0$. Alternatively, we can express it as: $f'(x) = 2x$.

1.5.2 Geometric interpretation of the derivative

The straight line passing through the distinct points $(x_0, f(x_0))$ and $(x, f(x))$ has a direction coefficient of $\frac{f(x)-f(x_0)}{x-x_0}$. Ultimately, we find that the directional derivative coefficient is the value $f'(x_0)$.

The equation of the tangent at the point $(x_0, f(x_0))$ is:

$$y = (x - x_0)f'(x_0) + f(x_0).$$



Proposition 1.5.1. *Let f be a function. Then,*

- *f is differentiable at x_0 if and only if the limit*

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

exists and finite.

f is differentiable at x_0 if and only if there exists $\ell \in \mathbb{R}$ (equal to $f'(x_0)$) and a function $\epsilon : I \to \mathbb{R}$ such that $\epsilon(x) \xrightarrow[x \to x_0]{} 0$ with the property that:

$$f(x) = f(x_0) + (x - x_0)\ell + (x - x_0)\epsilon(x).$$

Proposition 1.5.2. *Let I be an open interval and $x_0 \in I$. Let $f : I \to \mathbb{R}$ be a function.*

- *If f is differentiable at x_0, then f is continuous at x_0.*
- *If f is differentiable on I, then f is continuous on I.*

Example 1.15

Let c be a fixed real number. Consider the constant function f that takes the value c. We calculate the derivative of the constant function.

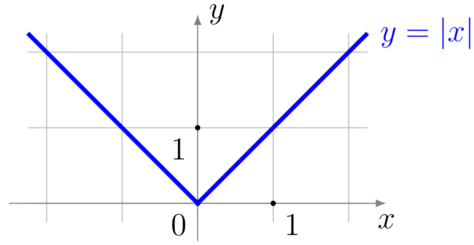
$$\forall x \in \mathbb{R}, \forall h \in \mathbb{R}^*, \frac{f(x+h) - f(x)}{h} = \frac{c - c}{h} = 0,$$

then:

$$\forall x \in \mathbb{R}, f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = 0.$$

Therefore, the derivative of the constant function is zero.

Remark 1.5.1. *The converse is incorrect: for example, the absolute value function* $f(x) = |x|$ *is continuous at* 0 *but not differentiable at* 0*.*



Indeed, the rate of increase at $x_0 = 0$ *achieves:*

$$\frac{f(x) - f(0)}{x - 0} = \frac{|x|}{x} = \begin{cases} +1 & \text{if } x > 0, \\ -1 & \text{if } x < 0. \end{cases}$$

1.5.3 Derivative calculation

Proposition 1.5.3. *Let* $f, g : I \to \mathbb{R}$ *be two differentiable functions on the interval* I*. Hence, for every* $x \in I$*, we have:*

- $(f + g)'(x) = f'(x) + g'(x)$
- $(\lambda f)'(x) = \lambda f'(x)$
  where λ *is a constant real number.*
- $(f \cdot g)'(x) = f'(x)g(x) + f(x)g'(x)$
- $\left(\frac{1}{f}\right)'(x) = -\frac{f'(x)}{f(x)^2}$
  (if $f(x) \neq 0$*)*
- $\left(\dfrac{f}{g}\right)'(x) = \dfrac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$

  (if $g(x) \neq 0$*)*

Remark 1.5.2. *It is easier to remember the following equation:*

$$(f + g)' = f' + g' \qquad (\lambda f)' = \lambda f' \qquad (f \cdot g)' = f'g + fg'$$

$$\left(\frac{1}{f}\right)' = -\frac{f'}{f^2}, \qquad \left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

$$\left(f^{-1}\right)' = \frac{1}{f' \circ f^{-1}}.$$

Proposition 1.5.4. *If f is a function that is differentiable at x and g is a function that is differentiable at $f(x)$, then the composition $g \circ f$ is a function that is differentiable at x, and its derivative is given by:*

$$\big(g \circ f\big)'(x) = g'\big(f(x)\big) \cdot f'(x).$$

Example 1.16

Let's calculate the derivative of the function

$$\ln(1 + x^2).$$

We have $g(x) = \ln(x)$ with $g'(x) = \frac{1}{x}$ and $f(x) = 1 + x^2$ with $f'(x) = 2x$. Then, the derivative of the composition

$$\ln(1 + x^2) = g \circ f(x)$$

is

$$\big(g \circ f\big)'(x) = g'\big(f(x)\big) \cdot f'(x) = g'\big(1 + x^2\big) \cdot 2x = \frac{2x}{1 + x^2}.$$

Differentiation of some common functions

- Constant function: If $f(x) = c$, where c is a constant, then $f'(x) = 0$.
- Power function: If $f(x) = x^n$, where n is a constant, then f'^{n-1}.
- Exponential function: If $f(x) = e^x$, then f'^x.
- Logarithmic function: If $f(x) = \log_b(x)$, where b is the base of the logarithm, then $f'(x) = \frac{1}{x \ln(b)}$.
- Trigonometric functions:

  Sine function: If $f(x) = \sin(x)$, then $f'(x) = \cos(x)$.

  Cosine function: If $f(x) = \cos(x)$, then $f'(x) = -\sin(x)$.

  Tangent function: If $f(x) = \tan(x)$, then $f'^2(x)$.

  where

  $$\sec(x) = \frac{1}{\cos(x)}$$

  Hyperbolic functions:

Hyperbolic sine function: If $f(x) = \sinh(x)$, then $f'(x) = \cosh(x)$.

Hyperbolic cosine function: If $f(x) = \cosh(x)$, then $f'(x) = \sinh(x)$.

Hyperbolic tangent function: If $f(x) = \tanh(x)$, then $f'(x) = \text{sech}^2(x)$.

where:

$$\text{sech}(x) = \frac{1}{\cosh(x)}$$

1.5.4 Successive derivatives

Let $f : I \to \mathbb{R}$ be a differentiable function, and let f' be its derivative. If the derivative function $f' : I \to \mathbb{R}$ is also differentiable, then $f'' = (f')'$ is the second derivative of the function f. In general:

$$f^{(0)} = f, \quad f^{(1)} = f'^{(2)} = f'' \quad \text{and....} \quad f^{(n+1)} = \left(f^{(n)}\right)'$$

If the nth derivative, $f^{(n)}$, exists, we say that f is differentiable n times.

Theorem 1.5.1 (Leibniz's rule)**.**

$$(f \times g)^{(n)} = f^{(n)} \times g + C_n^1 f^{(n-1)} \times g^{(1)} + \cdots + C_n^k \ f^{(n-k)} \times g^{(k)} + \cdots + f \times g^{(n)}$$

In other words:

$$\left(f \times g\right)^{(n)} = \sum_{k=0}^{n} C_n^k \ f^{(n-k)} \times g^{(k)}.$$

To prove the correctness of the Leibniz formula by induction: For $n = 0$, we have:

$$(f \times g)^{(0)}(x) = (f \cdot g)(x) = \sum_{k=0}^{0} C_0^k f^{(k)}(x) g^{(0-k)}(x) = f\,(x)\,g\,(x)$$

So, the property is true for $n = 0$. We assume that:

$$(f \times g)^{(n)}(x) = \sum_{k=0}^{n} C_n^k f^{(k)}(x) g^{(n-k)}(x)$$

and let's demonstrate that:

$$(f \times g)^{(n+1)}(x) = \sum_{k=0}^{n+1} C_{n+1}^k f^{(k)}(x) g^{(n+1-k)}(x)$$

we have

$$(f \times g)^{(n+1)}(x) = ((f \times g)^{(n)})'(x).$$

Therefore

$$(f\times g)^{(n+1)}(x) = \left(\sum_{k=0}^{n} C_n^k f^{(k)}(x) g^{(n-k)}(x)\right)'$$

so

$$(f\times g)^{(n+1)}(x) = \sum_{k=0}^{n} C_n^k \left(f^{(k+1)}(x) g^{(n-k)}(x) + f^{(k)}(x) g^{(n+1-k)}(x)\right)$$

Therefore

$$(f\times g)^{(n+1)}(x) = \sum_{k=0}^{n} C_n^k f^{(k+1)}(x) g^{(n-k)}(x) + \sum_{k=0}^{n} C_n^k f^{(k)}(x) g^{(n+1-k)}(x).$$

We substitute the variable in the first sum: $p = k + 1$

$$\sum_{k=0}^{n} C_n^k f^{(k+1)}(x) g^{(n-k)}(x) = \sum_{p=1}^{n+1} C_n^{p-1} f^{(p)}(x) g^{(n+1-p)}(x)$$

so

$$(f\times g)^{(n+1)}(x) = \sum_{k=1}^{n+1} C_n^{k-1} f^{(k)}(x) g^{(n+1-k)}(x) + \sum_{k=0}^{n} C_n^k f^{(k)}(x) g^{(n+1-k)}(x)$$

Therefore

$$\begin{aligned}(f\times g)^{(n+1)}(x) &= \left(\sum_{k=1}^{n} (C_n^{k-1} + C_n^k)\times \left(f^{(k)}(x)\ g^{(n+1-k)}(x)\right)\right) \\ &\quad + C_n^n f^{(n+1)}(x) g^{(0)}(x) + C_n^0 f^{(0)}(x) g^{(n+1)}(x)\end{aligned}$$

Note that:

$$C_n^{k-1} + C_n^k = C_{n+1}^k \text{ and } C_n^n = C_n^0 = 1$$

Therefore:

$$(f\times g)^{(n+1)}(x) = \left(\sum_{k=1}^{n} C_{n+1}^k f^{(k)}(x) g^{(n+1-k)}(x)\right) + f^{(n+1)}(x) g^{(0)}(x) + f^{(0)}(x) g^{(n+1)}(x)$$

Note that we can include the last two terms in the sum:

$$C_{n+1}^0 f^{(0)}(x) g^{(n+1-0)}(x) = f^{(0)}(x)\ g^{(n+1)}(x)$$

and

$$C_{n+1}^{n+1} f^{(n+1)} g^{(n+1-(n+1))}(x) = f^{(n+1)}(x) g^{(0)}(x).$$

Therefore:

$$(f\times g)^{(n+1)}(x) = \sum_{k=0}^{n+1} C_{n+1}^k f^{(k)}(x) g^{(n+1-k)}(x)$$

Therefore, according to the proof by induction, we have:

$$(\forall n \in \mathbb{N},\, n \leq p)(\forall x \in I)\ :\ (f\times g)^{(n)}(x) = \sum_{k=0}^{n} C_n^k f^{(k)}(x) g^{(n-k)}(x).$$

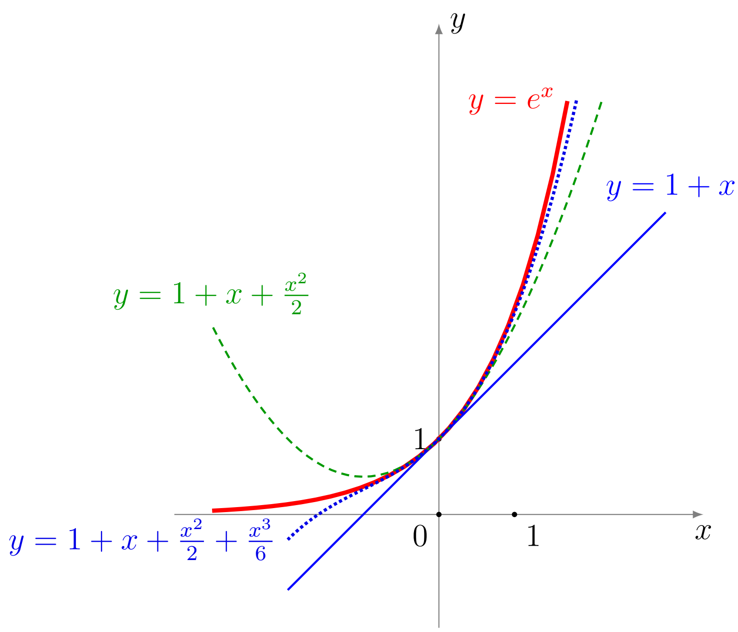
1.6 Limited Expansion

We take the example of the exponential function. You can give an idea of the behavior of the function $f(x) = e^x$ around the point $x = 0$ using its shadow, which has the equation $y = 1 + x$. We have approximated the graph with a straight line.

If we want to find a better approximation, we can take, for example, the equation $y = c_0 + c_1 x + c_2 x^2$. The graph of the function f near the point $x = 0$ is like the equation $y = 1 + x + \frac{1}{2}x^2$.

This equation has a special property: $g(x) = \exp x - \left(1 + x + \frac{1}{2}x^2\right)$, and then $g(0) = 0$, $g'(0) = 0$, and $g''(0) = 0$. We can find the equation of the equivalent parabola, meaning we find a second-degree approximation for the function f.

Of course, if we wanted to be more precise, we would continue to approximate using the third and fourth degrees...



In this part of the chapter, we will look for the nth-degree polynomial approximation for any function that provides a better fit. The results are valid only in the vicinity of a fixed point xx (often near 0). This polynomial approximation will be computed from the successive derivatives at the point under consideration.

1.6.1 Taylor formula

The Taylor formula, named after the mathematician Brook Taylor who developed it in 1712, allows for approximating a differentiable function multiple times around a point using power series, whose coefficients depend solely on the derivatives of the function at that point.

Theorem 1.6.1. *Let $f : I \to \mathbb{R}$ be a function of the class $\mathcal{C}^{n+1}(\mathbb{R})$ $(n \in \mathbb{N})$ and let $x_0, x \in I$, then we have*

$$\begin{aligned} f(x) &= f(x_0) + \frac{(x-x_0)}{1!}f'(x_0) + \frac{(x-x_0)^2}{2!}f''(x_0) + \dots \\ &\quad + \frac{(x-x_0)^n}{n!}f^{(n)}(x_0) + (x-x_0)^n \varepsilon(x-x_0), \end{aligned}$$

where

$$\lim_{x \to x_0} \varepsilon(x - x_0) = 0.$$

Example 1.17

Let the function f be defined as follows:

$$\begin{aligned} f :]-1, +\infty[&\to \mathbb{R} \\ x &\mapsto \ln(1+x) \end{aligned}$$

Differentiable infinitely many times, we will compute the Taylor series at the point 0 up to the first three orders.

We have $f(0) = 0$. Then, when we calculate:

$$f'(x) = \frac{1}{1+x} \Longrightarrow f'(0) = 1$$

Afterwards, we calculate:

$$f''(x) = -\frac{1}{(1+x)^2} \Longrightarrow f''(0) = -1.$$

Finally, we calculate:

$$f^{(3)}(x) = \frac{2}{(1+x)^3} \Longrightarrow f^{(3)}(0) = 2.$$

We can demonstrate by induction that:

$$f^{(n)}(x) = \frac{(-1)^{n-1}(n-1)!}{(1+x)^n}$$

Where the value can be calculated:

$$f^{(n)}(0) = (-1)^{n-1}(n-1)!.$$

Thus for $n > 0$ we have:

$$\frac{f^{(n)}(0)}{n!}x^n = \frac{(-1)^{n-1}(n-1)!}{n!}x^n = \frac{(-1)^{n-1}}{n}x^n.$$

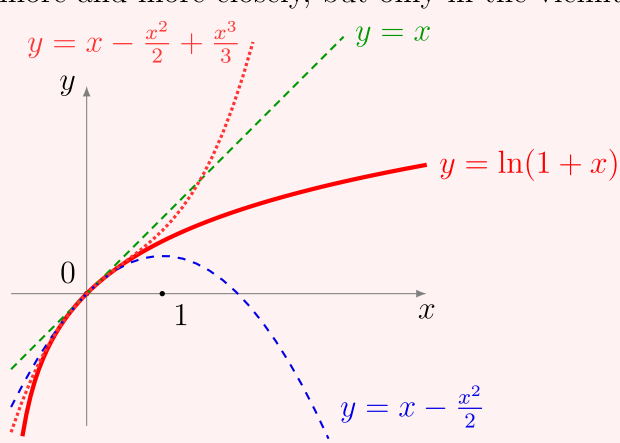
In general, the Taylor polynomial of the function f at the point 0 is

$$P_n(x) = \sum_{k=1}^{n} \frac{(-1)^{k-1}x^k}{k} = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + \frac{(-1)^{n-1}x^n}{n}.$$

Here are the first three Taylor series expansions:

$$\begin{aligned} P_1(x) &= x, \\ P_2(x) &= x - \frac{x^2}{2}, \\ P_3(x) &= x - \frac{x^2}{2} + \frac{x^3}{3}. \end{aligned}$$

In the graph below, the plots of the Taylor series P_1, P_2, and P_3 approach the graph of f more and more closely, but only in the vicinity of 0.



1.6.2 Mac-Laurent formula

Theorem 1.6.2. *Let $f : I \to \mathbb{R}$ be a function of the class $\mathcal{C}^{n+1}(\mathbb{R})$ $(n \in \mathbb{N})$ and let $x \in I$ Then have, by applying Taylor's formula at the point $x_0 = 0$, we find the Mack-Laurent formula:*

$$f(x) = f(0) + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(0) + \ldots + \frac{x^n}{n!}f^{(n)}(0) + \frac{x^n}{n!}\varepsilon(x).$$

Example 1.18

$$1)\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots + (-1)^n \frac{x^{2n}}{(2n)!} + x^{2n+1}\varepsilon(x)$$

$$2)\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + x^{2n+2}\varepsilon(x)$$

$$3)(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \ldots + \frac{\alpha(\alpha-1)\ldots(\alpha-n+1)}{n!}x^n + x^n\varepsilon(x)$$

$$3.1)\quad \alpha = -1 \Longrightarrow \frac{1}{1+x} = 1 - x + x^2 + \ldots + (-1)^n x^n + x^n\varepsilon(n)$$

$$3.2)\quad \alpha = -\frac{1}{2} \Longrightarrow \frac{1}{\sqrt{1+x}} = 1 - \frac{1}{2}x + \frac{3}{8}x^2 + \ldots + (-1)^n \frac{1*3*5\ldots(2n-1)}{2*4*6\ldots 2n}x^n + x^n\varepsilon(x)$$

$$4)e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + x^n\varepsilon(x)$$

$$5)\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \ldots + (-1)^{n-1}\frac{x^n}{n} + x^n\varepsilon(x)$$

1.6.3 Limited expansion of some common functions

-

★ $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + o(x^4)$

★ $\frac{1}{1-x} = 1 + x + x^2 + ... + x^n + o(x^n)$

★ $ch(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + ... + \frac{x^{2n}}{2n!} + o(x^{2n+1})$

★ $sh(x) = 1 + \frac{x^3}{3!} + \frac{x^5}{5!} + ... + \frac{x^{2n+1}}{(2n+1)!} + o(x^{2n+1})$

1.6.4 Operations on limited expansions

We saw previously from Taylor's and the Mac-Loran formula that we can change the limited expansion of a function at the point $a \in \mathbb{R}$ to a limited expansion at the point 0. Therefore, we will explain the operations on the limited expansion only at the point 0.

Let $n \in \mathbb{N}$ and let f and g be functions defined at 0 that accept in the neighborhood of 0 the limited expansion of degree n where:

$$\begin{aligned} f(x) &= p_0 + p_1 x + \cdots + p_n x^n + x^n \epsilon_1(x) \\ &= P_n(x) + x^n \epsilon_1(x) \end{aligned}$$

and

$$\begin{aligned} g(x) &= q_0 + q_1 x + \cdots + q_n x^n + x^n \epsilon_2(x) \\ &= Q_n(x) + x^n \epsilon_2(x) \end{aligned}$$

Proposition 1.6.1. .

- *$f + g$ accepts a limited expansion of degree n at 0 and represents the sum of the two limited expansions of the functions f and g:*

$$(f+g)(x) = f(x) + g(x) = P_n(x) + Q_n(x) + x^n \epsilon(x).$$

- *fg accepts a limited expansion of degree n at 0 and represents the product of the limited expansion of the functions f and g, leaving only the terms with degree less than or equal to n:*

$$(f \cdot g)(x) = f(x) \cdot g(x) = T_n(x) + x^n \epsilon(x)$$

Where $T_n(x)$ is the polynomial $(P_n(x) \cdot Q_n(x))$ stopping at degree n.

- *If $g(0) = 0$ (i.e. $q_0 = 0$) then the function $f \circ g$ accepts a limited expansion at 0 of degree n where the part of the polynomial stopping at degree n is defined by the structure $P(Q(x))$.*

- *If $q_0 \neq 0$ then we have:*

$$\frac{1}{g(x)} = \frac{1}{q_0} \frac{1}{1 + \frac{q_1}{q_0} x + \cdots + \frac{q_n}{q_0} x^n + \frac{x^n \epsilon_2(x)}{q_0}}.$$

- *If F is a primitive function of the function f, then F accepts a limited expansion at a of degree $n+1$ and is written:*

$$F(x) = P_{n+1}(x-a) + (x-a)^{n+1} \eta(x)$$

where: $\lim_{x \to a} \eta(x) = 0$.

Example 1.19

Calculate the limited expansion of the function $\arctan(x)$.

We know that:

$$\arctan'(x) = \frac{1}{1+x^2}.$$

We set:

$$f(x) = \frac{1}{1+x^2}$$

and $F(x) = \arctan(x)$ and we write:

$$\arctan' x = \frac{1}{1+x^2} = \sum_{k=0}^{n} (-1)^k x^{2k} + x^{2n}\epsilon(x).$$

because $\arctan(0) = 0$, then:

$$\arctan x = \sum_{k=0}^{n} \frac{(-1)^k}{2k+1} x^{2k+1} + x^{2n+1}\epsilon(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

Example 1.20

- The limited expansion of the function $\tan x$ at 0 is of order 5.

  Firstly:

  $$\sin x = x - \frac{x^3}{6} + \frac{x^5}{120} + x^5\epsilon(x).$$

  On the other hand

  $$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} + x^5\epsilon(x) = 1 + u$$

  we set

  $$u = -\frac{x^2}{2} + \frac{x^4}{24} + x^5\epsilon(x).$$

  In the calculation we need u^2 and u^3:

  $$u^2 = \left(-\frac{x^2}{2} + \frac{x^4}{24} + x^5\epsilon(x)\right)^2 = \frac{x^4}{4} + x^5\epsilon(x)$$

  then

  $$u^3 = x^5\epsilon(x).$$

so:

$$\begin{aligned}\frac{1}{\cos x} &= \frac{1}{1+u} = 1 - u + u^2 - u^3 + u^3\epsilon(u)\\ &= 1 + \frac{x^2}{2} - \frac{x^4}{24} + \frac{x^4}{4} + x^5\epsilon(x)\\ &= 1 + \frac{x^2}{2} + \frac{5}{24}x^4 + x^5\epsilon(x).\end{aligned}$$

Finely

$$\begin{aligned}\tan x &= \sin x \cdot \frac{1}{\cos x}\\ &= \big(x - \frac{x^3}{6} + \frac{x^5}{120} + x^5\epsilon(x)\big)\cdot\big(1 + \frac{x^2}{2} + \frac{5}{24}x^4 + x^5\epsilon(x)\big)\\ &= x + \frac{x^3}{3} + \frac{2}{15}x^5 + x^5\epsilon(x).\end{aligned}$$

The limited expansion of the function $\frac{1+x}{2+x}$ at 0 of order 4.

$$\begin{aligned}\frac{1+x}{2+x} &= (1+x)\frac{1}{2}\frac{1}{1+\frac{x}{2}}\\ &= \frac{1}{2}(1+x)\left(1 - \frac{x}{2} + \left(\frac{x}{2}\right)^2 - \left(\frac{x}{2}\right)^3 + \left(\frac{x}{2}\right)^4 + o(x^4)\right)\\ &= \frac{1}{2} + \frac{x}{4} - \frac{x^2}{8} + \frac{x^3}{16} - \frac{x^4}{32} + o(x^4),\end{aligned}$$

Example 1.21

Calculate the limited expansion of the function $h(x) = \sin\big(\ln(1+x)\big)$ at 0 of order 3.

- We set $f(u) = \sin u$ and $g(x) = \ln(1+x)$, from which:

$$f\circ g(x) = \sin\big(\ln(1+x)\big) \text{ and } g(0) = 0.$$

We write the limited expansion of order 3 for the function

$$f(u) = \sin u = u - \frac{u^3}{3!} + u^3\epsilon_1(u)$$

for u in the vicinity of 0.

- We set

$$u = g(x) = \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + x^3\epsilon_2(x)$$

for x in the vicinity of 0.

- We calculate u^2:

$$u^2 = \big(x - \frac{x^2}{2} + \frac{x^3}{3} + x^3\epsilon_2(x)\big)^2 = x^2 - x^3 + x^3\epsilon_3(x)$$

and u^3 :

$$u^3 = x^3 + x^3\epsilon_4(x).$$

then:

$$\begin{aligned} h(x) &= f \circ g(x) = f(u) \\ &= u - \frac{u^3}{3!} + u^3\epsilon_1(u) \\ &= \left(x - \frac{1}{2}x^2 + \frac{1}{3}x^3\right) - \frac{1}{6}x^3 + x^3\epsilon(x) \\ &= x - \frac{1}{2}x^2 + \frac{1}{6}x^3 + x^3\epsilon(x). \end{aligned}$$