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DEPARTMENT OF MATHEMATICS

COURSE HANDOUT

LEVEL: SECOND YEAR PHYSICAL LISCENCE

Titled by:

SERIES AND DIFFERENTIAL EQUATIONS

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FOREWORD

This course is intended for 2^{nd} year students of Material Sciences (MS). It has nine (09) main chapters, which expose the methods for calculating infinite sums such as numerical sequences, sequences of functions, series of functions, integer series, and diff equations,...etc. The aim of this course is to generalize the notion of finite sum of terms by studying how the latter behaves when we consider an infinite succession of terms. The key will be to consider these infinite sums, also called series, as the limit of sequences. In other words, when we remember the course on sequences, it will be easier to assimilate the course on the series This is why the first two chapters concerning reminders should not be neglected.

One of the key points of this course will be the study of Fourier series whose applications are quite numerous in other areas of mathematics (notably differential equations and partial differential equations). To reach the chapter concerning Fourier series, however, it will be necessary to take a short path which will take us there in a less abrupt way. As we wrote above, we will recall the structure of $\mathbb{R}$, then the notion of sequences in $\mathbb{R}$ or $\mathbb{C}$. We will then consider the series in their generality, then the sequences and series of functions, to then move on to integer series, to functions developable in integer series and finally Fourier series. We can then solve some differential equations using this theory. The objective of the other chapters of the course will be to solve differential equations using Laplace transforms. This mathematical tool cannot be applied rigorously without a little preliminary work on integrals depending on a parameter. Once these concepts are assimilated, you will have solid tools to solve several types of differential equations and partial differential equations but also problems a little more theoretical.

I hope that this handout will help students in the preparation for the exams and make it easier for them to read other works.

OUAAR Fatima

Chapter 1

SINGLE, DOUBLE AND TRIPLE INTEGRALS

In this chapter I designates a non-trivial interval of $\mathbb{R}$ and $\mathbb{K}$ the set of real numbers or complexes.

1.1 Simple integrals

1.1.1 Reminders

Definition 1.1.1 (Primitive) *Let f and F be functions of I in K, F is a primitive of f on I when F is differentiable on I and $\forall x \in I, \acute{F}(x) = f(x)$.*

Proposition 1.1.1 *If f admits a primitive on I then it admits an infinity of them all equal to a constant.*

Proposition 1.1.2 *Let f and $g \in \mathcal{F}(I, \mathbb{K})$, F be a primitive of f on I and G be a primitive of g on I.*

1) $\forall \alpha, \beta \in \mathbb{K}, (\alpha F + \beta G)$ is a primitive of $(\alpha f + \beta g)$ on I.

2) $\mathcal{R}e(F)$ (resp. $\mathcal{I}m(F)$) is a primitive on I of $\mathcal{R}e(f)$ (resp. $\mathcal{I}m(f)$). [1]

Example 1.1.1 (Search for a primitive of f by transforming expressions) *1) $f : t \longrightarrow \dfrac{1}{t^4 - 1}$ we decompose into simple elements*

2) $f : t \longrightarrow \tan^2 t$ we reveal a usual primitive

3) $f : t \longrightarrow \sin^4 t$ we linearize

4) $f : t \longrightarrow t\cos(\omega x)e^{\alpha x}$ use $f(x) = \mathcal{R}e(e^{\alpha + i\omega x})$

Example 1.1.2 *$f : t \longrightarrow \cos^2 t \sin^3 t$ we make úu^n appear. This method can replace linearization for products of the type $\cos^p x \sin^q x$ with p or q impairs.*

Definition 1.1.2 (Integral) *Let $a, b \in \mathbb{R}, a \leq b$ and $f : [a, b] \to \mathbb{R}$ be a continuous function. The integral from a to b of f is the real denoted $\int_a^b f(t)dt$ which is equal to the algebraic area of the domain delimited by the curve representative of f, the axis (Ox) and the lines $x = a$ and $x = b$, expressed in area unit*

- Extension to any two reals a and b: If $b < a$, we set $\int_a^b f(t)dt = -\int_b^a f(t)dt$.

- Extension to functions with complex values: Let $f \subset \mathcal{F}(I, \mathbb{C})$ is continuous, for all real numbers a and b of I, we set $\int_a^b f(t)dt = \int_a^b \mathcal{R}e(f)(t)dt + i\int_a^b \mathcal{I}m(f)(t)dt$.[10]

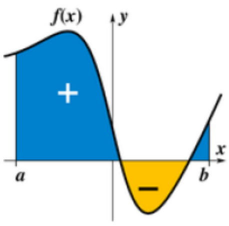

Figure 1.1: Integral Definition.

Remark 1.1.1 *The integration variable is silent i.e. $\int_a^b f(t)dt = \int_a^b f(x)dx = \int_a^b f(u)du = ...$*

Proposition 1.1.3 (Properties of the integral) *Let f and g be continuous on I with values in $\mathbb{K}$ and a, b and c three real numbers of I.*

$\int_a^a f(t)dt = 0$ et $\int_a^b f(t)dt = -\int_b^a f(t)dt$.

Chasles relation: $\int_a^b f(t)dt = \int_a^c f(t)dt + \int_c^b f(t)dt$

Linearity: $\forall \alpha, \beta \in \mathbb{R}, \int_a^b [\alpha f(t) + \beta g(t)]\, dt = \alpha \int_a^b f(t)dt + \beta \int_a^b g(t)dt$

Positivity: Si $a \leq b$ et $f \geq 0$ sur $[a, b]$ alors $\int_a^b f(t)dt \geq 0$

Growth of the integral: if $a \leq b$ and $f \leq g$ on $[a, b]$ then $\int_a^b f(t)dt \leq \int_a^b g(t)dt$

Triangle inequality: If $a \leq b$ *then* $\left|\int\limits_a^b f(t)dt\right| \leq \int\limits_a^b |f(t)|\, dt$

1.1.2 Link between integrals and primitives of a function

Theorem 1.1.1 (Fundamental theorem of analysis) *Let* f *be a continuous function on* I *with values in* $\mathbb{K}$, $F : x \longrightarrow \int\limits_a^x f(t)dt$ *is the unique primitive of* f *on* I *which cancels out at* a.

Consequences:

- Any continuous function on I admits an infinity of primitives on I. Let a be fixed in I, the set of primitives of f on I is $\{\ x \longrightarrow \int\limits_a^x f(t)dt + k\ ,\ k$ describes $\mathbb{K}\}$.
- Let G be any primitive of f on I and $a \in I$, we have: $\forall x \in I, G(x) = G(a) + \int\limits_a^x f(t)dt.$
- The notation: $\int\limits_I f = \int\limits_I f(t)dt$ denotes any primitive of f on I. For example: $\int_{\mathbb{R}} xdx = \frac{x^2}{2} + k$, where $k \in \mathbb{R}$. Attention, here the integration variable is no longer silent.

corollary 1.1.1 *Let* f *be a continuous function on* I,$\forall a, b \in I, \int\limits_a^b f(t)dt = [F(t)]_a^b = F(b) - F(a)$ *where* F *is any primitive of* f *on* I.*[11]*

1.1.3 Integration by parts formula

Definition 1.1.3 *Let* $f : I \to \mathbb{K}$, *we say that* f *is of class* C^1 *on* I *when* f *is differentiable on* I *with* f *continues on* I. *The set of functions of class* C^1 *on* I *with values in* $\mathbb{K}$ *is denoted* $C^1(I, \mathbb{K})$.

Theorem 1.1.2 *If* u *and* v *are two functions of class* C^1 *on* I *then* $\forall a, b \in I, \int\limits_a^b u(t)\acute{v}(t)dt = [u(t)v(t)]_a^b - \int\limits_a^b \acute{u}(t)v(t)dt$

Example 1.1.3 (Classic examples for the calculation of integral) $\int\limits_0^1 xe^x dx$ *and* $\int\limits_1^e t^2 \ln t dt$

Example 1.1.4 (Classic examples for calculating primitive) $\int\limits_1^x \ln t dt$ *et* $\int x \arctan x dx$. *We can directly use the IPP formula when* u *and* v *are of class* C^1 *on* I*:* I*:* $\forall x \in I, \int u(t)\acute{v}(t)dt = u(t)v(t) - \int \acute{u}(t)v(t)dt$ *(Be careful to validate the hypotheses of the theorem).*

1.1.4 Variable change formula

Theorem 1.1.3 *Let f continue on I and $\varphi : [a, b] \to I$, of class C^1 on $[a, b]$. We have*

$$\int_a^b f(\varphi(x))\acute{\varphi}(x)dx \overset{\overset{(1)}{\longrightarrow}}{\underset{\underset{(2)}{\longleftarrow}}{=}} \int_{\varphi(a)}^{\varphi(b)} f(t)dt.$$

1.1.5 Applications

Application to the calculation of integrals

<u>1^{st} case:</u> We want to use the change of variable in the sense (1): We set $\varphi(x) = t$

Method: • We replace $\varphi(x)$ by t

- We replace $\acute{\varphi}(x)dx$ by dt.
- We modify the limits of the integral.
- Example: $\int_0^1 \frac{dx}{chx}$ by setting $e^x = t$

<u>2^{nd} case:</u> We want to use the change of variable in the direction (2): We set $t = \varphi(x)$

Method: • We determine a and b and we verify that φ is of class C^1 on $[a, b]$.

- We replace $\varphi(x)$ by t.
- We replace dt by $\acute{\varphi}(x)dx$.
- We modify the limits of the integral.
- Example: $\int_0^1 \sqrt{1 - t^2}dt$ by setting $t = \sin x$. Be careful to validate the hypotheses of the theorem.[11]

Application to the calculation of primitives

<u>1^{st} case:</u> We want to use the change of variable in the direction (1)

Méthode : • On pose le changement de variable choisi: avec de classe C^1 sur un intervalle de $\mathbb{R}$, à valeurs dans I

- We then have: $dt = \acute{\varphi}(x)dx$.
- We obtain: $\int f(\varphi(x))\acute{\varphi}(x)dx = \int f(t)dt = F(t) = F(\varphi(x))$ where F is an antiderivative of f on I.

Example 1.1.5 $\int \dfrac{dx}{1-\sin x}$ *on* $]0;\pi[$ *by setting* $\tan(x/2) = t$.

2^{nd} case: We want to use the change of variable in the sense (2): We must use a bijective change of variable in order to be able to return to the initial variable.

Method: • We set: $t = \varphi(x)$ with φ bijective of J on I, where J interval of and of class C^1 on J.

• On a alors: $dt = \acute{\varphi}(x)dx$

• We obtain: $\int f(t)dt = \int f(\varphi(x))\acute{\varphi}(x)dx = G(x) = G(\varphi^{-1}(t))$ where G is an antiderivative of $(fo\varphi)x\acute{\varphi}$ on J.

Example 1.1.6 $\int \sqrt{t^2-3}$ *on* $I = \left]-\sqrt{3}, \sqrt{3}\right[$, *setting* $t - 3\sin x = \varphi(x)$.

To know how to do without help: Primitive of $f : x \mapsto \dfrac{1}{ax^2+bx+c}$ on an interval I where $ax^2+bx+c \neq 0$

1^{sr} case: ax^2+bx+c has two real roots x_1 and x_2: We decompose f into simple elements: $\forall x \in I$, $\dfrac{A}{x-x_1} + \dfrac{B}{x-x_2}$ with real A and B. We obtain $\forall x \in I, \int f(x)dx = A\ln|x-x_1| + B\ln|x-x_1|$.

Example 1.1.7 $\int \dfrac{dx}{x^2-1}$ *on* $]-1,1[$.

2^{nd} case: ax^2+bx+c has a double real root x_0: $\forall x \in I, f(x) = \dfrac{A}{(x-x_0)^2}$ with real A. We obtain $\forall x \in I, \int f(x)dx = \dfrac{-A}{(x-x_0)}$

Example 1.1.8 $\int \dfrac{dx}{4x^2+4x+1}$ *on* $]0,+\infty[$

3^{rd} case: ax^2+bx+c has no real roots: $\Delta = b^2-4ac < 0$. We write ax^2+bx+c in canonical form

$$ax^2+bx+c = a\left[\left(x+\frac{b}{2a}\right)^2 - \frac{\Delta}{4a^2}\right] - a\left[\left(x+\frac{b}{2a}\right)^2 - A^2\right]$$

with A real, $A > 0$. We set $x + \dfrac{b}{2a} = t$, change of variable therefore affines C^1 and bijective of $\mathbb{R}$ on $\mathbb{R}$. We obtain $\forall x \in I$,

$$\int f(x)dx = \frac{1}{a}\int \frac{dt}{1+t^2} = \frac{1}{aA}\arctan\left(\frac{t}{A}\right) - \frac{1}{aA}\arctan\left(\frac{x+\frac{b}{2a}}{A}\right)$$

1.2 Double integrals

Multiple integrals constitute the generalization of so-called simple integrals: that is to say the integrals of a function of a single real variable. Here we focus on generalization to functions with a greater number of variables (two or three). Recall that a real function f, defined on an interval $[a, b]$, is said to be Riemann integrable if it can be framed between two staircase functions; hence any continuous function is integrable. The integral of f over $[a, b]$, denoted $\int_a^b f(t)dt$, is interpreted as the area between the graph of f, the axis (XoX) and the lines of equations $x = a, a = b$. By subdividing $[a, b]$ into n subintervals $[x_{i-1}, x_i]$ of the same length $\Delta x = \dfrac{b-a}{n}$, we define the integral of f over $[a, b]$ by:[3]

$$\int_a^b f(x)dx = \lim_{n \longrightarrow +\infty} \sum_{i=1}^{n} f(a_i)(x_i - x_{i-1}), \qquad a_i \in [x_{i-1}, x_i]$$

where $f(a_i)(x_i - x_{i-1})$ represents area of the base rectangle $[x_{i-1}, x_i]$ and height $f(a_i)$:

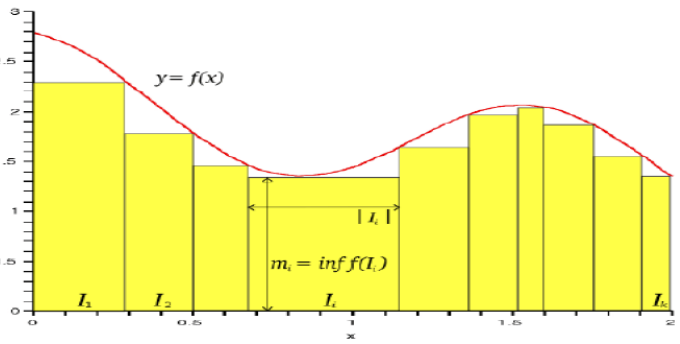

Figure 1.2: Principle of double integral.

1.2.1 Principle of the double integral on a rectangle

Let f be the real function of the two variables x and y, continuous on a rectangle $D = [a, b] \times [c, d]$ of $\mathbb{R}^2$. Its representation is a surface S in the space provided with the reference $\left(O, \vec{i}, \vec{j}, \vec{k}\right)$. We divide D into sub-rectangles, in each sub-rectangle $[x_{i-1}, x_i] \times [y_{i-1}, y_i]$ we choose a point $M(x, y)$ and we calculate the image of (x, y) for the function f. The sum of the volumes of the columns whose base is sub-rectangles and the height $f(x, y)$ is an approximation of the volume between the plane $Z = 0$ and the surface S. When the grid becomes sufficiently "fine" so that the diagonal of each sub-rectangle tends towards 0, this volume will be the limit of the Riemann sums and we note it:[12]

$$\iint\limits_{D} f(x,y)\,dxdy = \lim_{n\longrightarrow+\infty} \frac{(b-a)(d-c)}{n^2} \sum_{1\le i\le n;1\le j\le n} f\left(a+i\frac{(b-a)}{n}, c+j\frac{(d-c)}{n}\right).$$

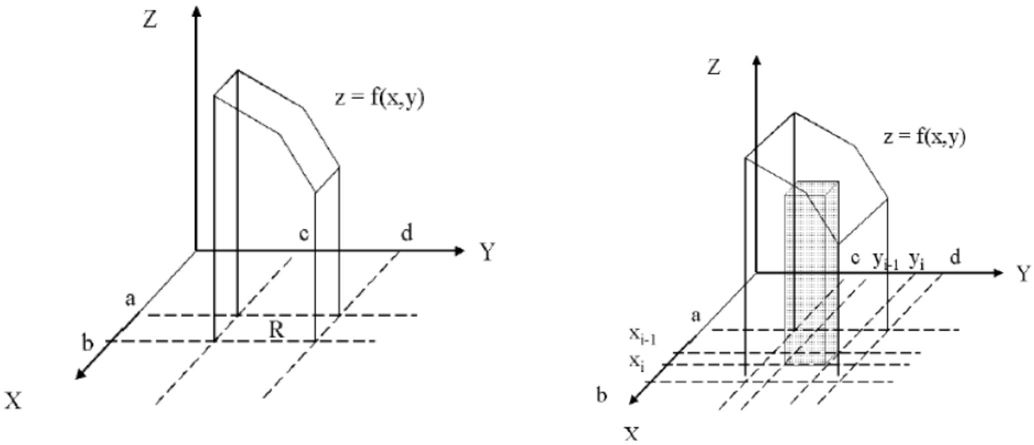

Figure 1.3: Double integral.

Example 1.2.1 *Using the definition, calculate* $\iint\limits_{[0,1]\times[0,1]} (x+2y)\,dxdy$.

Remark 1.2.1 *A priori, the double integral is made to calculate volumes, just as the simple integral was made to calculate an area.*

In a double integral, the terminals at x and y must always be arranged in ascending order.

Theorem 1.2.1 *Let D be a bounded domain of $\mathbb{R}^2$. Then any continuous function $f : D \longrightarrow \mathbb{R}$ is integrable in the Riemann sense.[5]*

1.2.2 Properties of double integrals

1. The double integral over a domain D is linear:

$$\iint\limits_{D} (\alpha f + \mu g)(x,y)\,dxdy = \alpha \iint\limits_{D} f(x,y)\,dxdy + \mu \iint\limits_{D} g(x,y)\,dxdy.$$

2. If D and $\acute{D}$ are two domains such that $D \cap \acute{D} = \left\{ \begin{array}{ll} \emptyset, & \text{or} \\ \text{a curve}, & \text{or} \\ \text{isolated points}, & \text{or} \end{array} \right\}$, then:

$$\iint\limits_{D \cup \acute{D}} f(x,y)\, dxdy = \iint\limits_{D} f(x,y)\, dxdy + \iint\limits_{\acute{D}} f(x,y)\, dxdy.$$

3. If $f(x,y) \geq 0$ at any point in D, with f not identically zero, then $\iint\limits_{D} f(x,y)\, dxdy$ is strictly positive.

4. If $\forall (x,y) \in D, f(x,y) \leq g(x,y)$, then $\iint\limits_{D} f(x,y)\, dxdy \leq \iint\limits_{\acute{D}} g(x,y)\, dxdy$.

5. $\left| \iint\limits_{D} f(x,y)\, dxdy \right| \leq \iint\limits_{D} |f(x,y)|\, dxdy$.

1.2.3 Fubini formulas

Theorem 1.2.2 *Let f be a continuous function on a rectangle $D = [a,b] \times [c,d]$ of $\mathbb{R}$. We have:*

$$\iint\limits_{D} f(x,y)\, dxdy = \int\limits_{c}^{d} \left[\int\limits_{a}^{b} f(x,y)\, dx \right] dy.$$

So we calculate a double integral over a rectangle by calculating two single integrals:

- By first integrating with respect to x between a and b (leaving y constant). The result is a function of y.
- By integrating this expression of y between c and d. Alternatively, we can do the same by integrating first at y and then at x.

Special case: If $g : [a,b] \longrightarrow \mathbb{R}$ and $h : [c,d] \longrightarrow \mathbb{R}$ are two continuous functions, then $\iint\limits_{[a,b]\times[c,d]} g(x)h(y)dxdy = \left(\int\limits_{a}^{b} g(x)dx \right) \left(\int\limits_{c}^{d} h(y)dy \right)$.

Example 1.2.2 *Calculation of $I = \iint\limits_{[0,\frac{\pi}{2}]\times[0,\frac{\pi}{2}]} \sin(x+y)\, dxdy$. According to Fubini, we have:*

$$I = \int\limits_{0}^{\frac{\pi}{2}} \left[\int\limits_{0}^{\frac{\pi}{2}} \sin(x+y)\, dx \right] dy = \int\limits_{0}^{\frac{\pi}{2}} \left[\int\limits_{0}^{\frac{\pi}{2}} \sin(x+y)\, dy \right] dx = \int\limits_{0}^{\frac{\pi}{2}} (\cos y + \sin y)\, dy = [\sin y - \cos y]_{0}^{\frac{\pi}{2}} = 2.$$

In this example x and y play the same role.

Example 1.2.3 *Calculation of* $I = \iint\limits_{[0,1]\times[2,5]} \frac{1}{(1+x+2y)^2}dxdy$. *Let's calculate*

$$I = \int_2^5 \left[\int_0^1 \frac{1}{(1+x+2y)^2}dx\right] dy = \int_2^5 \left[\frac{1}{(1+x+2y)}\right]_0^1 dy$$
$$= \frac{1}{2}\left[\ln(1+2y) - \ln(2+2y)\right]_2^5 = \frac{1}{2}\ln\frac{11}{10}.$$

Example 1.2.4 *Calculate the integral* $I = \iint\limits_{[0,\frac{\pi}{2}]\times[0,\frac{\pi}{2}]} \sin(x)\cos(y)dxdy$.

Theorem 1.2.3 *Let f be a continuous function on a bounded domain D of $\mathbb{R}^2$. The double integral $I = \iint\limits_D f(x,y)\,dxdy$ is calculated in one of the following ways:*
- If we can represent the domain D in the form $D = \left\{(x,y) \in \mathbb{R}^2 / f_1(x) \le y \le f_2(x), a \le x \le b\right\}$ then

$$\iint\limits_D f(x,y)\,dxdy = \int_a^b \left[\int_{f_1(x)}^{f_2(x)} f(x,y)\,dy\right] dx.$$

- If we can represent the domain D in the form $D = \left\{(x,y) \in \mathbb{R}^2 / g_1(x) \le x \le g_2(x), c \le y \le d\right\}$, then:

$$\iint\limits_D f(x,y)\,dxdy = \int_c^d \left[\int_{g_1(x)}^{g_2(x)} f(x,y)\,dx\right] dy.$$

- If both representations are possible, the two results are obviously equal.

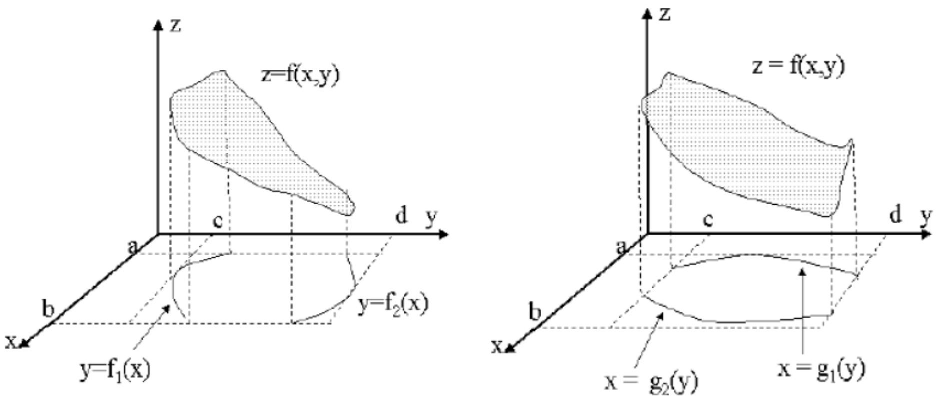

Figure 1.4: Theorem ilustration.

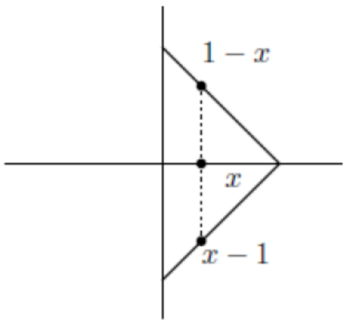

Figure 1.5: The domain D.

Example 1.2.5 *Calculate the integral* $\iint\limits_D (x^2+y^2)\,dxdy$ *with* D *is the triangle with vertices* $(0,1), (0,-1)$ *and* $(1,0)$. *For this we will define* D *analytically by the inequalities:*

$$D = \left\{(x,y) \in \mathbb{R}^2 / x-1 \le y \le 1-x, 0 \le x \le 1\right\}$$

$$\iint\limits_D (x^2+y^2)\,dxdy = \int\limits_0^1 \left[\int\limits_{x-1}^{1-x} (x^2+y^2)\,dy\right] dx = \int\limits_0^1 \left[x^2 y + \frac{y^3}{3}\right]_{x-1}^{1-x} dx = \frac{1}{3}.$$

Example 1.2.6 *Calculate* $I = \iint\limits_D (x+2y)\,dxdy$ *on the domain* D *formed by the union of the left part of the unit disk and the triangle of vertices* $(0,-1), (0,1)$ *and* $(2,1)$. *We have*

$$I = \int\limits_{-1}^{1} \left[\int\limits_{-\sqrt{1-y^2}}^{y+1} (x+2y)\,dx\right] dy = \int\limits_{-1}^{1} \left(3y + 3y^2 + 2y\sqrt{1-y^2}\right) dy = 2.$$

Example 1.2.7 *Calculate the integral* $I = \iint\limits_D e^{x^2} dxdy$ *where* $D = \left\{(x,y) \in \mathbb{R}^2 / 0 \le y \le x \le 1\right\}$. *The domain is the interior of the triangle limited by the* x *axis, the line* $x = 1$ *and the line* $y = x$. *In this case we are obliged to integrate first with respect to* y *then with respect to* x, *because the primitive of the function is not expressed using the usual functions. Hence* $I = \int\limits_0^1 \left[\int\limits_0^x e^{x^2} dy\right] dx = \int\limits_0^1 x e^{x^2} dx = \frac{e-1}{2}$.

Example 1.2.8 *Calculate* $I = \int\limits_0^4 \left[\int\limits_{2x}^8 \sin\left(y^2\right) dy\right] dx = \int\limits_0^8 \left[\int\limits_0^{\frac{y}{2}} \sin\left(y^2\right) dx\right] dy = \frac{1}{4}\int\limits_0^8 2y \sin\left(y^2\right) dy = \frac{1-\cos 64}{4}$.

1.2.4 Change of variable

We will have a result similar to that of the simple integral, where the change of variable $x = \varphi(t)$ required us to replace the "dx" by $\acute{\varphi}(t)$. It is the Jacobian which will play the role of the derivative[1].[7]

Theorem 1.2.4 *Let $(u, v) \in \Delta \longrightarrow (x, y) = \varphi(u, v) \in D$ be a bijection of class C^1 from domain Δ to domain D. Let $|J_\varphi|$ the absolute value of the determinant of the Jacobian matrix of φ. So, we have:*

$$\iint\limits_D f(x, y)\, dxdy = \iint\limits_\Delta f \circ \varphi(u, v)\, |J_\varphi|\, dudv.$$

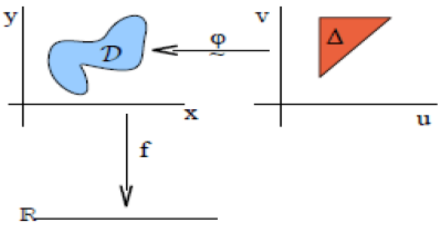

Figure 1.6: Changement de variable pour les intégrales doubles.

Example 1.2.9 *Calculate $\iint\limits_D (x-1)^2\, dxdy$ on the domain with*

$$D = \left\{(x, y) \in \mathbb{R}^2 / -1 \leq x + y \leq 1, -2 \leq x - y \leq 2\right\}.$$

By changing the variable $u = x + y, v = x - y$. The domain D in (u, v) is therefore the rectangle $\{-1 \leq u \leq 1, -2 \leq v \leq 2\}$. We also have $x = \dfrac{u+v}{2}, y = \dfrac{u-v}{2}$. The Jacobian of this change of

[1] We call the Jacobian matrix $\varphi : \mathbb{R}^n \longrightarrow \mathbb{R}^p$ of the matrix with p rows and n columns:

$$J_\varphi = \begin{pmatrix} \frac{\partial \varphi_1}{\partial x_1} & \frac{\partial \varphi_1}{\partial x_2} & \cdots\cdots & \frac{\partial \varphi_1}{\partial x_n} \\ \frac{\partial \varphi_2}{\partial x_1} & \frac{\partial \varphi_2}{\partial x_1} & \cdots\cdots & \frac{\partial \varphi_2}{\partial x_n} \\ . & . & \cdots\cdots & . \\ . & . & \cdots\cdots & . \\ \frac{\partial \varphi_p}{\partial x_1} & \frac{\partial \varphi_p}{\partial x_2} & \cdots\cdots & \frac{\partial \varphi_p}{\partial x_n} \end{pmatrix}$$

The first column contains the partial derivatives of the coordinates of φ with respect to the first variable x_1, the second column contains the partial derivatives of the coordinates of φ with respect to the second variable x_2 and so on.

variables is $J = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix}$ *whose determinant is* $\frac{-1}{2}$. *And so*

$$I = \frac{1}{8} \int_{-2}^{2} \left[\int_{-1}^{1} (u+v-2)^2 \, du \right] dv = \frac{136}{3}.$$

Remark 1.2.2 *- If* $|\det(J_\varphi)| = 1$, *we obtain* $\iint\limits_D f(x,y)\, dxdy = \iint\limits_\Delta f[\varphi(u,v)]\, dudv$

- This allows us to use symmetries: if for example $\forall (x,y) \in D, (-x,y) \in D$ *et* $f(-x,y) = f(x,y)$ *then* $\iint\limits_D f(x,y)\, dxdy = 2 \iint\limits_{\acute{D}} f(x,y)\, dxdy$, *where* $\acute{D} = D \cap (\mathbb{R}^+ \times \mathbb{R})$.

Changing variable to polar coordinates

Let $\varphi : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be such that $(r,\theta) \longrightarrow (r\cos\theta, r\sin\theta)$. Then φ is of class C^1 on $\mathbb{R}^2$, and its Jacobian is $J_\varphi(r,\theta) = \begin{vmatrix} r\cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$. Then

$$I = \iint\limits_D f(x,y)\, dxdy = \iint\limits_\Delta g(r,\theta)\, r dr d\theta.$$

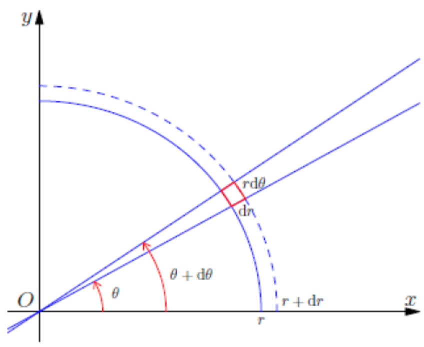

Figure 1.7: Changing variable to polar coordinates.

Example 1.2.10 *1) Calculate by passing in polar coordinates* $I = \iint\limits_D \frac{1}{x^2+y^2} dxdy$ *where* $D = \{(x,y) : 1 \le x^2+y^2 \le 4, x \ge 0, y \ge 0\}$ *which represents a quarter of the part between the two circles centered at the origin and with radii 1 and 2 (ring). From where*

$$I = \iint\limits_D \frac{1}{x^2+y^2} dxdy = \int_0^{\frac{\pi}{2}} \int_1^2 \frac{r}{r^2} dr d\theta = \frac{\pi}{2} \ln 2.$$

2) Calculate the volume of a sphere $V = \iint\limits_{x^2+y^2<R^2} \sqrt{R^2 - x^2 - y^2}dxdy$ *and since the function is even with respect to the two variables,* $V = 8\int\limits_0^{\frac{\pi}{2}}\int\limits_0^R \sqrt{R^2 - r^2}rdrd\theta = \frac{4}{3}\pi R^2$*.[12]*

1.2.5 Applications

1. **Calculation of area of a domain D:** We have seen that $\iint\limits_D f(x,y)\,dxdy$ measures the volume under the representation of f and above D. We also have the possibility of using the double integral to calculate the area itself of domain D. To do this, simply take $f(x,y) = 1$. Thus, the area A of the domain is $A = \iint\limits_D dxdy = \iint\limits_\Delta rdrd\theta$.

   Example 1.2.11 *Calculate the area delimited by the ellipse with equation* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$*. Let us note the area ofthis ellipse A, therefore* $A = \iint\limits_{\frac{x^2}{a^2}+\frac{y^2}{b^2}<1} dxdy$*. By symmetry and passing to generalized polar coordinates:* $x = ar\cos\theta, y = br\sin\theta$*, we obtain* $A = 4\int\limits_0^{\frac{\pi}{2}}\int\limits_0^1 abrdrd\theta = \pi ab$*.*

2. **Calculation of the area of a surface:** We call D the region of the XOY plane delimited by the projection onto the XOY plane of the surface representative of a function f, denoted $\sum$. The surface area of $\sum$ delimited by its projection D on the plane XOY is given by $A = \iint\limits_D \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1}dxdy$

   Example 1.2.12 *Let's calculate the area of the paraboloid* $\sum = \left\{(x,y,z) : z = x^2 + y^2, 0 \le z \le h\right\}$*. Since the surface* $\sum$ *is equal to the graph of the function* $f(x,y) = x^2 + y^2$ *defined above the domain* $D = \left\{(x,y) : x^2 + y^2 \le h\right\}$*. From where:*

$$Aire\left(\sum\right) = \iint\limits_D \sqrt{4x^2 + 4y^2 + 1}dxdy = 2\pi\int_0^{\sqrt{h}} \sqrt{4r^2 + 1}rdr = \frac{\pi}{6}(4h+1)^{3/2}.$$

3. **Mass and centers of inertia:** If we note $\rho(x,y)$ s the surface density of a plate Δ, its mass is given by the formula $M = \iint\limits_\Delta \rho(x,y)\,dxdy$. And its center of inertia $G = (x_G, y_G)$

is such that:

$$x_G = \frac{1}{M}\iint\limits_{\Delta} x\rho(x,y)\,dxdy$$
$$y_G = \frac{1}{M}\iint\limits_{\Delta} y\rho(x,y)\,dxdy$$

Example 1.2.13 *Determine the center of mass of a thin triangular metal plate whose vertices are at* $(0,0),(1,0)$ *et* $(0,2)$*, knowing that its density is* $\rho(x,y) = 1+3x+y$.

$$M = \iint\limits_{\Delta} \rho(x,y)\,dxdy = \int\limits_{0}^{1}\int\limits_{0}^{2-2x} (1+3x+y)\,dxdy = \frac{8}{3}$$
$$x_G = \frac{1}{M}\iint\limits_{\Delta} x\rho(x,y)\,dxdy = \int\limits_{0}^{1}\int\limits_{0}^{2-2x} x(1+3x+y)\,dxdy = \frac{3}{8}$$
$$y_G = \frac{1}{M}\iint\limits_{\Delta} y\rho(x,y)\,dxdy = \int\limits_{0}^{1}\int\limits_{0}^{2-2x} y(1+3x+y)\,dxdy = \frac{11}{16}$$

1.3 Triple integrals

The principle is the same as for double integrals, If $(x,y,z) \longrightarrow f(x,y,z) \in \mathbb{R}$ is a continuous function of three variables on a domain D of $\mathbb{R}^3$, we define $\iiint\limits_{D} f(x,y,z)\,dxdydz$ as sum limit of the form:

$$\sum_{i,j,k} f(u_i,v_j,w_k)(x_i - x_{i-1})(y_j - y_{j-1})(z_k - z_{k-1})$$

Remark 1.3.1 *We have the same algebraic properties of double integrals: linearity, ...*

1.3.1 Fubini formulas

1. **On a parallelepiped:** Fubini's theorem applies quite naturally when $D = [a,b]\times[c,d]\times[e,f]$, we come down to calculating three simple integrals:

$$\iiint\limits_{D} f(x,y,z)\,dxdydz = \int_a^b\left[\int_c^d\left[\int_e^f f(x,y,z)\,dz\right]dy\right]dx = \int_e^f\left[\int_c^d\left[\int_a^b f(x,y,z)\,dx\right]dy\right]dz = ...$$

   Example 1.3.1 *Calculate* $\iiint\limits_{[0,1]\times[1,2]\times[1,3]} (x+3yz)\,dxdydz$.

2. **On any bounded domain:** To establish the treatment of the search for the integration

bounds. For a certain fixed x, varying between $x_{\min}$ and $x_{\max}$, we cut out a surface D_x in D. We can then represent in the YOZ plane, then the treatment on D_x is done as with double integrals: $I = \int_{x_{\min}}^{x_{\max}} \left[\int_{y_{\min}}^{y_{\max}} \left[\int_{z_{\min}}^{z_{\max}} f(x,y,z)\, dz \right] dy \right] dx$. Of course, we can swap the roles of x, y and z.

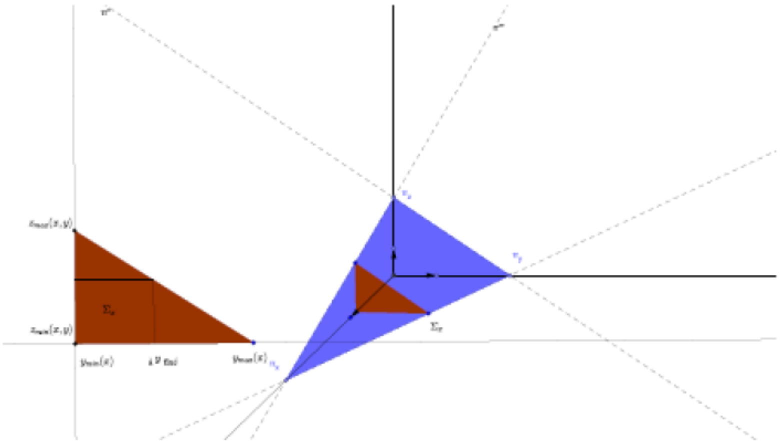

Figure 1.8: Triple integral.

Example 1.3.2 *Calculate* $I = \iiint\limits_D \left(x^2 + yz\right) dxdydz$ *in the domain*

$$D = \{(x,y,z) : x \geq 0, y \geq 0, z \geq 0, x + y + 2h \leq 1\}$$

$$I = \iiint\limits_D \left(x^2 + yz\right) dxdydz = \int_0^{1/2} \left[\int_0^{1-2z} \left[\int_0^{1-2z-x} \left(x^2 + yz\right) dy \right] dx \right] dz = \frac{1}{96}.$$

1.3.2 Changing variables

If we have a bijective map φ and class C^1 from domain Δ to domain D, defined by: $(u,v,w) \longrightarrow \varphi(u,v,w) = (x,y,z)$. The formula for changing variables is: $\iiint\limits_D f(x,y,z)\, dxdydz = \iiint\limits_\Delta f \circ \varphi(u,v,w) \left|J_\varphi(u,v,w)\right| dudvdw$. By noting $|J_\varphi|$ the absolute value of the determinant of the Jacobian.

1. **Calculation in cylindrical coordinates:** In dimension 3, the cylindrical coordinates are given by:

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$

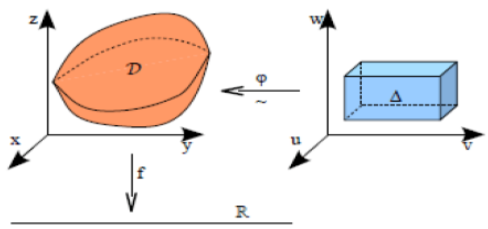

Figure 1.9: Changing variables for triple integrales.

The determinant of the Jacobian matrix of $\varphi(r,\theta,z) \longrightarrow (x,y,z)$ is:

$$|J_\varphi| = \begin{vmatrix} r\cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = rdrd\theta dz$$

So we have

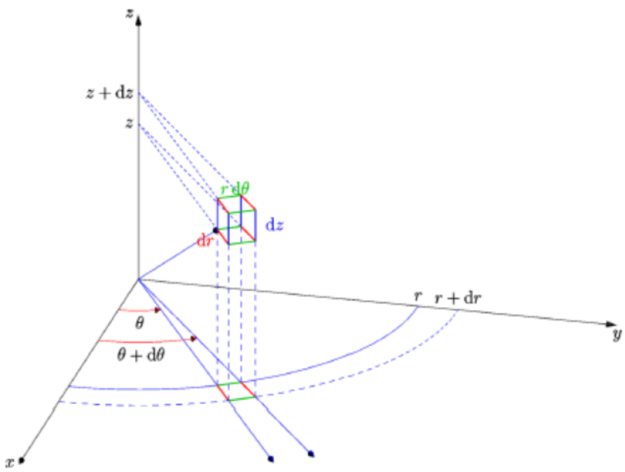

Figure 1.10: Principle of calculation of the Jacobian in cylindrical coordinates.

$$I = \iiint_D f(x,y,z)\,dxdydz = \iiint_\Delta g(r,\theta,z)\,rdrd\theta dz = \int_{\theta_{\min}}^{\theta_{\max}} \left[\int_{r_{\min}}^{r_{\max}} \left[\int_{z_{\min}}^{z_{\max}} g(r,\theta,z)\,rdz \right] dr \right] d\theta.$$

Example 1.3.3 *Calculate* $I = \iiint_V (x^2 + y^2 + 1)\,dxdydz$ *or*

$$D = \{(x,y,z) : x^2 + y^2 \le 1, \text{ and } 0 \le z \le 2\}$$

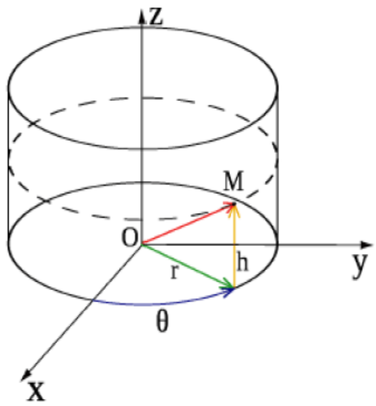

Figure 1.11: Cylindrical coordinates.

$$I=\int_0^2\int_0^{2\pi}\int_0^1 r\left(r^2+1\right)drd\theta dz=\int_0^{2\pi}d\theta\int_0^2 dz\left[\frac{1}{4}\left(r^2+1\right)^2\right]_0^1=4\pi.$$

2. **Calculation in spherical coordinates:** In dimension 3, the spherical coordinates are given by:

$$\begin{cases} x=r\sin\theta\cos\varphi \\ y=r\sin\theta\sin\varphi \\ z=r\cos\theta \end{cases}$$

The determinant of the Jacobian matrix of $\varphi(r,\theta,\varphi)\longrightarrow(x,y,z)$ is

$$|J_\varphi|=\begin{vmatrix} \sin\theta\cos\varphi & r\cos\theta\cos\varphi & -r\sin\theta\sin\varphi \\ \sin\theta\sin\varphi & r\cos\theta\sin\varphi & r\sin\theta\cos\varphi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}=r^2\sin\theta drd\theta d\varphi.$$

So we have

$$I=\iiint\limits_D f(x,y,z)\,dxdydz=\iiint\limits_\Delta g(r,\theta,z)\,r^2\sin\theta drd\theta d\varphi.$$

Example 1.3.4 *Calculate* $I=\iiint\limits_D zdxdydz$*, or*

$$D=\left\{(x,y,z):x^2+y^2+z^2\leq R^2,\ \textit{and}\ z\geq 0\right\}.$$

The domain is the upper hemisphere (centered at the origin and of radius R), passing to

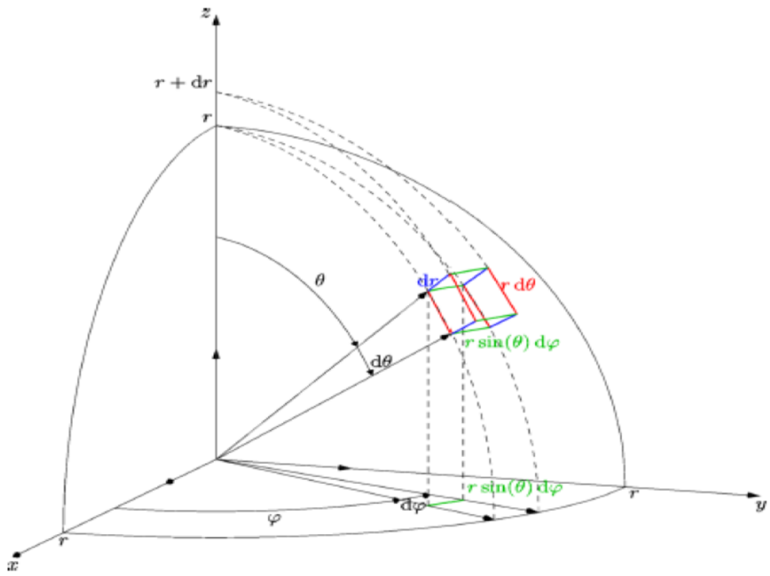

Figure 1.12: Principle of calculation of the Jacobian in spherical coordinates.

spherical coordinates:

$$I=\int_0^{2\pi}d\varphi\int_0^{\frac{\pi}{2}}\cos\theta\sin\theta d\theta\int_0^R r^2dr=\frac{\pi}{3}R^3$$

1.3.3 Applications

1. **Volume:** The volume of a body is given by $V=\iiint\limits_D dxdydz$ such that D is the domain delimited by this body.

   Example 1.3.5 *Calculate the volume of a sphere,* $V=\iiint\limits_{x^2+y^2+z^2<R^2} dxdydz$, *according to the property of symmetry:* $V=8\iiint\limits_D dxdydz$ *where*

   $$D=\left\{(x,y,z)\in\mathbb{R}^3 : x^2+y^2+z^2\leq R^2,\ \textit{and } x\geq 0, y\geq 0, z\geq 0\right\}$$

   from where $V=8\int_0^{\frac{\pi}{2}}d\varphi\int_0^{\frac{\pi}{2}}\sin\theta d\theta\int_0^R r^2dr=\dfrac{4\pi}{3}R^3.$

2. **Mass, center and moments of inertia**: Let μ be the density of a solid which occupies region V, then its mass is given by

   $$M=\iiint\limits_V \mu(x,y,z)\,dxdydz.$$

   The center of mass $G=(x_G,y_G,z_G)$ has coordinates.

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$$x_G = \frac{1}{M} \iiint\limits_V x\mu(x,y,z)\,dxdydz.$$
$$y_G = \frac{1}{M} \iiint\limits_V y\mu(x,y,z)\,dxdydz.$$
$$z_G = \frac{1}{M} \iiint\limits_V z\mu(x,y,z)\,dxdydz.$$

The moments of inertia with respect to the three axes are:

$$I_x = \iiint\limits_V \left(y^2+z^2\right)\mu(x,y,z)\,dxdydz.$$
$$I_y = \iiint\limits_V \left(x^2+z^2\right)\mu(x,y,z)\,dxdydz.$$
$$I_z = \iiint\limits_V \left(y^2+x^2\right)\mu(x,y,z)\,dxdydz.$$

Example 1.3.6 *Determine the center of mass of a solid of constant density, bounded by the parabolic cylinder* $x = y^2$ *and the planes* $x = z, z = 0$ *and* $x = 1$.
The mass is $= \int\limits_{-1}^{1}\left[\int\limits_{y^2}^{1}\left[\int\limits_{0}^{x}\mu dz\right]dx\right]dy = \frac{4\mu}{5}$, *due to symmetry of the domain and* μ *with respect to the OXZ plane, we has*

$$y_G = \frac{1}{M}\iiint\limits_V y\mu dxdydz = \frac{\mu}{M}\int\limits_{-1}^{1}\left[\int\limits_{y^2}^{1}\left[\int\limits_{0}^{x} xdz\right]dx\right]dy = \frac{5}{7}.$$
$$z_G = \frac{1}{M}\iiint\limits_V z\mu dxdydz = \frac{\mu}{M}\int\limits_{-1}^{1}\left[\int\limits_{y^2}^{1}\left[\int\limits_{0}^{x} zdz\right]dx\right]dy = \frac{5}{14}.$$

Chapter 2

IMPROPER INTEGRALES

In first year, we studied the integral of defined and continuous functions on a compact interval (closed bounded) $[a,b]$ with $-\infty < a < b < +\infty$, there existed a so-called primitive function F of f such that:[4]

$$\acute{F}(x) = f(x) \forall x \in [a,b] \text{ and } \int_a^b f(t)dt = F(b) - F(a).$$

Which represents the area delimited by the graph of the function f on $[a,b]$. In this chapter, we will learn how to calculate the integrals of unbounded domains, either because the integration interval is infinite (going up to $+\infty$ or $-\infty$), or because the function to be integrated tends towards infinite at the limits of the interval. These integrals are called improper integrals or generalized integrals.

We end our introduction by explaining the plan of this chapter. When we do not know how to calculate an antiderivative, we resort to two types of method: either the function has a constant sign in the vicinity of the uncertain point, or it changes sign an infinite number of times in this vicinity (we then say that it "oscillates"). We will also distinguish the case where the uncertain point is $\pm\infty$ or a finite value. There are therefore four distinct cases, depending on the type of the uncertain point, and the sign, constant or not, of the function to be integrated. These four types are schematized in the following figures:

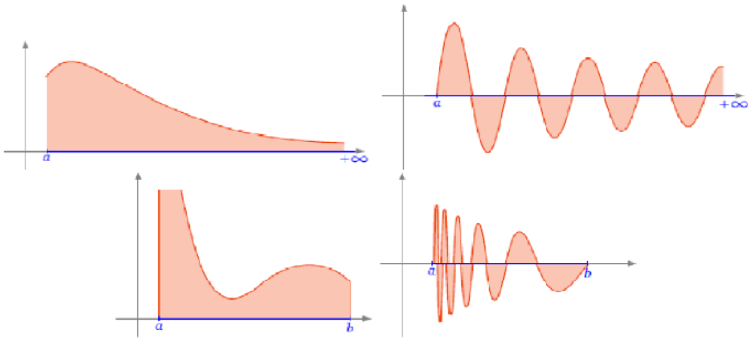

Figure 2.1: Different types of integrals.

2.1 Definitions and properties

2.1.1 Uncertain points

- We first start by identifying the uncertain points, either $+\infty$ or $-\infty$ on the one hand, and on the other hand the point(s) in the vicinity of which the function is not bounded.
- We then divide each integration interval into as many intervals as are necessary so that each of them contains only one uncertain point, placed at one of the two limits.
- the integral over the complete interval is the sum of the integrals over the intervals of the division.
- The only goal is to isolate the difficulties: the choices of cutting points are arbitrary.

2.1.2 Convergence/divergence

Definition 2.1.1 *Let f be a continuous function on $[a, +\infty[$. We say that the integral $\int_a^{+\infty} f(t)dt$ converges if the imitates, when $x \longrightarrow +\infty$, of the primitive $\int_a^x f(t)dt$ exists and is finished, i.e.*

$$\int_a^{+\infty} f(t)dt = \lim_{x \longrightarrow +\infty} \int_a^x f(t)dt.$$

Otherwise, we say that the integral diverges.

Let f be a continuous function on $]a, b]$. We say that the integral $\int_a^b f(t)dt$ converges if the right *limit, when* $x \longrightarrow a$*, of the primitive* $\int_x^b f(t)dt$ *exists and is finished, i.e.*

$$\int_a^b f(t)dt = \lim_{x \longrightarrow a^+} \int_a^b f(t)dt.$$

Otherwise, we say that the integral diverges.

Example 2.1.1 *The integral* $\int_0^{+\infty} e^{-t}dt = \lim_{x \longrightarrow +\infty} \int_0^x e^{-t}dt = \lim_{x \longrightarrow +\infty} \left[-e^{-t}\right]_0^x = 1$*. So the integral converges.*

The integral $\int_0^{+\infty} \sin(t)dt = \lim_{x \longrightarrow +\infty} \int_0^x \sin(t)dt = \lim_{x \longrightarrow +\infty} [-\cos(t)]_0^x = \nexists$ *because* $\lim_{x \longrightarrow +\infty} \cos(x) = \nexists$*. So the integral diverges.*

The integral $\int_0^1 \ln(t)dt = \lim_{x \longrightarrow +0} \int_x^1 \ln(t)dt = \lim_{x \longrightarrow +0} [t\ln(t) - t]_x^1 = -1$*. So the integral converges.*

The integral $\int_0^1 \frac{1}{t}dt = \lim_{x \longrightarrow +0} \int_x^1 \frac{1}{t}dt = \lim_{x \longrightarrow +0} [\ln(t)]_x^1 = +\infty$*. So the integral diverges.*

Remark 2.1.1 - *The generalized integral is considered as the limit of a definite integral.*

Remark 2.1.2 - *Convergence is therefore equivalent to finite limit. Divergence means either there is no limit or the limit is infinite.*

2.1.3 Relationship of Chasles

Proposition 2.1.1 (Relation de Chasles) *Let* $f : [a, +\infty[\longrightarrow \mathbb{R}$ *be a continuous function. For all* $c \in [a, +\infty[$ *the improper integrals* $\int_a^{+\infty} f(t)dt$ *and* $\int_c^{+\infty} f(t)dt$ *are of the same nature, and in the case of convergence we have:*

$$\int_a^{+\infty} f(t)dt = \int_a^c f(t)dt + \int_c^{+\infty} f(t)dt.$$

Proof. Using the Chasles relation for the usual Riemann integrals, with $a \le c \le x$:

$$\int_a^x f(t)dt = \int_a^c f(t)dt + \int_c^x f(t)dt.$$

Then passing to the limit $x \longrightarrow +\infty$. Now, if we are in the case of a continuous function

$f :]a,b] \longrightarrow \mathbb{R}, c \in]a,b]$, then we have a similar result, and in the case of convergence:

$$\int_x^b f(t)dt = \int_a^c f(t)dt + \int_c^b f(t)dt.$$

Then passing to the limit $x \longrightarrow a^+$ ■

Remark 2.1.3 - *"Being of the same nature" means that the two integrals are convergent at the same time or divergent at the same time.*

- The Chasles relation therefore implies that convergence does not depend on the behavior of the function on bounded intervals, but only on its behavior in the neighborhood of $+\infty$.[5]

2.1.4 Linearity

Proposition 2.1.2 (Linearity of the improper integral) *Let f and g be two continuous functions on $[a,+\infty[$, and λ, μ two real numbers. If the integrals $\int_a^{+\infty} f(t)dt$ and $\int_a^{+\infty} g(t)dt$. converge, then $\int_a^{+\infty} [\lambda f(t) + \mu g(t)]\, dt$ converges and we have:*

$$\int_a^{+\infty} [\lambda f(t) + \mu g(t)]\, dt = \lambda \int_a^{+\infty} f(t)dt + \mu \int_a^{+\infty} g(t)dt$$

The linearity relation is valid for the functions of an interval $]a,b]$, not bounded in a.

Remark 2.1.4 *The converse in the linearity relation is false, we can find two functions f, g such that*

$\int_a^{+\infty} (f(t) + g(t))\, dt$ converges, without $\int_a^{+\infty} f(t)dt$ nor $\int_a^{+\infty} g(t)dt$ converges.

2.1.5 Positivity

Proposition 2.1.3 (Positivity of the improper integral) *Let $f, g : [a,+\infty[\longrightarrow \mathbb{R}$ be continuous functions, having a convergent integral.*

$$\text{if } f \leq g \text{ then } \int_a^{+\infty} f(t)dt \leq \int_a^{+\infty} g(t)dt.$$

In particular, we also have:

$$\text{if } f \geq 0 \text{ then } \int\limits_{a}^{+\infty} f(t)dt \geq 0.$$

The positivity relation is valid for the functions of an interval $]a, b]$*, not bounded in* a*.*

Remark 2.1.5 *If we do not wish to distinguish the two types of improper integrals on an interval* $[a, +\infty[$ *(or* $]-\infty, b]$*) on the one hand and* $]a, b]$ *(or* $[a, b[$*) on the other hand, then it is practical to add the two ends to the number line:*

$$\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}.$$

Thus the interval $[a, b[$ *with* $a \in \mathbb{R}$ *and* $b \in \overline{\mathbb{R}}$ *designates the infinite interval* $[a, +\infty[$ *(if* $b = +\infty$*) or the finite interval* $[a, b[$ *(if* $b < +\infty$*). Likewise for the interval* $]a, b]$ *with* $a = +\infty$ *or* $a \in \mathbb{R}$*.[1]*

2.1.6 Cauchy criterion

Theorem 2.1.1 (Cauchy criterion) *Let* $f : [a, +\infty[\longrightarrow \mathbb{R}$ *be a continuous function. The improper integral* $\int\limits_{a}^{+\infty} f(t)dt$ *converges iff*

$$\forall \epsilon > 0, \exists M \geq a \qquad u, v \geq M \implies \left| \int\limits_{a}^{+\infty} f(t)dt \right| < \epsilon.$$

Proof. It is enough to apply the Cauchy criterion for the limits to the function $F(x) = \int\limits_{a}^{+\infty} f(t)dt$. Let $F : [a, +\infty[\longrightarrow \mathbb{R}$. Then $\lim\limits_{x \longrightarrow +\infty} F(x) = \lim\limits_{x \longrightarrow +\infty} \int\limits_{a}^{x} f(t)dt$ exists and is finite iff:

$$\forall \epsilon > 0, \exists M \geq a \qquad u, v \geq M \implies |F(u) - F(v) =| \left| \int\limits_{u}^{v} f(t)dt \right| < \epsilon.$$

■

2.1.7 Case of two uncertain points

When both ends of the definition interval are uncertain points. It is just a matter of reducing ourselves to two integrals each having a single uncertain point.

Definition 2.1.2 *Let* $a, b \in \overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ *with* $a < b$*. Let* $f :]a, b[\longrightarrow \mathbb{R}$ *be a continuous function. We say that the integral*

$\int_a^b f(t)dt$ converges if there exists $c \in]a,b[$ such that the two improper integrals $\int_a^c f(t)dt$ and $\int_c^b f(t)dt$ converge. The value of this doubly improper integral is then

$$\int_a^b f(t)dt = \int_a^c f(t)dt + \int_c^b f(t)dt.$$

Remark 2.1.6 - *Chasles' relations imply that the nature and value of this doubly improper integral do not depend on the choice of c, with $a < c < b$.*
- If one of the two integrals $\int_a^c f(t)dt$ diverges, or $\int_c^b f(t)dt$ diverge, alors $\int_a^b f(t)dt$ diverge.

Example 2.1.2

$$\int_{-\infty}^{+\infty} \frac{t}{(1+t^2)^2}dt = \int_{-\infty}^{2} \frac{t}{(1+t^2)^2}dt + \int_{2}^{+\infty} \frac{t}{(1+t^2)^2}dt.$$

We choose $c = 2$ at random. We start with the first integral

$$\begin{aligned}\int_{-\infty}^{2} \frac{t}{(1+t^2)^2}dt &= \lim_{x \longrightarrow -\infty} \int_x^2 \frac{t}{(1+t^2)^2}dt \\ &= -\frac{1}{2} \lim_{x \longrightarrow -\infty} \left[\frac{1}{(1+t^2)^2}\right]_x^2 \\ &= -\frac{1}{2} \lim_{x \longrightarrow -\infty} \left[\frac{1}{5} - \frac{1}{(1+x^2)}\right]. \\ &= -\frac{1}{10}.\end{aligned}$$

Then $\int_{-\infty}^{2} \frac{t}{(1+t^2)^2}dt$ converges. Likewise for $\int_{2}^{+\infty} \frac{t}{(1+t^2)^2}dt$ which converges to $\frac{1}{10}$. Thus the integral $\int_{-\infty}^{+\infty} \frac{t}{(1+t^2)^2}dt$ converges and is worth 0.

2.2 Improper integrals on an unbounded interval

2.2.1 Positive functions

We will assume that the function is positive or zero on the integration interval $[a, +\infty[$. The convergence criteria for positive functions are also valid for negative functions, you just need to replace f by $(-f)$. Recall that, by definition,

$$\int_a^{+\infty} f(t)dt = \lim_{x \longrightarrow +\infty} \int_a^x f(t)dt.$$

As f is positive, then the primitive is increasing, or else $\lim\limits_{x\longrightarrow+\infty}\int\limits_a^x f(t)dt$ is bounded, and therefore the integral $\int\limits_a^{+\infty} f(t)dt$ converges, or $\lim\limits_{x\longrightarrow+\infty}\int\limits_a^x f(t)dt$ tends towards $+\infty$ therefore diverges. If we cannot (or do not want to) calculate an primitive of f, we study the convergence using comparaison criteria which allow us to deduce its nature without explicitly calculating them.[8]

2.2.2 Comparison criterion

Theorem 2.2.1 *Let f and g be two positive and continuous functions on $[a,+\infty[$. As:*

$$\exists A\geq a, \forall t>A \qquad f(t)\leq g(t).$$

1. *If $\int\limits_a^{+\infty} g(t)dt$ converges $\Longrightarrow \int\limits_a^{+\infty} f(t)dt$ converges.*
2. *If $\int\limits_a^{+\infty} f(t)dt$ diverges $\Longrightarrow \int\limits_a^{+\infty} g(t)dt$ diverge.*

Proof. The convergence of the integrals does not depend on the left bound of the interval, and we can simply study $\int\limits_A^x f(t)dt$ and $\int\limits_A^x g(t)dt$. Now using the positivity of the integral, we obtain that, for all $x\geq A$,

$$\int\limits_A^x f(t)dt \leq \int\limits_A^x g(t)dt.$$

If $\int\limits_A^x g(t)dt$ converges, then $\int\limits_A^x f(t)dt$ is an increasing and bounded function and therefore converges. Conversely, if $\int\limits_A^x f(t)dt$ tends towards $+\infty$, then $\int\limits_A^x g(t)dt$ tends towards $+\infty$ too. ■

Example 2.2.1 *The integral $\int\limits_1^{+\infty} e^{-t^2}dt$ is convergent because: $\forall t\in[1,+\infty[, -t^2\leq t$, as e^t is an increasing function then $e^{-t^2}\leq e^t$. And since $\int\limits_1^{+\infty} e^{-t}dt = \lim\limits_{x\longrightarrow+\infty}\left[-e^{-t}\right]_1^x$, therefore the integral $\int\limits_1^{+\infty} e^{-t}dt$ converges.*

2.2.3 Equivalence criterion

Theorem 2.2.2 (Equivalence criterion) *Let f and g be two strictly positive and continuous functions on $[a,+\infty[$. Suppose they are equivalent in the neighborhood of $+\infty$, that is:*

$$\lim_{t\longrightarrow+\infty}\frac{f(t)}{g(t)}=1.$$

Then the integral $\int_a^{+\infty} f(t)dt$ *converges if is only if* $\int_a^{+\infty} g(t)dt$ *converges.*

Proof. To say that two functions are equivalent in the neighborhood of $+\infty$, is to say that their ratio tends towards 1, or again:

$$\forall \epsilon > 0 \qquad \exists A > a \qquad \forall t > A \qquad \left|\frac{f(t)}{g(t)} - 1\right| < \epsilon,$$

or again:

$$\forall \epsilon > 0 \qquad \exists A > a \qquad \forall t > A \qquad (1-\epsilon)\, g(t) < f(t) < (1+\epsilon)\, g(t).$$

Let us set $\epsilon < 1$, and apply the comparison theorem on the interval $[A, +\infty[$. If the integral $\int_A^{+\infty} f(t)dt$ converges, then the integral $\int_A^{+\infty} (1-\epsilon)\, g(t)dt$ converges, therefore the integral $\int_A^{+\infty} g(t)dt$ also converges by linearity. Conversely, if $\int_A^{+\infty} f(t)dt$ diverges, then $\int_A^{+\infty} (1+\epsilon)\, g(t)dt$ diverges, therefore $\int_A^{+\infty} g(t)dt$ also diverges. ■

Example 2.2.2 *The integral* $\int_1^{+\infty} \frac{1}{1+t^2}dt$ *converges because:* $\lim_{t\longrightarrow+\infty} \frac{\frac{1}{1+t^2}}{\frac{1}{t^2}} = 1 \implies \frac{1}{1+t^2} \sim \frac{1}{t^2}$ *and as* $\int_1^{+\infty} \frac{1}{t^2}dt = \lim_{x\longrightarrow+\infty} \int_1^{x} \frac{1}{t^2}dt = \lim_{x\longrightarrow+\infty} \left[-\frac{1}{t}\right]_1^x = 1$, *then the integral* $\int_1^{+\infty} \frac{1}{t^2}dt$ *converges by the equivalence criterion the integral* $\int_1^{+\infty} \frac{1}{1+t^2}dt$ *converges.*

Proposition 2.2.1 *Let f and g be two strictly positive and continuous functions on $[a, +\infty[$ such that:*

$$\lim_{t\longrightarrow+\infty} \frac{f(t)}{g(t)} = l.$$

If $l \neq 0$ and $l \neq +\infty$, $f(t) \underset{+\infty}{\sim} lg(t)$. Then the two integrals $\int_a^{+\infty} f(t)dt$ *and* $\int_a^{+\infty} g(t)dt$ *are of the same nature.*

If $l = 0$, $f(t) \leq g(t)$. Then if the integral $\int_a^{+\infty} g(t)dt$ *converges* $\implies \int_a^{+\infty} f(t)dt$ *converges.*

If $l = +\infty$, $f(t) \geq g(t)$. Then if the integral $\int_a^{+\infty} g(t)dt$ *diverges* $\implies \int_a^{+\infty} f(t)dt$ *diverges.*

Example 2.2.3 *The integral* $\int_1^{+\infty} \frac{\ln(t)}{1+t^2}dt$ *converges because:* $\lim_{t\longrightarrow+\infty} \frac{\frac{\ln(t)}{1+t^2}}{\frac{1}{t^{\frac{3}{2}}}} = 0$. *So* $\frac{\ln(t)}{1+t^2} \leq \frac{1}{t^{\frac{3}{2}}}$, *and as* $\int_1^{+\infty} \frac{1}{t^{\frac{3}{2}}}dt = \lim_{x\longrightarrow+\infty} \left(\frac{-2}{\sqrt{t}}\right)_1^x = 2$ *converges* $\implies \int_1^{+\infty} \frac{\ln(t)}{1+t^2}dt$ *converges.*

2.2.4 Riemann integrals

Definition 2.2.1 *A Riemann integral is an integral which is written in the form:*

$$\int_1^{+\infty} \frac{1}{t^\alpha} dt, \ \text{où } \alpha \in \mathbb{R}_+^*.$$

In this case, the primitive is explicit:

$$\int_1^{+\infty} \frac{1}{t^\alpha} dt = \begin{cases} \lim\limits_{x \longrightarrow +\infty} \left[\frac{1}{-\alpha+1} \frac{1}{t^{\alpha-1}}\right]_1^x & \text{if } \alpha \neq 1 \\ \\ \lim\limits_{x \longrightarrow +\infty} [\ln(t)]_1^x & \text{if } \alpha = 1 \end{cases}.$$

So we deduce the nature of Riemann integrals

$$\text{if } \alpha > 1 \text{ then } \int_1^{+\infty} \frac{1}{t^\alpha} dt \text{ converges.}$$

$$\text{if } \alpha \leq 1 \text{ then } \int_1^{+\infty} \frac{1}{t^\alpha} dt \text{ diverges.}$$

Proposition 2.2.2 *Let f be a positive and continuous function on $[a, +\infty[$.*

- If $f(t) \underset{+\infty}{\sim} \frac{1}{t^\alpha}$ where ($l \neq 0$ and $l \neq +\infty$) then $\int_a^{+\infty} f(t)dt \begin{cases} \text{converges if } \alpha > 1 \\ \text{diverges if } \alpha \leq 1 \end{cases}$.

- If $\lim\limits_{t \longrightarrow +\infty} t^\alpha f(t)dt = 0$ then $\int_a^{+\infty} f(t)dt$ converges if $\alpha > 1$.

- If $\lim\limits_{t \longrightarrow +\infty} t^\alpha f(t)dt = +\infty$ then $\int_a^{+\infty} f(t)dt$ diverges if $\alpha \leq 1$.

Example 2.2.4 *Let $\int_1^{+\infty} \frac{|\sin t|}{t^2} dt$. The function $\frac{|\sin t|}{t^2}$ is continuous and positive on $[1, +\infty[$*

$\lim\limits_{t \longrightarrow +\infty} t^{3/2} \frac{|\sin t|}{t^2} = 0$, so $\int_1^{+\infty} \frac{|\sin t|}{t^2} dt$ converges because $\alpha = \frac{3}{2}$. [2]

2.2.5 Bertrand integral

Definition 2.2.2 *A Bertrand Integral is an integral of the form:*

$$\int_e^{+\infty} \frac{1}{t^\alpha (\ln t)^\beta} dt, \ \text{where } \alpha \in \mathbb{R}_+^*, \beta \in \mathbb{R}.$$

- *If* $\alpha > 1$, *the integral converges.*

- *If* $\alpha < 1$, *the integral diverges.*

- *If* $\alpha = 1$ *and* $\begin{cases} \beta > 1, \text{ the integral converges} \\ \beta > 1, \text{ the integral diverges.} \end{cases}$

Theorem 2.2.3

$$\int_e^{+\infty} \frac{1}{t^\alpha (\ln t)^\beta} dt \text{ converges} \Leftrightarrow (\alpha > 1) \text{ or } (\alpha = 1 \text{ and } \beta > 1).$$

$$\int_e^{+\infty} \frac{1}{t^\alpha |\ln t|^\beta} dt \text{ diverges} \Leftrightarrow (\alpha < 1) \text{ or } (\alpha = 1 \text{ and } \beta > 1).$$

Example 2.2.5 *Let* $\int_1^{+\infty} \frac{1}{t}\sin(\frac{1}{t})dt$. *The function* $\frac{1}{t}\sin(\frac{1}{t})$ *is continuous and positive on* $[1, +\infty[$ $\frac{1}{t}\sin(\frac{1}{t}) \underset{+\infty}{\sim} \frac{1}{t^2}$, *and as* $\int_1^{+\infty} \frac{1}{t^2}dt$ *is Riemann* $\alpha = 2 > 1$ *therefore converges by equivalence the integral* $\int_1^{+\infty} \frac{1}{t}\sin(\frac{1}{t})dt$ *converges.*

Example 2.2.6 *Let* $\int_1^{+\infty} \sqrt{t^2+3t}\ln\left[\cos(\frac{1}{t})\right]\sin^2\left(\frac{1}{\ln t}\right)dt$. *The function* $\sqrt{t^2}+3t\ln\left[\cos(\frac{1}{t})\right]\sin^2\left(\frac{1}{\ln t}\right)$ *is continuous and positive on* $[2, +\infty[$.

$$\begin{array}{ll} \sqrt{t^2+3t} = t\sqrt{1+\frac{3}{t}} & \underset{+\infty}{\sim} t \\ \ln\left[\cos(\frac{1}{t})\right] & \underset{+\infty}{\sim} -\frac{1}{2t^2} \\ \sin^2\left(\frac{1}{\ln t}\right) & \underset{+\infty}{\sim} \left(\frac{1}{\ln t}\right)^2 \end{array}.$$

So

$$\sqrt{t^2+3t}\ln\left[\cos(\frac{1}{t})\right]\sin^2\left(\frac{1}{\ln t}\right) \underset{+\infty}{\sim} -\frac{1}{2t(\ln t)^2}$$

and as $\int_2^{+\infty} -\frac{1}{2t(\ln t)^2}dt$ *is a Bertrand integral* $\alpha = 1, \beta = 2$ *therefore converges by equivalence the integral* $\int_1^{+\infty} \sqrt{t^2+3t}\ln\left[\cos(\frac{1}{t})\right]\sin^2\left(\frac{1}{\ln t}\right)dt$ *converges.*

2.2.6 Absolute convergence of an improper integral

Definition 2.2.3 *Let a real function f, locally integrable on an interval* $[a, +\infty[$. *We say that the integral* $\int_a^{+\infty} f(t)dt$ *s absolutely convergent if the integral* $\int_a^{+\infty} |f(t)|\,dt$ *is convergent.*

Theorem 2.2.4 *An absolutely convergent integral is convergent.*

Example 2.2.7 *Study of the integral* $\int_{0}^{+\infty} \frac{\sin t}{1+\cos t+e^t}dt$. *The function* $\frac{\sin x}{1+\cos x+e^x}$ *is locally integrable on the interval* $[a,+\infty[$. *When* x *tends towards* $+\infty$, *on a:* $\left|\frac{\sin x}{1+\cos x+e^x}\right| \leq \frac{1}{1+\cos x+e^x}$ *and* $\frac{1}{1+\cos x+e^x} \sim e^{-x}$. *We have:* $\int_{0}^{+\infty} \frac{\sin t}{1+\cos t+e^t}dt$ *is therefore absolutely convergent, therefore convergent.*

Example 2.2.8 *Study of* $\int_{1}^{+\infty} \frac{\sqrt{t}\sin\left(\frac{1}{t^2}\right)}{\ln(1+t)}dt$. *When* x *tends towards* $+\infty$, *we have:* $0 < f(x) \leq \sqrt{x}\sin\left(\frac{1}{x^2}\right)$ *and* $\sqrt{x}\sin\left(\frac{1}{x^2}\right) \sim \frac{1}{x^{3/2}}$. *The integral* $\int_{1}^{+\infty} \frac{\sqrt{t}\sin\left(\frac{1}{t^2}\right)}{\ln(1+t)}dt$ *is absolutely convergent.* [6]

2.3 Integration by parts

Theorem 2.3.1 (Integration by parts) *Let* u *and* v *two functions of class* C^1 *on the interval* $[a,+\infty[$. *Suppose that* $\lim_{t\longrightarrow+\infty} u(t)v(t)$ *exists and is finite. Then the integrals* $\int_{a}^{+\infty} u(t)\acute{v}(t)dt$ *and* $\int_{a}^{+\infty} \acute{u}(t)v(t)dt$ *are of the same nature. In case of convergence we have:*

$$\int_{a}^{+\infty} u(t)\acute{v}(t)dt = \left[\lim_{t\longrightarrow+\infty} u(t)v(t) - u(a)v(a)\right] - \int_{a}^{+\infty} \acute{u}(t)v(t)dt.$$

Proof. This is the usual formula for integration by parts

$$\int_{a}^{+\infty} u(t)\acute{v}(t)dt = [u(t)v(t)]_a^x - \int_{a}^{x} \acute{u}(t)v(t)dt.$$

noting that by hypothesis that $\lim_{x\longrightarrow+\infty} uv$ has a finite limit. ■

Example 2.3.1 *Let the integral be* $\int_{0}^{+\infty} \lambda t e^{-\lambda t}dt$ *ou* $\lambda > 0$. *We carry out the integration by parts* *with* $u = \lambda t, \acute{v} = e^{-\lambda t}$. *We therefore have* $\acute{u} = \lambda, v = -\frac{1}{\lambda}e^{-\lambda t}$. *Also*

$$\int_0^x \lambda t e^{-\lambda t} dt = \left[-te^{-\lambda t}\right]_0^x + \int_0^x e^{-\lambda t} dt$$
$$= -xe^{-\lambda x} - \frac{1}{\lambda}\left(e^{-\lambda x} - 1\right)$$
$$\int_0^{+\infty} \lambda t e^{-\lambda t} dt = \lim_{x \longrightarrow +\infty} \int_0^x \lambda t e^{-\lambda t} dt = \frac{1}{\lambda} \text{ (then the integral converges).}$$

2.4 Change of variable

Theorem 2.4.1 (Change of variable) *Let f be a function defined on an interval $I = [a, +\infty[$. Let $J = [\alpha, \beta[$ be an interval with $\alpha, \beta \in \mathbb{R}$ or $\beta = +\infty$. Let $\varphi : J \longrightarrow I$ be a diffeomorphism*[1] *of class C^1. The integrals $\int_a^{+\infty} f(x)dx$ and $\int_\alpha^\beta f(\varphi(t))\acute{\varphi}(t)\, dt$ sare of the same nature. In case of convergence, we have:*

$$\int_a^{+\infty} f(x)dx = \int_\alpha^\beta f(\varphi(t))\acute{\varphi}(t)\, dt$$

Example 2.4.1 *The following example is very interesting: the function $f(t) = \sin t^2$ has a convergent integral, but does not tend to 0 (when $t \longrightarrow +\infty$). This is to be contrasted with the case of series: for a convergent series the general term always tends towards 0. Let the integral be $\int_1^{+\infty} \sin(t^2)\, dt$. We carry out the change of variable $u = t^2$, which gives $t = \sqrt{u}$, $dt = \frac{1}{2\sqrt{u}}du$ with φ is a diffeomorphism.*

$$\begin{array}{cccc} \varphi : & [1, x^2] & \longrightarrow & [1, x] \\ & u & \longrightarrow & t \end{array}$$

$$\int_1^{+\infty} \sin(t^2)\, dt = \int_1^{x^2} \sin(u) \frac{1}{2\sqrt{u}} du$$

Now by Abel's theorem $\int_1^{+\infty} \sin(u) \frac{1}{2\sqrt{u}} du$ converges, therefore $\int_1^{x^2} \sin(u) \frac{1}{2\sqrt{u}} du$ admits a finite limit, which proves that $\int_1^x \sin(t^2)\, dt$ also admits a finite limit. Then $\int_1^{+\infty} \sin(t^2)\, dt$ converges.

Example 2.4.2 *Let the integral be $\int_1^2 \frac{dt}{\sqrt{t-1}}$. We carry out the change of variable $u = \sqrt{t-1}$,*

[1] We recall that $\varphi : J \longrightarrow I$ a diffeomorphism of class $\varphi : J \longrightarrow I$ if φ is a bijective C^1 map, whose reciprocal bijection is also C^1.

which gives $t = u^2 + 1$, $dt = 2udu$ *with* φ *is a diffeomorphism:*

$$\begin{array}{cccc} \varphi: & [\sqrt{x-1}, 1] & \longrightarrow & [x, 2] \\ & u & \longrightarrow & t \end{array}$$

$$\lim_{x \longrightarrow 1} \int_1^2 \frac{dt}{\sqrt{t-1}} = \lim_{x \longrightarrow 1} \int_1^2 2udu = 2\,[u]_{\sqrt{x-1}}^1 = \lim_{x \longrightarrow 1} 2\left[1 - \sqrt{x-1}\right] = 2$$

$\int_0^1 2du$ *converges, which proves that* $\int_1^2 \frac{dt}{\sqrt{t-1}}$ *also admits a finite limit. Then* $\int_1^2 \frac{dt}{\sqrt{t-1}}$ *converges.*

Example 2.4.3 *We will calculate the value of the following two improper integrals:*

$$I = \int_0^{\pi/2} \ln(\sin(t))dt, \qquad J = \int_0^{\pi/2} \ln\left[\cos(t)\right] dt.$$

- Show that the integral I converges: *As* $\ln(\sin(t)) \underset{0^+}{\sim} \ln(t) \leq \frac{1}{\sqrt{t}}$, *let us carry out an integration by parts of* $\int_0^{\pi/2} \ln(t)dt$ *the integral converges. By equivalence* $\int_0^{\pi/2} \ln(\sin(t))dt$ *converges.*

- Check that $I = J$*:* *We carry out the change of variable* $t = \frac{\pi}{2} - u$. *We have* $dt = -du$ *and a diffeomorphism between* $t \in \left[x, \frac{\pi}{2}\right]$ *and* $u \in \left[\frac{\pi}{2} - x, 0\right]$. *So*

$$\int_x^{\pi/2} \ln(\sin(t))dt = \int_{\frac{\pi}{2}-x}^{0} \ln(\sin(\frac{\pi}{2} - x))\,(-du) = \int_0^{\frac{\pi}{2}-x} \ln\left[\cos(u)\right] du$$

$$I = \int_0^{\pi/2} \ln\left[\sin(t)\right] dt = \lim_{x \longrightarrow 0^+} \int_x^{\pi/2} \ln\left[\sin(t)\right] dt = \lim_{x \longrightarrow 0^+} \int_0^{\frac{\pi}{2}-x} \ln\left[\cos(u)\right] du$$

Cela prouve $I = J$. *Donc* J *converge.*

- Calculer $I + J$*:* *This proves* $I = J$. *So* J *converges.*

$$\begin{aligned} I + J &= \int_0^{\pi/2} \ln\left[\sin(t)\right] dt + \int_0^{\pi/2} \ln\left[\cos(t)\right] dt = \int_0^{\pi/2} \ln\left[\sin(t)\right] + \ln\left[\cos(t)\right] dt \\ &= \int_0^{\pi/2} \ln\left[\sin(t).\cos(t)\right] dt = -\frac{\pi}{2} \ln 2 + \int_0^{\pi/2} \ln\left[\sin(2t)\right] dt \end{aligned}$$

And since $I = J$, *we have* $2I = -\frac{\pi}{2}\ln 2 + I$. *We still have to evaluate* $L = \int\limits_0^{\pi/2} \ln\left[\sin(2t)\right] dt$*:*

Let us change the variable $u = 2t$, *the integral* L *becomes:*

$$\begin{aligned} L &= \frac{1}{2}\int\limits_0^{\pi} \ln\left[\sin(u)\right] du \\ &= \frac{1}{2}I + \frac{1}{2}\int\limits_{\pi/2}^{\pi} \ln\left[\sin(u)\right] du \end{aligned}$$

we carry out the change of variable $v = \pi - u$, *we will have*

$$\begin{aligned} L &= \frac{1}{2}I + \frac{1}{2}\int\limits_{\pi/2}^{0} \ln\left[\sin(\pi - u)\right] (-dv) \\ &= \frac{1}{2}I + \frac{1}{2}\int\limits_0^{\pi/2} \ln\left[\sin(v)\right] dv \\ &= \frac{1}{2}I + \frac{1}{2}I \\ &= I. \end{aligned}$$

So, as $2I = \dfrac{\pi}{2}\ln 2 + L$ *and* $L = I$, *we find:*

$$I = J = \int\limits_0^{\pi/2} \ln\left[\sin(t)\right] dt = -\frac{\pi}{2}\ln 2.$$

Chapter 3

DIFFERENTIAL EQUATIONS

Differential equation is an equation: whose unknown is a function (generally denoted $y(x)$ or simply y) and in which appear some of the derivatives of the function (first derivative $\acute{y}$, or derivatives of higher orders $\H{y}$, $y^{(3)}$,...). [5]

3.1 General information on 1st order differential equations

Let's move on to the complete definition of a differential equation and especially a solution of a differential equation.

3.1.1 Definition and Examples

Definition 3.1.1 *Given a function of three variables F, we call a 1^{st} order differential equation any relation of the form:*

$$F(x, y, \acute{y}) = 0, \tag{3.1}$$

between the variable x, the function $y(x)$ and its derivative $\acute{y}(x)$. The function φ, differentiable, is called the solution or integral of the differential equation (3.1) on a set I of $\mathbb{R}$ if

$$\forall x \in I, F\left(x, \varphi(x), \acute{\varphi}(x)\right) = 0.$$

Example 3.1.1 *$\acute{y} + y = x$ admits on $\mathbb{R}$ the solution $\varphi(x) = x - 1$. $x\acute{y} - 1 = 0$ admits on $\mathbb{R}^*$ the solution $\varphi(x)(x) = \ln|x|$.*

Integrating a differential equation means determining all the solutions, specifying, if necessary, the definition set of each.

3.1.2 Cauchy's theorem

If f is continuous and has a continuous derivative with respect to y on an open set Ω de $\mathbb{R}^2$, whatever the point (x_0, y_0) of Ω, there exists a unique solution $\varphi(x)$ of the equation $\acute{y} = f(x, y)$ defined in the neighborhood of x_0 and such that $y(x_0) = y_0$. For given x_0 the solution depends on y_0. The set of solutions of a 1^{st} order differential equation depends on an arbitrary constant λ, $y_\lambda = \varphi(x, \lambda)$. This set of solutions will be called **General Integral**. By giving particular values for λ we obtain particular solutions. The condition $y(x_0) = y_0$ is called **initial condition.**

Example 3.1.2 *Integrate the differential equation $y - \acute{y} = 0$, such that $y(1) = 1$. $y - \acute{y} = 0 \iff \frac{dy}{y} = dx$ hence $y = \lambda e^x$, or $y(1) = 1 \iff 1 = \lambda e$ which gives $\lambda = \frac{1}{e}$. So the solution to the general equation is given $y = e^{x-1}$.*

3.2 Integration of 1st order differential equations

3.2.1 Equation with separable variables

A differential equation with separable variables is a 1^{st} order equation that can be written in the form

$$\acute{y} = \frac{f(x)}{g(x)} \qquad \text{or} \qquad g(y)dy = f(x)dx,$$

the functions f and g are assumed to be continuous, from where $\int g(y)dy = \int f(x)dx + \lambda$ we obtain $G(y) = F(x) + \lambda$

Example 3.2.1

$$\begin{aligned} \acute{y} + y = a, &\qquad a \in \mathbb{R} \\ \frac{dy}{dx} = a - y &\implies \frac{dy}{a - y} = -dx \\ \ln|y - a| = x + \lambda &\qquad \textit{we obtain} \qquad y = a + Ce^{-x} \end{aligned}$$

3.2.2 Homogeneous equation

We call a homogeneous differential equation of the 1^{st} order an equation of the form $F(x, y, \acute{y}) = 0$, in which the change of x into λx and y into λy leaves $\acute{y}$ invariant.

Geometric interpretation: Let (M, MT) be a contact element. The equation being homogeneous, the tangent in $\acute{M}(\lambda x, \lambda y)$ is parallel to (MT), that is to say that the set of integral curves is globally invariant in all homothety of center o, $\acute{y}$ therefore depends on $\frac{y}{x}$. We assume that $\acute{y} = f(\frac{y}{x})$.

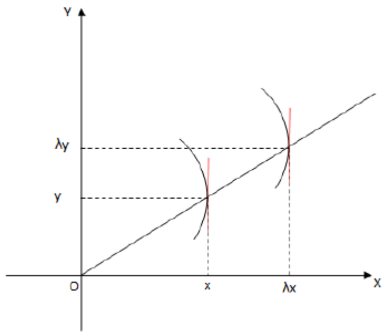

Figure 3.1: Geometric interpretation of homogeneous equation.

Integration method: We set $\frac{y}{x} = t$ or $y = tx$, we obtain $dy = tdx + xdt$, from where $\frac{dy}{dx} = t + x\frac{dt}{dx} = f(t)$. By separating the variables, we obtain $\frac{dx}{x} = \frac{dt}{f(t) - t}$ for $f(t) - t \neq 0$. Then $\ln x = \int \frac{dt}{f(t) - t} = \varphi(t)$. We find a parametric representation of the integral curves in the form:

$$\begin{cases} x = \lambda e^{\varphi(t)} \\ y = \lambda t e^{\varphi(t)} \end{cases}$$

Example 3.2.2 *Let the differential equation be* $\acute{y} = \frac{y^2 - x^2}{2xy}$*, We set* $\frac{y}{x} = t$*, then* $\acute{y} = \frac{y^2 - x^2}{2xy}$ *therefore* $y = xt \implies t + x\frac{dt}{dx} = \frac{t^2-1}{2t}$ *hence* $x\frac{dt}{dx} = \frac{-t^2 + 1}{2t}$ *and* $\frac{-2tdt}{t^2 + 1} = \frac{dx}{x}$ *or* $\frac{-d(t^2 + 1)}{t^2 + 1} = \frac{dx}{x}$*. So*

$$\begin{cases} x = \frac{\lambda}{t^2 + 1} \\ y = \frac{\lambda t}{t^2 + 1} \end{cases}$$

3.2.3 Linear equations

We call a linear differential equation of the 1^{st} order an equation in the form

$$a(x)\acute{y} + b(x) = f(x) \tag{3.2}$$

in which the functions a, b and f are assumed to be continuous on the same subset $I \subset \mathbb{R}$. a and b are the coefficients of the equation. We call an equation without an associated second member the equation

$$a(x)\acute{y} + b(x) = 0 \tag{3.3}$$

Theorem 3.2.1 (Fundamental theorem) *The general solution of the linear differential equation of the 1^{st} order (3.2) is obtained by adding to a particular solution of the complete equation (3.2) the general solution of the equation without an associated second member(3.3).*

Integration of the equation (3.3): The equation $a(x)\acute{y} + b(x)y = 0$ has separable variables, we can write: $\frac{dy}{y} = \frac{-b(x)}{a(x)}dx$ from where $\ln y = -\int \frac{b(x)}{a(x)}dx$, i.e. $y(x) = \lambda y_1(x)$ with $y_1(x) = e^{-\int \frac{b(x)}{a(x)}dx}$.

Example 3.2.3 *Let the linear differential equation be $(1 + x^2)\acute{y} - xy = 1$. It is easy to verify that $y = x$ is a particular solution of the given equation. Solving the equation without a second member, $(1 + x^2)\acute{y} - xy = 0$. We have*

$$\frac{dy}{dx} = \frac{xdx}{1 + x^2} = \frac{1}{2}\left(\frac{d(1 + x2)}{1 + x^2}\right),$$

hence $y = \sqrt{1 + x^2}$. The general solution is $x + \lambda\sqrt{1 + x^2}$.

3.2.4 Method of variation of the constant

In the case where we do not know the particular solution of the linear differential equation we use the method of variation of the constant. Let $y = \lambda y_1(x)$ be the general solution of the equation without a second member, we propose to seek if there exist solutions of the equation of the form $y = \lambda(x)y_1(x)$, $\lambda(x)$ now represents a function differentiable from the variable x, and we have [5]

$$\acute{y} = \acute{\lambda}(x)y_1(x) + (x)\acute{y}_1(x)$$
$$a(x)\acute{y} + b(x)y = a(x)[\acute{\lambda}(x)y_1(x) + \lambda(x)\acute{y}_1(x)] + b(x)\lambda(x)y_1(x) = f(x)$$
$$\lambda(x)a(x)\acute{y}_1(x) + \lambda(x)[a(x)\acute{y}_1(x) + b(x)y_1(x)] = f(x)$$

Or $a(x)\acute{y}_1(x) + b(x)y_1(x) = 0$ puis donc $\acute{\lambda}(x) = \dfrac{f(x)}{a(x)y_1(x)}$ d'où $\lambda(x) = \varphi(x) + C$ avec $\varphi(x) = \int \dfrac{f(x)}{a(x)y_1(x)}dx$. C'est-à-dire $y = \varphi(x)y_1(x) + Cy_1(x)$.

Example 3.2.4 *Integrate the equation:* $\acute{y}\cos x + y\sin x = x$. *The equation without a second member:* $\acute{y}\cos x + y\sin x = 0$ *admits as a general solution* $y = \lambda\cos x$. *We put* $y(x) = \lambda(x)\cos x$, *then* $\acute{y} = \acute{\lambda}(x)\cos x - \lambda(x)\sin x = x$ *therefore* $[\acute{\lambda}(x)\cos x - \lambda(x)\sin x]\cos x + \lambda(x)\cos x\sin x = x$, *therefore* $\acute{\lambda}(x) = \dfrac{x}{\cos^2 x}$ *from where* $\lambda(x) = \int \dfrac{x}{\cos^2 x}dx$ *after integration by parts we obtain* $\lambda(x) = x\tan x - \int \tan x dx = x\tan x + \ln|\cos x| + C$. *Eventually*

$$y = C\cos x + x\sin x + \cos x\ln|\cos x|.$$

3.2.5 Equation reducing to a linear equation

1. **Bernoulli equation:** Bernoulli equation is a 1^{st} order differential equation of the form

$$a(x)\acute{y} + b(x)y = f(x)y^{\alpha}$$

- If $\alpha = 1$ the equation is linear.

- If $\alpha \neq 1$, by dividing both sides of the equation by y^{α}, we obtain $a(x)\acute{y}y^{-\alpha} + b(x)y^{1-\alpha} = f(x)$. We set $z = y^{\alpha}$ then $\acute{z} = (1-\alpha)y^{-\alpha}\acute{y}$, hence the linear equation $\dfrac{a(x)}{1-\alpha}\acute{z} + b(x)z = f(x)$.

Example 3.2.5 *Either*

$$y - x\acute{y} = 2xy^2 \tag{3.4}$$

By dividing by y^2 *we obtain,* $\dfrac{1}{y} - x\dfrac{\acute{y}}{y^2} = 2x$, *we set* $z = \dfrac{1}{y}$ *soit* $\acute{z} = -\dfrac{\acute{y}}{y^2}$, *the equation becomes:*

$$z + x\acute{z} = 2x \tag{3.5}$$

we notice that $z = x$ *is a particular solution of (3.4), the equation without a second member gives* $\dfrac{dz}{z} = -\dfrac{dx}{x}$, *we obtain* $z = \dfrac{\lambda}{x}$. *Then* $z = x + \dfrac{\lambda}{x}$ *is the general solution of (3.5), and* *consequently the general solution of (3.4) is* $\dfrac{1}{x+\frac{\lambda}{x}}$.

2. **Riccati equation:** We call Riccati equation a differential equation of the 1^{st} order of the form

$$\acute{y} = a(x)y^2 + b(x)y + c(x)$$

We can only integrate this equation when we know a particular solution. Suppose y_1 is a particular solution, then

$$\acute{y}_1 = a(x)y_1^2 + b(x)y_1 + c(x)$$
$$\acute{y} - \acute{y}_1 = a(x)\left[y^2 - y_1^2\right] + b(x)(y - y_1).$$

By setting $y - y_1 = z$ we obtain

$$\acute{z} = a(x)z(2y_1 + z) + b(x)z$$
$$\acute{z} = a(x)z^2 + [2a(x) + y_1 + b(x)]z,$$

we have led to a Bernoulli equation.

Example 3.2.6 *Integrate* $\acute{y} = y^2 - 2xy + x^2 + 1$. *We notice that* $y = x$ *is a particular solution, we put* $y = x + z$, *then* $1 + \acute{z} = (x+z)^2 - 2x(x+z) + x^2 + 1$, *we obtain* $\acute{z} = z^2$. *We integrate this equation, We write* $\dfrac{\acute{z}}{z^2} = 1$ *i.e.* $\left(\dfrac{1}{z}\right)' = -1$ *hence* $\dfrac{1}{z} = -x + \lambda$, *finally* $y = x + \dfrac{1}{\lambda - x}$.

3.3 Second order differential equations

3.3.1 Definition and Examples

Definition 3.3.1 *We call a* 2^{nd} *order differential equation any relation of the form:* $F(x, y, \acute{y}, \acute{\acute{y}}) = 0$ *between the variable* x, *the function* $y(x)$ *and its first and second derivatives. The function* φ, *twice differentiable, is then called solution or integral over* I *subset of* $\mathbb{R}$ *if* $\forall x \in I, F(x, \varphi(x), \acute{\varphi}(x), \varphi(x)) = 0$

Example 3.3.1 *The equation* $\acute{\acute{y}} + \omega^2 y = 0$, *admits for solution on* $\mathbb{R}$, $\varphi_1(x) = \cos x$ *and* $\varphi_2(x) = \sin x$

Example 3.3.2 *The equation* $\acute{\acute{y}}= 0$, *admits for solution on* $\mathbb{R}$, *any polynomial of the* 1^{er} *degree* $\varphi(x) = ax + b$ *with* (a, b) *arbitrary.*

We will admit without demonstration that, under certain hypotheses, a differential equation of the 2^{nd} order admits an infinity of solutions depending on two arbitrary constants λ_1 and $\lambda_2 : y = \varphi(x, \lambda_1, \lambda_2)$, all of these solutions constitutes the general integral and represents the equation of a family of curves of two parameters C_{λ_1,λ_2} called integral curves. [7]

3.3.2 Equation reducing to 1st order

1. **Equation not containing y:** Consider a differential equation of 2^{nd} order $F(x, y, \acute{y}, \ddot{y}) = 0$. By setting $z = \acute{y}$ the equation becomes $F(x, z, \acute{z}) = 0$.

   Example 3.3.3 *Let the equation* $\acute{\acute{y}}+\acute{y}^2 = 0$, *by setting* $z = \acute{y}$ *we obtain* $\acute{z} + z^2 = 0$, *then* $-\frac{dz}{z^2} = dx \implies \frac{1}{z} = x - x_0$ *(*x_0 *constant), therefore* $z = \frac{dy}{dx} = \frac{1}{x - x_0}$, *hence* $dy = \frac{dx}{x - x_0} \implies y - y_0 = \ln|x - x_0|$, *the solution depends on two constants* x_0, y_0.

2. **Equation not containing x:** Consider a differential equation of 2^{nd} order $F(x, y, \acute{y}, \ddot{y}) = 0$. If we consider that $\acute{y}$ as a function of y, by setting $\acute{y} = z(y)$ we obtain

$$\ddot{y} = \frac{d\acute{y}}{dx} = \frac{dz}{dx} = \frac{dz}{dy}.\frac{dy}{dx} = \frac{dz}{dy}.z$$

   y therefore plays the role of variable, and the equation becomes $F(y, z, z\frac{dz}{dy}) = 0$, it is a 1^{er} order equation for z. Let $z = \varphi(y, \lambda_1)$ be the integral of this equation, then $z = \frac{dy}{dx} = \varphi(y, \lambda_1)$ or $\frac{dy}{\varphi(y, \lambda_1)} = dx$, then by integrating $x = f(y, \lambda_1)+\lambda_2$ with $f(y, \lambda_1) = \int \frac{dx}{\varphi(y, \lambda_1)}$.

   Example 3.3.4 *Consider the equation* $y^2\acute{\acute{y}}+\acute{y} = 0$. *By setting* $\acute{y} = z(y)$ *or* $\acute{\acute{y}}= z\acute{z}$, *the equation becomes* $y^2z\acute{z} + z = 0$, *discarding the banal solution* $z = 0$ *(corresponding to* $y = k$*), we have* $y^2\acute{z} + 1 = 0 \implies \acute{z} = \frac{1}{y} + \lambda_1$, *we have brought to the first order equation* $\frac{dy}{dx} = \frac{1}{y} + \lambda_1$, *then*

$$dx = \frac{dy}{\frac{1}{y} + \lambda_1} = \frac{ydy}{\lambda_1 y + 1} = \frac{1}{\lambda_1}(1 - \frac{1}{\lambda_1 y + 1})dy.$$

   From where $x = \frac{1}{\lambda_1}y - \frac{1}{\lambda_1^2}\ln|\lambda_1 y + 1| + \lambda_2$.

3.3.3 Second order linear differential equation

Definition 3.3.2 *We call a linear differential equation of the 2^{nd} order an equation of the form*

$$a(x)\acute{\acute{y}} + b(x)\acute{y} + c(x)y = f(x). \tag{3.6}$$

a, b, c, f are functions on $I \subset \mathbb{R}$. (a, b, c are called coefficients of the equation). We associate with this equation the so-called equation without a second member

$$a(x)\acute{\acute{y}} + b(x)\acute{y} + c(x)y = 0 \tag{3.7}$$

Theorem 3.3.1 (Fundamental theorem) *is obtained by adding to a particular integral of the complete equation the integral of the equation without a second member. If y is the general solution of (3.6) and y_0 is a particular solution of (3.6), and Y is the general solution of (3.7), then $y = y_0 + Y$.*

Integration of the equation without a second member

1. **Case where we know two particular solutions**: If y_1, y_2 are two solutions of (3.6), then $y_1 + y_2$ and λy_1 with ($\lambda \in \mathbb{R}$ are solutions of (3.7). y_1 and y_2 are said to be linearly independent if there do not exist two non-zero constants λ_1, λ_2 such that: $\forall x \in I, \lambda_1 y_1(x) + \lambda_2 y_2(x) = 0$ this results in $\lambda_1 \acute{y}_1(x) + \lambda_2 \acute{y}_2(x) = 0$. this results in

$$\begin{cases} \lambda_1 y_1(x) + \lambda_2 y_2(x) = 0 \\ \lambda_1 \acute{y}_1(x) + \lambda_2 \acute{y}_2(x) = 0 \end{cases}$$

admits only $(\lambda_1, \lambda_2) = (0, 0)$ as a solution. So the determinant

$$w(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ \acute{y}_1(x) & \acute{y}_2(x) \end{vmatrix}$$

called Wronskian of y_1, y_2 is not zero; On the contrary if $w(x) = 0$ then y_1 and y_2 are linearly dependent.

Theorem 3.3.2 *The dimension of the vector space of the solutions of the equation $a(x)\acute{\acute{y}} + b(x)\acute{y} + c(x)y = 0$ is equal to 2.*

Consequence: If y_1 and y_2 are two linearly independent solutions of the equation without a second member, the general solution is written $Y(x) = \lambda_1 y_1(x) + \lambda_2 y_2(x)$, λ_1, λ_2 are two arbitrary constants.

Example 3.3.5 *$\acute{\acute{y}}+wy = 0, y_1 = \cos wx$ and $\sin wx$ are two independent solutions because*

$$w(x) = \begin{vmatrix} \cos wx & \sin wx \\ -w\sin wx & w\cos wx \end{vmatrix} = w \neq 0$$

$Y = \lambda_1 \cos wx + \lambda_2 \sin wx.$

Example 3.3.6 *$\acute{\acute{y}}-wy = 0$, $y_1 = \cosh wx$ and $\sinh wx$ are two independent solutions* $Y = \lambda_1 \cosh wx + \lambda_2 \sinh wx.$

2. **Case where we only know a particular solution:** Let y_1 be a solution of (3.7), we set $y = y_1 z$ therefore $\acute{y} = \acute{y}_1 z + y_1 \acute{z}$ and $\ddot{y} = \ddot{y}_1 z + 2\acute{y}_1 \acute{z} + y_1 \ddot{z}$. Let $a(x)[\ddot{y}_1 z + 2\acute{y}_1 \acute{z} + y_1 \ddot{z}] + b(x)[\acute{y}_1 z + y_1 \acute{z}] + c(x)yz = 0$ taking into account $a\ddot{y}_1 + b\acute{y}_1 + cy_1 = 0$, then we obtain $ay_1\ddot{z} + (2a\acute{y}_1 + by_1)\acute{z} = 0$, is an equation that can be integrated easily.

Example 3.3.7 *Let the equation $x^2\acute{\acute{y}}+x\acute{y} - y = 0, y_1 = x.y = xz \implies \acute{y} = z + x\acute{z} \implies \acute{\acute{y}} = 2\acute{z} + xz$, then $x^2(2\acute{z} + x\acute{\acute{z}}) + x(z + x\acute{z}) - xz = 0$ where $x\acute{\acute{z}}+3\acute{z} = 0$ or $\frac{\acute{\acute{z}}}{\acute{z}} = -3x$, which gives $\acute{z} = \frac{\lambda_1}{x^3}$, so $z = -\frac{\lambda_1}{x^2} + \lambda_2$, we obtain $y = \frac{\lambda_1}{2}\frac{1}{x} + \lambda_2 x$ or even $y = C_1\frac{1}{x} + C_2 x$.*

3.3.4 Integration of the full equation

1. **In the case where we know a particular solution y_0:** Simply apply the fundamental theorem, $y = y_0 + Y$.

Example 3.3.8 *Given the equation $x^2\acute{\acute{y}}+x\acute{y}-y = x^3$, the solution of the equation without a second member is $Y = C_1\frac{1}{x} + C_2 x$, we are looking for a particular solution in the form of a polynomial of $3^{ème}$ degree $y_0 = ax^3$ therefore $\acute{y}_0 = 3ax^2$, and $\acute{\acute{y}}_0 = 6ax$ which gives $a = \frac{1}{8}$, therefore the general solution of the given equation is $y = \frac{x^3}{8} + C_1\frac{1}{x} + C_2 x$.*

2. **If we do not know a particular solution:** We apply the method of variation of constants. Let y_1, y_2 be two independent solutions of (3.7) $y = \lambda_1 y_1 + \lambda_2 y_2$ the general

solution of the equation without a second member. We set $y = \lambda_1(x)y_1 + \lambda_2(x)y_2$ where λ_1, λ_2 are functions, then $\acute{y} = \acute{\lambda}_1(x)y_1 + \acute{\lambda}_2(x)y_2 + \lambda_1(x)\acute{y}_1 + \lambda_2(x)\acute{y}_2$, by imposing the condition $\acute{\lambda}_1(x)y_1 + \acute{\lambda}_2(x)y_2 = 0$ on obtient $\H{y} = \acute{\lambda}_1(x)\acute{y}_1 + \acute{\lambda}_2(x)\acute{y}_2 + \lambda_1(x)\H{y}_1 + \lambda_2(x)\H{y}_2$, or by reporting in (3.6)

$$a\left[\acute{\lambda}_1(x)y_1 + \acute{\lambda}_2(x)y_2 + \lambda_1(x)\acute{y}_1(x) + \lambda_2(x)\acute{y}_2(x)\right] + b\left[\lambda_1(x)\acute{y}_1 + \lambda_2(x)\acute{y}_2\right] + c\left[\lambda_1(x)y_1 + \lambda_2(x)y_2\right] = f(x),$$

but we have

$$\begin{cases} a\H{y}_1 + b\acute{y}_1 + cy_1 = 0 \\ a\H{y}_2 + b\acute{y}_2 + cy_2 = 0 \end{cases}$$

we obtain

$$\begin{cases} \acute{\lambda}_1(x)y_1(x) + \acute{\lambda}_2(x)y_2(x) = 0 \\ \acute{\lambda}_1(x)\acute{y}_1(x) + \acute{\lambda}_2(x)\acute{y}_2(x) = \dfrac{1}{a(x)}f(x) \end{cases}.$$

$w(x) = \begin{vmatrix} y_1 & y_2 \\ \acute{y}_1 & \acute{y}_2 \end{vmatrix} \neq 0$ because y_1, y_2 are linearly independent.

Example 3.3.9 *Consider the equation* $\H{y} + y = \tan x$. *We have* $\H{y} + y = 0 \implies y = \lambda_1 \cos x + \lambda_2 \sin x$. *Then* $\acute{y} = -\lambda_1 \sin x + \lambda_2 \cos x$ *if*

$\acute{\lambda}_1 \cos x + \acute{\lambda}_2 \sin x = 0$. *And* $\H{y} = -\acute{\lambda}_1 \sin x + \acute{\lambda}_2 \cos x - \lambda_1 \cos x + \lambda_2 \sin x$. *By reporting in the equation we obtain:* $\H{y} + \acute{y} = -\acute{\lambda}_1 \sin x + \acute{\lambda}_2 \cos x = \tan x$, *we have the system:*

$$\begin{cases} \acute{\lambda}_1 \cos x + \acute{\lambda}_2 \sin x = 0 \\ -\acute{\lambda}_1 \sin x + \acute{\lambda}_2 \cos x = \tan x = 0 \end{cases}$$

hence $\acute{\lambda}_1 = -\dfrac{\sin x}{\cos x} = \cos x - \dfrac{1}{\cos x}$, et $\acute{\lambda}_2 = \sin x$, Then

$$\lambda_1 = \sin x - \ln|\tan(\frac{x}{2} + \frac{\pi}{4})| + C_1$$
$$\lambda_2 = \cos x + C_2.$$

The general solution is written: $y = C_1 \cos x + C_2 \sin x - \cos x \ln|\tan(\frac{x}{2} + \frac{\pi}{4})|$.

3.4 Linear equation with constant coefficients

Definition 3.4.1 *We call a linear differential equation of the second order with constant coefficients a differential equation of the form*

$$a\ddot{y} + b\acute{y} + cy = f(x) \tag{3.8}$$

in which a, b, c are constants.

We associate with (3.8) the equation without a second member

$$a\ddot{y} + b\acute{y} + cy = 0 \tag{3.9}$$

If y_0 is a solution of (3.9) and Y the general solution of (3.9), then $y = y_0 + Y$.

3.4.1 Integration

We set $y = e^{rx}$, then $\acute{y} = re^{rx}$ et $\ddot{y}= r^2e^{rx}$, by reporting in (3.9) we obtain $ar^2e^{rx}+bre^{rx}+ce^{rx} = 0$ or $e^{rx} \neq 0$ so we obtain

$$ar^2 + br + c = 0. \tag{3.10}$$

Then $y = e^{rx}$ is solution of the differential equation if and only if r is root of (3.10).

Discussion:

1. If (3.10) admits two different roots $r_1 \neq r_2$, then $y_1 = e^{r_1x}$ and $y_2 = e^{r_2x}$ are two particular integrals of (3.10) linearly independent because $r_1 \neq r_2$, the general integral is written $y = \lambda_1e^{r_1x} + \lambda_2e^{r_2x}$.

2. If $\Delta < 0$, r_1 and r_2 are complex conjugates, the general solution is written $y = \lambda_1e^{r_1x} + \lambda_2e^{r_2x}$, we choose $\lambda_2 = \overline{\lambda_1}$ then $y = \lambda_1e^{r_1x} + \overline{\lambda_1}e^{\overline{r_1x}} = 2\mathcal{R}e(\lambda_1e^{r_1x})$.

   Example 3.4.1 *a) $\ddot{y}–2\acute{y} - 3y = 0$, we have $r^2 - 2r - 3 = 0 \implies r_1 = -1, r_2 = 3$, then $y = \lambda_1e^{-x} + \lambda_2e^{3x}$.*

   b) $\ddot{y}–2\acute{y} + 5y = 0$, so $r^2 - 2r + 5 = 0 \implies r_1 = 1 - 2i, r_2 = 1 + 2i$, hence $y = e^x(\mu_1 \cos 2x + \mu_2 \sin 2x)$.

3. If (3.10) admits a double root $r_1 = r_2 = \dfrac{-b}{2a}$, $y = e^{rx}$ is a particular solution. We look for the general solution in the form $y = e^{rx}z$, where z is an unknown function of x, we have $y' = e^{rx}(rz + z')$ and $y''= e^{rx}(r^2z + 2rz' + z'')$, by transferring into the equation we obtain: $e^{rx}a\left[(r^2z + 2rz' + z'') + b(rz + z') + cz\right] = 0$, therefore $e^{rx}\left[ar^2 + br + cz + (2ar + b)z' + az''\right] = 0$, then $z''= 0$, hence $z = \lambda_1 x + \lambda_2$, he general solution is given by $y = e^{rx}(\lambda_1 x + \lambda_2)$.

Example 3.4.2 *$y''+4y' + 4y = 0$, on a $r^2 + 4r + 4 = 0 \implies r_1 = r_2 = -2$, then $y = e^{-2x}(\lambda_1 x + \lambda_2)$.*

3.4.2 Integration of the full equation

The solution of the equation without a second member being assumed to be known, we can:

- Either use the constant variation method.

- Either look for a particular solution of degree n of (3.8).

1. **$f(x) = P_n(x)$ where P_n is a polynomial of degree n.** It is natural to look for a particular solution in the form of a polynomial:

   1) of degree n if $c \neq 0$.

   2) of degree $n + 1$ if $c = 0$ and $b \neq 0$.

   3) of degree $n + 2$ if $c = 0$ and $b = 0$.

   Example 3.4.3 *$y''-2y' - 3y = 3x^2 + 1$, we look for the particular solution in the form $y = \alpha x^2 + \beta x + \gamma$, then $y' = 2\alpha x + \beta$ and $y''= 2\alpha$, we find $\alpha = -1, \beta = \dfrac{4}{3}, \gamma = \dfrac{-17}{3}$, the general solution is $y = \lambda_1 e^{-x} + \lambda_2 e^{3x} - x^2 + \dfrac{4}{3}x - \dfrac{-17}{3}$.*

2. **$f(x) = e^{mx}P_n(x)$, with $(m \in \mathbb{C})$**, we look for a particular integral in the form $y = e^{mx}z(x)$, then $y' = e^{mx}(mz+z')$, and $y''= e^{mx}(m^2z^2+2mz'+z'')$, or after simplification by e^{mx}, $az''+(2am+b)z' + (am^2 + bm + c)z = P_n(x)$. We find ourselves brought back to the previous case, we will therefore take for $z(x)$ a polynomial:

   - of degree n if $am^2 + bm + c \neq 0$ that is to say if m is not the root of the characteristic equation.

   - of degree $n + 1$ if, $am^2 + bm + c = 0$ and $2am + b \neq 0$ (m simple root).

   - of degree $n + 2$ if, $am^2 + bm + c = 0$ and $2am + b = 0$ (m double root).

Example 3.4.4 *$y''-y' = 2xe^x$, here $m = 1$, we have $m = 1$ is a simple root of the characteristic equation $r^2 - 1$. We are looking for a particular solution in the form $y = e^x z(x)$ where $z(x)$ is a polynomial of degree 2, n here equal to 1 since $P_n(x) = 2x$, it is the case $(n+1)$. $z(x) = \alpha x^2 + \beta x + \gamma$, therefore $z' = 2\alpha x + \beta$ and $z'' = 2\alpha$, by entering into the equation, and after calculation we find $\alpha = \frac{1}{2}, \beta = -\frac{1}{2}$, the particular solution is $y = e^x(\frac{x^2}{2} - \frac{x}{2})$. The general solution is therefore $y = \lambda_1 e^x + \lambda_2 e^{-x} + \frac{e^x}{2}(x^2 - x)$. [4]*

Chapter 4

NUMERICAL SERIES

The notion of sequence is closely linked to that of "series", that is to say linked to the problem of the summation of an infinity of terms. It was the Greeks in the 5^{th} century BC who began to glimpse the notion of infinity. It was only in the 16^{th} century that infinity took on its full meaning. The notion of series comes from the fact that for an infinite list of real numbers, that is to say for a sequence $(U_n)_{n\in\mathbb{N}}$, we pose the problem of considering "the sum" of all the elements of this sequence of numbers: $U_0 + U_1 + ... + U_n + ...$How can we give meaning to a sum of an infinite number of terms.

4.1 General information on numerical series

Definition 4.1.1 *To any sequence $(U_n)_{n\in\mathbb{N}}$ of real or complex numbers, it is possible to associate a sequence $(S_n)_{n\in\mathbb{N}}$ defined by: $\forall n \in \mathbb{N}, S_n = \sum_{k=0}^{n} U_k$. Conversely, to any sequence $(S_n)_{n\in\mathbb{N}}$, it is possible to associate a sequence $(U_n)_{n\in\mathbb{N}}$ defined by: $U_0 = Sl$ and $\forall n \in \mathbb{N}, U_{n+1} = S_{n+1} - S_n$.*

We call series the sequence $(S_n)_{n\in\mathbb{N}}$ and we note: $\sum_{n\in\mathbb{N}} U_n$ or by $\sum_n U_n$ or by $\sum U_n$.

Definition 4.1.2 *Let $(U_n)_{n\in\mathbb{N}}$ be a sequence of real or complex numbers. The sum $S_n = \sum_{k=0}^{n} U_k$ is called the partial sum of rank n of the series $\sum U_n$.*

- The sequence $(S_n)_{n\in\mathbb{N}}$ is called the sequence of partial sums of the series $\sum U_n$.

- The sequence $(U_n)_{n\in\mathbb{N}}$ is called a sequence associated with the series $\sum U_n$. U_n is said to be a general term of the series $\sum U_n$.

4.1.1 Convergence of a series

Definition 4.1.3 - *If the sequence of partial sums* $(S_n)_{n\in\mathbb{N}}$ *of a series* $\sum U_n$ *is convergent with finite limit* S*, we will say that the series is convergent and we will note* $\sum_{n\in\mathbb{N}} U_n = S$.

- *If the sequence* $(S_n)_{n\in\mathbb{N}}$ *tends towards* $+\infty$ *or towards* $-\infty$*, we will say that the series is divergent of the first kind and we will note* $\sum_{n\in\mathbb{N}} U_n = \pm\infty$.

- *If the sequence* $(S_n)_{n\in\mathbb{N}}$ *does not converge towards a finite limit, nor towards* $\pm\infty$*, owe will say that the series is divergent of the second kind.*

Remark 4.1.1 *Any finite series is convergent since the sequence of partial sums is constant from a certain rank.* $S_n = U_0 + U_1 + ... + U_p \qquad \forall n \geq p.$

Remark 4.1.2 *(a) We say that* $\sum_{n\in\mathbb{N}} U_n$ *and* $\sum_{n\geq n_0} U_n$*, where* $n \in \mathbb{N}$*, are of the same nature.*

(b) In the case where the series is convergent, the symbol $S = \sum_{n\in\mathbb{N}} U_n$ designates both the series and also the result of this series.

(c) The series $\sum_{n\in\mathbb{N}} U_n$ converges to S if: $\forall\epsilon > 0, \exists N_0 \in \mathbb{N}/, n \geq N_0 \Longrightarrow |S - S_n| \leq \epsilon$.

We use the definition of the convergence of a sequence because the partial sum $(S_n)_{n\in\mathbb{N}}$ is none other than a sequence, and we can write $\forall\epsilon > 0, \exists N_0 \in \mathbb{N}/, n \geq N_0 \Longrightarrow |\sum_{k=n}^{+\infty}| \leq \epsilon$.

Remark 4.1.3 *The sequence of partial sums* (S_n) *can be divergent by not having a limit when* $n \longrightarrow \infty$ *or by not having a finite limit for example* $(S_n) = \exp(in)$ *and* $S_n = \ln n$.

The nature of a series (convergence or divergence) does not depend on the first terms of the series, which means that the series $\sum_{n>0} u_n$ *and* $\sum_{n>N} u_n$ *converge (or diverge) at the same time.*

Proposition 4.1.1 *For the series* $\sum_{n\in\mathbb{N}} U_n$ *to be convergent, it is necessary that the associated sequence* $(U_n)_{n\in\mathbb{N}}$ *be convergent with zero limit. However, this condition is not sufficient, there exist divergent series whose associated sequence* $(U_n)_{n\in\mathbb{N}}$ *is convergent to* 0.

Example 4.1.1 *We consider the series* $\sum_{n>0} e^{\frac{1}{n}}$*, we know that* $\lim_{n\longrightarrow+\infty} e^{\frac{1}{n}} = 1 \neq 0$*, therefore the given series is divergent.*

Definition 4.1.4 *a) We call the sum of two numerical series* $\sum_{n\in\mathbb{N}} U_n$ *et* $\sum_{n\in\mathbb{N}} V_n$*, the general term series* $\forall n \in \mathbb{N}, W_n = U_n + V_n$ *from where* $\sum_{n\in\mathbb{N}} W_n = \sum_{n\in\mathbb{N}} U_n + \sum_{n\in\mathbb{N}} V_n$.

b) We call multiplication of a numerical series $\sum_{n\in\mathbb{N}} U_n$ by a non-zero scalar λ, ($\lambda \in \mathbb{R}^*$ or $\lambda \in \mathbb{C}^*$) the general term series $\forall n \in \mathbb{N}, W_n = \lambda U_n$ hence $\sum_{n\in\mathbb{N}} W_n = \sum_{n\in\mathbb{N}} (\lambda U_n)$.

c) We call the product of two series $(U_n)_{n\in\mathbb{N}}$ and $(V_n)_{n\in\mathbb{N}}$ the general term series; $\forall n \in \mathbb{N}, W_n = \sum_{k=0}^{n} U_k V_{n-k}$ hence $\sum_{n\in\mathbb{N}} W_n = \sum_{n\in\mathbb{N}} \left(\sum_{k=0}^{n} U_k V_{n-k}\right)$.

Proposition 4.1.2 (Vector space of convergent series) *Let $(U_n)_{n\in\mathbb{N}}$ et $(V_n)_{n\in\mathbb{N}}$ be two real sequences. If the series $\sum_{n\in\mathbb{N}} U_n$ and $\sum_{n\in\mathbb{N}} V_n$ are convergent with respective results U and V, then:*

- The series $\sum_{n\in\mathbb{N}} (U_n + V_n)$ is convergent with sum $U + V$.

- The series $\sum_{n\in\mathbb{N}} (\lambda U_n + \mu V_n)$ with ($\lambda, \mu \in \mathbb{R}$ or $\mathbb{C}$) is convergent with sum $\lambda U + \mu V$.

<u>*Consequence*</u>: We note that the complex series $\sum_{n\in\mathbb{N}}(a_n + ib_n)$ converges if and only if the two real series $\sum_{n\in\mathbb{N}} a_n$ and $\sum_{n\in\mathbb{N}} b_n$ converge. Then $\sum_{n\in\mathbb{N}}(a_n + ib_n) = \sum_{n\in\mathbb{N}} a_n + i \sum_{n\in\mathbb{N}} b_n$.

Proposition 4.1.3 *- The sum of two series, one of which is convergent and the other divergent of the first kind, is divergent of the first kind.*

- The sum of two series, one of which is convergent and the other is divergent of the second kind, is divergent of the second kind.

- The series $\sum_{n\in\mathbb{N}} \lambda U_n$ with ($\lambda \in \mathbb{R}$ or $\mathbb{C}$) is of the same nature as the series $\sum_{n\in\mathbb{N}} U_n$.

- The product of two convergent series is not necessarily a convergent series. [8]

4.1.2 Séries de Cauchy

Proposition 4.1.4 *A series $\sum_{n\in\mathbb{N}} U_n$ is convergent if and only if*

$$\forall \epsilon > 0, \exists N_0 \in \mathbb{N}, /p \geq n \geq N_0 \Longrightarrow |\sum_{k=n}^{p} U_k| \leq \epsilon$$

or:

$$\forall \epsilon > 0, \forall p > 0, \exists N_0 \in \mathbb{N}, /n \geq N_0 \Longrightarrow |\sum_{k=n}^{n+p} U_k| \leq \epsilon$$

4.2 Positive term series

The interest in studying series with positive terms (i.e. with terms in $\mathbb{R}^+$) is that the sequence $(S_n)_{n\in\mathbb{N}}$ of partial sums defined by its general term: $\forall n \in \mathbb{N}, \sum_{k=0}^{n} U_k$ is real and increasing.

Indeed; $S_{n+1} - S_n = \sum_{k=0}^{n+1} U_k - \sum_{k=0}^{n} U_k = U_k = U_n \geq 0$.

Proposition 4.2.1 *A series with positive terms $\sum_{n\in\mathbb{N}} U_n$ is either convergent or divergent of the first kind (of limit $+\infty$). Furthermore, for this series to be convergent, it is necessary and sufficient that the sequence $(S_n)_{n\in\mathbb{N}}$ of partial sums defined by its general term: $\forall n \in \mathbb{N}, S_n = \sum_{k=0}^{n} U_k$ is bounded.*

4.2.1 Comparison criterion

Proposition 4.2.2 *Let $\sum_{n\in\mathbb{N}} U_n$ and $\sum_{n\in\mathbb{N}} V_n$ be two series with positive terms, satisfying, $\forall n \in \mathbb{N}, U_n \leq V_n$.*

a) If $\sum_{n\in\mathbb{N}} V_n$, is convergent, then $\sum_{n\in\mathbb{N}} U_n$ is convergent.

b) If $\sum_{n\in\mathbb{N}} U_n$, is divergent of the first kind, then $\sum_{n\in\mathbb{N}} V_n$ is divergent of the first kind.

Proof. a) If we have $U_n \leq V_n, \forall n \in \mathbb{N}$, we also have: $U_0 + U_1 + ... + U_n = S_n \leq V_0 + V_1 + ... + V_n = \acute{S}_n \leq \sum_{n=0}^{+\infty} V_n = \acute{S}$. If $\sum_{n\in\mathbb{N}} V_n$ converges, then $\acute{S}$ exists and $\sum_{n\in\mathbb{N}} U_n$ is increased and therefore convergent.

b) obvious. ■

Proposition 4.2.3 (Use of equivalences) *Let $\sum_{n\in\mathbb{N}} U_n$ and $\sum_{n\in\mathbb{N}} V_n$ be two series with positive terms. If $U_n \underset{+\infty}{\sim} V_n$ then $\sum_{n\in\mathbb{N}} U_n$ and $\sum_{n\in\mathbb{N}} V_n$ are of the same nature.*

Example 4.2.1 (Harmonic series) *The harmonic series is given by its general term $U_n = \dfrac{1}{n}$. The general term tends towards zero, but the series diverges.*

Example 4.2.2 $U_n = \dfrac{1}{n\cos^2 n}$ *and* $V_n = \dfrac{1}{n}$ *we notice that $U_n \leq V_n, \forall n \in \mathbb{N}$ therefore $\sum_{n\in\mathbb{N}} V_n$ diverges hence the divergence of $\sum_{n\in\mathbb{N}} U_n$.*

Example 4.2.3 $Un = arc\sin\dfrac{2n}{4n^2+1}, Vn = \dfrac{2n}{4n^2+1}$. *We have $U_n \sim V_n$ and we set $\acute{V}_n = \dfrac{1}{2n}$ then $V_n \sim \acute{V}_n$ we see that $\sum_{n\in\mathbb{N}} \acute{V}_n$ diverges, so $\sum_{n\in\mathbb{N}} V_n$ diverges and finally $\sum_{n\in\mathbb{N}} U_n$ diverges.*

Example 4.2.4 $U_n = \dfrac{\sin^2 n}{n^2}$. *On a $U_n < \dfrac{1}{n^2} = Vn$, we know that $\sum_{n\in\mathbb{N}} V_n$ converges (Riemann serie's) so $\sum_{n\in\mathbb{N}} U_n$ converges.*

Example 4.2.5 $Un = \dfrac{\cos h\frac{1}{n}}{n}$, *and* $V_n = \dfrac{1}{n}$. *We see that $U_n \sim V_n$, but $\sum_{n\in\mathbb{N}} V_n$ diverges, therefore $\sum_{n\in\mathbb{N}} U_n$ diverges.*

4.2.2 Cauchy criterion

Proposition 4.2.4 *Let $\sum_{n\in\mathbb{N}} U_n$ be a series with positive terms. We consider $\lim_{n\longrightarrow+\infty} \sqrt[n]{U_n} = l$ where $l \in \mathbb{R}_+ \cup \{+\infty\}$.*

a) If $0 \leq l \leq 1$, then the series $\sum_{n\in\mathbb{N}} U_n$ is convergent.

b) If $l > 1$ then the series $\sum_{n\in\mathbb{N}} U_n$ is convergent.

c) If $l = 1$ we cannot conclude

We note that if $\lim_{n\longrightarrow+\infty} \sqrt[n]{U_n} \xrightarrow{>1} 1$, then we can conclude that $\sum_{n\in\mathbb{N}} U_n$ diverges.

Example 4.2.6 *Let the series be $\sum_{n\in\mathbb{N}} U_n = \sum_{n\in\mathbb{N}} \left(\frac{n+a}{n+b}\right)^{n^2}$. We have $\sqrt[n]{U_n} = \left(\frac{n+a}{n+b}\right)^{n} = \left(\frac{1+\frac{a}{n}}{1+\frac{b}{n}}\right)^{n} \longrightarrow \frac{e^a}{e^b} = e^{a-b} = l$*

Discussion: If $b > a$, then $l < 1$, the series converges.

If $b < a$, then $l > 1$, the series diverges.

If $a = b$, then $U_n = 1 \nrightarrow 0$ the series diverges. [5]

4.2.3 D'Alembert criterion

Lemma 4.2.1 *Let $\sum_{n\in\mathbb{N}} U_n$ and $\sum_{n\in\mathbb{N}} V_n$ be two real series with positive terms. If we have $\forall n \in \mathbb{N}, \frac{U_{n+1}}{U_n} \leq \frac{V_{n+1}}{V_n}$, such that; $\exists \alpha \in \mathbb{R}$, then $\forall n \in \mathbb{N}, U_n \leq \alpha V_n$.*

Theorem 4.2.1 *Let $\sum_{n\in\mathbb{N}} U_n$ and $\sum_{n\in\mathbb{N}} V_n$ be two real series with positive term such that: $\lim_{n\longrightarrow+\infty} \frac{U_n}{V_n} = l \neq 0$. Then the two series converge or diverge at the same time (of the same nature).*

Theorem 4.2.2 *Let $\sum_{n\in\mathbb{N}} U_n$ be a series with positive terms. We consider the series $\sum_{n\in\mathbb{N}} V_n$ defined such that, $\forall n \in \mathbb{N}, V_n = \frac{U_{n+1}}{U_n}$ moreover we consider $\lim_{n\longrightarrow+\infty} V_n = \lim_{n\longrightarrow+\infty} \frac{U_{n+1}}{U_n} = l$ où $l \in \mathbb{R}_+ \cup \{+\infty\}$.*

a) If $0 l 1$, then the series $\sum_{n\in\mathbb{N}} U_n$ is convergent.

b) If $l > 1$ then the series $\sum_{n\in\mathbb{N}} U_n$ is divergent.

c) If $l = 1$ we cannot conclude.

We assume that $\lim_{n\longrightarrow+\infty} \frac{U_{n+1}}{U_n} \xrightarrow{>1} 1$, then we can conclude that $\sum_{n\in\mathbb{N}} U_n$ diverges.

Example 4.2.7 *Let the series* $\sum_{n\geq 1}\frac{n^3}{n!}$. *We set* $\forall n \geq 1, U_n = \frac{n^3}{n!}$, *then* $\lim\limits_{n\longrightarrow+\infty}\frac{U_{n+1}}{U_n} = \lim\limits_{n\longrightarrow+\infty}\frac{(n+1)^3}{(n+1)!}.\frac{n!}{n^3} = 0 < 1$. *The series converges.*

Proposition 4.2.5 *Let* $(U_n)_{n\in\mathbb{N}}$ *be a sequence of positive real numbers. Suppose that* $\lim\limits_{n\longrightarrow+\infty}\sqrt[n]{U_n} = l$, $\lim\limits_{n\longrightarrow+\infty}\frac{U_{n+1}}{U_n} = \acute{l}$, *then* $l = \acute{l}$.

4.2.4 Series, Integrals and Riemann criterion

Proposition 4.2.6 *Let* f *be a decreasing function, defined from* $\mathbb{R}^+$ *into* $\mathbb{R}^+$ *such that:* $U_n = f(n)$. *Then the series* $\sum_{n\in\mathbb{N}} U_n$ *and* $\int_0^{+\infty} f(x)dx$ *are of the same nature.*

Definition 4.2.1 *We call a Riemann series a series of the form* $\sum_{n>1}\frac{1}{n^a}$ *where* $a \in \mathbb{R}_+^*$.

Proposition 4.2.7 *a) If* $0 < a < 1$, *the Riemann series diverges.*

b) If $a > 1$, *the Riemann series converges.*

4.3 Series with any terms

In this paragraph we will study the case of series $\sum_{n\in\mathbb{N}} U_n$ where U_n s complex or real of any sign.

4.3.1 Definition et proposition

Definition 4.3.1 *We will say that the general term series* $U_n \in \mathbb{C}$ *is absolutely convergent if the series with positive terms of general term* $|U_n|$ *converges.*

Proposition 4.3.1 *Any absolutely convergent series is convergent.*

Remark 4.3.1 *There exist convergent series which are not absolutely convergent.*

Example 4.3.1 *Let* $\sum_{n\in\mathbb{N}} U_n$ *such that* $U_{2n-1} = \frac{1}{2n}$ *and* $U_{2n} = \frac{-1}{2n}$ *with* $n > 1$. *Then* $|U_n| \sim \frac{1}{n}$ *therefore* $\sum_{n\in\mathbb{N}} |U_n|$ *diverges, the series* $\sum_{n\in\mathbb{N}} U_n$ *is not absolutely convergent. However,* $S_{2n} = 0$ *and* $S_{2n-1} = \frac{1}{2n}$ *therefore the series* $\sum_{n\in\mathbb{N}} U_n$ *cconverges with zero sum. We will say that a series which converges without being absolutely convergent is semi-convergent.*

4.3.2 Abel's Sum

Theorem 4.3.1 *Let $\sum_{n\in\mathbb{N}} U_n$ be a series whose general term, real or complex, is written in the form $U_n = a_n b_n$; Suppose that:*

1) The sequence $(a_n)_{n\in\mathbb{N}}$ is in positive terms decreasing and tends to 0 when $n \longrightarrow +\infty$.

2) There exists $M \in \mathbb{R}$ such that $\forall x \in \mathbb{N} |\sum_{k=1}^{n} b_k| \leq M$.

Then the series $\sum_{n\in\mathbb{N}} U_n$ is convergent.

4.3.3 Alternating series

Definition 4.3.2 *We call an alternating series a numerical series whose general term U_n is of the form $U_n = (-1)^n V_n$, where $(V_n)_{n\in\mathbb{N}}$ denotes a sequence of constant sign.*

Example 4.3.2 *The series $\sum_{n=0}^{\infty} \dfrac{(-1)^n}{n+1}$ is an alternating series. It is called alternating harmonic series.*

Theorem 4.3.2 *If the sequence $(V_n)_{n\in\mathbb{N}}, V_n > 0$, is decreasing and converges to zero, then the series $\sum_{n\in\mathbb{N}} U_n = \sum_{n\in\mathbb{N}} (-1)^n V_n$ is convergent and its sum S satisfies the inequality: $S_{2p+1} \leq S \leq S_{2p}$.*

Chapter 5

SEQUENCES AND SERIES OF FUNCTIONS

Many functions appear as boundaries of other simpler functions. This is the case, for example, of the exponential function, which can be defined by one of the following two formulas. $e^x = \lim\limits_{n \longrightarrow +\infty} (1 + x^n)^n$ ou $e^x = \sum_{n=0}^{+\infty} \frac{x^n}{n!}$.

This is also the case for more theoretical problems, such as when we construct solutions to equations (for example differential), we often construct by induction towards an approximate solution which converge towards an exact solution.

5.1 Function suites

5.1.1 General notions

In the following, $\mathbb{K}$ designates one of the fields $\mathbb{R}$ or $\mathbb{C}$. Let E be a $\mathbb{K}$-vector space and I be any set, the set $\mathcal{F}_E(I)$ of maps defined on I, with values in E, is equipped with the following two scalar operations: addition $(f, g) \longrightarrow f + g$ and multiplication by a scalar $(\lambda, f) \longrightarrow \lambda f$, defined by: if $f, g \in_{FE(I)}$ and $\lambda \in K, (f + g)(x) = f(x) + g(x)$ and $(\lambda f)(x) = \lambda f(x).(\mathcal{F}_E(I), +, \times)$ is a K-vector space.

Definition 5.1.1 *A sequence in $\mathcal{F}_E(I)$ is a map of $\mathbb{N}$ in $\mathcal{F}_E(I)$ which associates with each natural number n a function f_n. It is noted $(f_n)_{n>0}$ or simply $(f_n)_n$.*

Remark 5.1.1 • *$(f_{n+n_0})_n$ is also denoted $(f_n)_{n>n_0}$.*

• In a sequence $(f_n)_n$, the f_n are assumed to have the same definition set.

• The sequence $(f_n)_n$ can be seen as a numerical sequence $(f_n(x))_n$ dependent on the parameter x, traversing a given set.

5.1.2 Simple convergence

The notion of convergence of a sequence of real or complex numbers naturally leads to that of convergence at each point for the sequences of functions defined as follows.

Definition 5.1.2 *A sequence $(f_n)_n$ of maps $f_n : I \longrightarrow \mathbb{K}$ is said to simply converge on I if there exists a map $f : I \longrightarrow \mathbb{K}$ such that $\forall x \in I; \lim_{n\longrightarrow+\infty} f_n(x) = f(x)$*

- f is called simple limit of $(f_n)_n$

- If f exists it is unique.

We write:

$$\left(\forall x \in I; \lim_{n\longrightarrow+\infty} f_n(x) = f(x)\right) \iff (\forall x \in I; \forall\epsilon > 0; \exists N(x,\epsilon)/n \geq N) \implies |f_n(x) - f(x)| \leq \epsilon) .$$

Remark 5.1.2 *It should be noted that N depends on x and ϵ.*

Example 5.1.1 *A numerical sequence is a very particular case of sequences of functions, here the functions are constants.*

Example 5.1.2 *Let $I =]0,1[$ and $\mathbb{K} = \mathbb{R}, f_n(x) = \dfrac{1}{(n+1)x}$. Then $\lim_{n\longrightarrow+\infty} f_n(x) = f(x) = 0, \forall x \in I$.*

Example 5.1.3 *We consider $\forall x \in \mathbb{R}, f_n(x) = \left(1 + \dfrac{x}{n}\right)^n = e^{n\ln\left(1+\frac{x}{n}\right)}$. Then $\forall x \in \mathbb{R}; f_n(x) = e^x$.*

Theorem 5.1.1 *Let $(f_n)_n$ et $(g_n)_n$ two sequences in $\mathcal{F}_E(I)$ simply convergent to f and g respectively, and $\lambda \in \mathbb{K}$, Then*

1. *The sequence $(f_n + g_n)_n$ simply converges to $f + g$.*

2. *The sequence $(\lambda f_n)_n$ simply converges to λf.*

3. *The sequence $(f_n g_n)_n$ simply converges to fg.*

Theorem 5.1.2 (Cauchy criterion) *For a sequence $(f_n)_n$ of applications $f_n : I \longrightarrow \mathbb{K}$ to simply converge, it is necessary and sufficient that:*

$$\forall x \in I, \forall \epsilon > 0, \exists N(x,\epsilon)/m \geq N(x,\epsilon); n \geq N(x,\epsilon) \implies |f_m(x) - f_n(x)| \leq \epsilon.$$

5.1.3 Uniform convergence

Example 5.1.4 *Let $I = [0,1]$ and $\forall n \in \mathbb{N}, f_n(x) = x^n$, it is clear that $\lim_{n \longrightarrow +\infty} f_n(x) = f(x)$ such as,*

$$f(x) = \begin{cases} 0 & if\ x \in [0,1[\\ 1 & if\ x = 1 \end{cases}$$

We conclude that:

- Each function f_n is continuous whatever n.

- Each $(f_n)_n$ simply converges to f.

- f is not continuous.

This is why it is necessary to use a more precise notion which preserves continuity by passing to the limit, this is uniform convergence. [1]

Definitions

Definition 5.1.3 *We call the norm of uniform convergence the norm for $(f_n)_n$ and f of $\mathcal{F}_E(I)$; $\|f_n - f\| = \sup_{x \in I} |f_n(x) - f(x)|$.*

Definition 5.1.4 *A sequence of maps $f_n : I \longrightarrow \mathbb{K}$ is said to be uniformly convergent on I if there exists a map $f : I \longrightarrow \mathbb{K}$ such that,*

$$\lim_{n \longrightarrow +\infty} \left(\sup_{x \in I} |f_n(x) = f(x)| \right) = 0.$$

or,

$$\forall \epsilon > 0; \exists N; \forall x; \exists n[n > N \implies |f_n(x) - f(x)| < \epsilon].$$

or

$$\lim_{n \longrightarrow +\infty} \|f_n - f\| = 0.$$

Interpretation*:* We have $|f_n(x) - f(x)| < \epsilon \implies f(x) - \epsilon < fn(x) < f(x) + \epsilon$. We say that for $n > N$, the graph of f_n is contained in a band of width 2ϵ symmetrical with respect to the graph of f.

Proposition 5.1.1 *Uniform convergence implies simple convergence. Indeed;* $\forall x, \forall n : |f_n(x) - f(x)| \leq |f_n - f| \underset{+\infty}{\longrightarrow} 0$*. The converse is false*

Example 5.1.5 *Let* $f_n(x) = \dfrac{nx}{1+nx}$ *with* $x \in [0, +\infty[$*. We have shown that this sequence simply converges to*

$$f(x) = \begin{cases} 0 & if\ x \in [0, 1[\\ 1 & if\ x = 1 \end{cases}.$$

as; $\|f_n - f\| = \sup_{x\in I} |f_n(x) - f(x)| = \sup_{x\in I} \dfrac{1}{1+nx} = \lim_{x\longrightarrow 0} \dfrac{1}{1+nx} = 1$ *i.e.* $\|f_n - f\| \nrightarrow 0$*, hence the convergence is not uniform. [13]*

Cauchy criterion for uniform convergence

Theorem 5.1.3 *Let* $(f_n)_n$ *be a sequence of functions on* I*. For the sequence* $(f_n)_n$ *to be uniformly convergent on* I *towards a function* f *it is necessary and sufficient that:*

$$\forall \epsilon > 0; \exists N; \forall n, \forall m; \forall x \subset I[n, m > N \implies |f_n(x) - f_m(x)| < \epsilon].$$

or

$$\forall \epsilon > 0; \exists N; \forall n, \forall m; \forall x \subset I[n, m > N \implies \|f_n(x) - f_m(x)\| < \epsilon].$$

Example 5.1.6 *Let* $f_n(x) = \dfrac{2xn}{1+n^2x^2}$ *sur* $I = [0, 1]$*. We have* $\lim_{n\longrightarrow+\infty} f_n = f = 0$*. As the upper bound of the function* $y \longrightarrow \dfrac{2y}{1+y^2}$ *on* $[0, +\infty[$ *is equal to* $\dfrac{1}{2}$ *for* $y = 1$*, we have* $\|f_n\| = \sup_I |f| = \dfrac{1}{2}$*. Then* $f_n \longrightarrow 0$ *simply but not uniformly.*

How to show that a sequence of functions converges uniformly

To show that a sequence of functions $(f_n)_n$ is uniformly convergent:

1. We show that it is simply convergent, which allows us to define f.

2. We seek to increase $|f_n(x) - f(x)|$ by a sequence $(\varepsilon_n)_n$ of positive real numbers which converges to 0 such that $(\varepsilon_n)_n$ does not depend on x.

To determine $(\varepsilon_n)_n$, we have two methods.

1. Majoer $|f_n(x) - f(x)|$ independently of x.

2. Calculate $\sup_{x\in I} |f_n(x) = f(x)|$ using the study of the function $|f_n(x) = f(x)|$.

Operations and uniform convergence

Theorem 5.1.4 *Let $(f_n)_n$ and $(g_n)_n$ be two sequences in $\mathcal{F}_E(I)$ uniformly convergent to f and g respectively, and $\lambda \in \mathbb{K}$. So*

(i) The sequence $(f_n + g_n)_n$ converges uniformly $f + g$.

(ii) The sequence $(\lambda f_n)_n$ converges uniformly λf.

(iii) If the maps f and g are bounded, the sequence $(f_n g_n)_n$ converges uniformly to fg.

Remark 5.1.3 *In (iii), the assumption (f and g bounded) cannot be omitted, otherwise the theorem is false. [2]*

5.1.4 Properties of uniform convergence

Continuity

Theorem 5.1.5 *Let $(f_n)_n \in \mathcal{F}_E(I)$, be a sequence of functions uniformly convergent to f. If all functions f_n are continuous at x_0, then f is continuous at x_0.*

Proof. Either

$$\epsilon > 0; \exists N/\forall n > N, \forall x \in I; |f_n(x) - f(x)| < \frac{\epsilon}{3} \tag{5.1}$$

Let us fix an integer $n > N$, f_n being continuous in x_0, then It exists

$$\delta > 0/\forall x \in I; |x - x_0| < \delta \implies |f_n(x) - f_n(x_0)| < \frac{\epsilon}{3} \tag{5.2}$$

From (5.1) and (5.2) we obtain:

$$\forall x \in I; |x-x_0| < \delta \implies |f_n(x)-f_n(x_0)| \leq |f(x)-f_n(x)|+|f_n(x)-f_n(x_0)|+|f_n(x_0)-f(x_0)| < \epsilon.$$

■

corollary 5.1.1 *If $f_n \longrightarrow f$ uniformly and if all functions f_n are continuous on I, then f is continuous on I.*

Remark 5.1.4 *1) If a sequence of continuous functions converges to a non-continuous function, the convergence is not uniform.*

2) The theorem gives a sufficient condition for $f = \lim f_n$ to be continuous, but this condition is not necessary. It can happen that the functions f_n being continuous, f is continuous, without the convergence being uniform.

Example 5.1.7 *Let $(f_n)_n$ be such that $I = [0,2]$.*

$$f_n(x) = \begin{array}{ll} nx^2 & \text{if } 0 \leq x \leq \dfrac{1}{n} \\ nx^2 + 2n & \text{if } \dfrac{1}{n} \leq x \leq \dfrac{2}{n} \\ 0 & \text{if } x \leq \dfrac{2}{n} \end{array} .$$

The functions f_n are continuous, and $\lim\limits_{n \longrightarrow +\infty} f_n = f = 0$ simply. Because:

If $x = 0$, $f_n(0) = 0$.

If $x > 0, f_n(x) = 0$ for all $n > \dfrac{2}{x}$.

However the convergence is not uniform since: $\|f_n(x) - f(x)\| = f_n(\dfrac{1}{n}) = n \longrightarrow +\infty$ when $n \longrightarrow +\infty$.

Example 5.1.8 *If a sequence of functions $(f_n)_n$ simply converges to f, then f is not necessarily continuous. Let $I = [0,1]$, and let $f_n(x) = x^n$. It is clear that $f_n(x) \longrightarrow f(x)$ such that*

$$f(x) = \begin{cases} 0 & \text{if } x \in [0,1[\\ 1 & \text{if } x = 1 \end{cases} .$$

f is not continuous.

Example 5.1.9 *There exist sequences $(f_n)_n$ which converge to f continues. Either*

$$f_n(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ nx & \text{if } 0 \leq x \leq \frac{1}{n} \\ nx+2 & \text{if } \frac{1}{n} \leq x \leq \frac{2}{n} \\ 0 & \text{if } x \geq \frac{2}{n} \end{cases} .$$

- For $x < 0$, we have $\lim\limits_{n\longrightarrow+\infty} f_n = 0$

- For $x > 0$, we have $n > \dfrac{2}{n} \implies f_n(x) = 0$ therefore $\lim\limits_{n\longrightarrow+\infty} f_n = 0$, so the limit is continuous on R although the convergence is not uniform there since: $\|f_n\| = \sup\limits_{x\in\mathbb{R}} |f_n(x)| = 1$ does not tend towards zero. [12]

Integration

Theorem 5.1.6 *Let $(f_n)_n$ be a sequence of integrable functions on $I = [a, b]$ converging uniformly to f. So:*

a) f is integrable on $[a, b]$

b) $\lim\limits_{n\longrightarrow+\infty} \int_a^b f_n(x)dx = \int_a^b f(x)dx$.

Proof. a) Let $\epsilon > 0$, there exists n such that:

$$\forall x_i \in \mathbb{R}, n \in [a, b], f_n(x) - \frac{\epsilon}{2(b-a)} \leq f(x) \leq f_n(x) + \frac{\epsilon}{2(b-a)} \tag{5.3}$$

because $f_n \longrightarrow f$ uniformly. The function f_n being integrable, there exists a subdivision $d = \{x_0, x_l, ..., x_k\}$ of $[a, b]$ such that

$$\sum_{i=1}^{k} (M_{n_i} - m_{ni})(x_i - x_{i-1}) < \frac{\epsilon}{2} \tag{5.4}$$

or

$$M_{n_i} = \sup_{[x_{i-1},x_i]} f_n \text{ et } m_{n_i} = \inf[x_{i-1}, x_i] f_n.$$

Noting $M_i = \sup\limits_{[x_{i-1},x_i]} f$ and $m_i = \inf\limits_{[x_{i-1},x_i]} f$.

Then (5.3) implies for $1 \leq i \leq k$ that

$$m_{n_i} - \frac{\epsilon}{2(b-a)} \leq m_i \leq M_{n_i} + \frac{\epsilon}{2(b-a)}$$

Which gives taking into account (5.4

$$\sum_{i=1}^{k}(M_i - m_i)(x_i - x_{i-1}) \leq \sum_{i=1}^{k}(M_{n_i} - m_{n_i})(x_i - x_{i-1}) + \frac{\epsilon}{2} < \epsilon.$$

Thus, f is integrable. ■

b) We have for all n;

$$|\int_a^b f_n(x)dx - \int_a^b f(x)dx| \leq \int_a^b |f_n - f|dx \leq \|f_n - f\|(b-a) \longrightarrow 0.$$

Remark 5.1.5 *The condition $f_n \longrightarrow f$ uniformly is sufficient but not necessary for $\lim\limits_{n\longrightarrow+\infty} \int_a^b f_n(x)dx = \int_a^b f(x)dx$.*

Example 5.1.10 *Either*

$$f_n(x) = \begin{cases} 0 & \text{if } x = 0 \\ 0 & \text{if } \frac{1}{n} \leq x \leq 1 \\ 1 & \text{if } 0 < x < \frac{1}{n} \end{cases}$$

For $n > \dfrac{1}{x}$ we have $f_n(x) = 0$ therefore $\lim\limits_{n\longrightarrow+\infty} f_n(x) = f(x) = 0$. But this convergence is not uniform because: $\sup\limits_{[0,1]} f_n = 1 \forall n$, however

$\int_0^1 f_n(x)dx = \dfrac{1}{n} \longrightarrow 0 = \int_0^1 f(x)dx$ that is to say, although $\int_0^1 f_n(x)dx = \int_0^1 f(x)dx$, but the convergence is not uniform.

Derivation

We would like to be able to give a theorem similar to that of continuity or that of integration but unfortunately, this is not possible.

Theorem 5.1.7 *Let $(f_n)_n$ be a sequence of continuously differentiable functions on $[a,b]$($f_n \in C^1[a,b],\mathbb{R}$) satisfying the following properties:*

a) the sequence $(f'_n)_n$ converges uniformly to g on $[a,b]$.

b) There exists a point $x_0 \in [a,b]$ such that the sequence $(f_n(x_0))$ converges to a limit l. Then $(f_n)_n$ converges uniformly to f on $[a,b]$ with ($f \in C^1[a,b],\mathbb{R}$) and we have $\forall x \in [a,b] \acute{f}(x) = g(x)$. In other words: $\lim\limits_{n\longrightarrow+\infty} \acute{f_n}(x) = \left(\lim\limits_{n\longrightarrow+\infty} f_n(x)\right)'$.

Proof. We have $f_n(x) = f_n(x_0) + \int_{x_0}^{x} \acute{f_n}(t)dt$. According to (3.1.24,a) we have $\lim\limits_{n\longrightarrow+\infty} \int_{x_0}^{x} \acute{f_n}(t)dt = \int_{x_0}^{x} g(t)dt$ uniformly on $[a,b]$. Thus $(f_n)_n$ converges uniformly to f defined by:

$$f(x) = \lim_{n\longrightarrow+\infty} f_n(x_0) + \int_{x_0}^{x} g(t)dt = l + \int_{x_0}^{x} g(t)dt$$

as f continuously differentiable, then $\acute{f} = g$. ■

5.2 Series of functions

5.2.1 Definitions and properties

Definition 5.2.1 *Let $(f_n)_n$ be a sequence of functions with real or complex values defined on a non-empty set $I \subset \mathbb{R}$, let us associate with this sequence the sequence of functions $(S_n)_n$ defined by: $\sum_{k=0}^{n} f_k$.*

We call a series of functions on (I) with general term f_n the pair $((f_n)_n, (S_n)_n)$. The sequence $(S_n)_n$ is called n^{th} partial sum of the series of functions $((f_n)_n, (S_n)_n)$ and will be denoted, $\sum f_n$ or $\sum_{n=0}^{+\infty} f_n$ or $f_0 + f_1 + ...f_n + ...$

Definition 5.2.2 *The series $\sum_{k=0}^{+\infty} f_k$ is called the remainder of order n of the series $\sum_{n=0}^{+\infty} f_n$.*

Definition 5.2.3 *We will say that $\sum_{k=0}^{+\infty} f_k$ is convergent in $x_0 \in I$ if the numerical series $\sum_{k=0}^{+\infty} f_k(x_0)$ is convergent. We will say that $\sum_{k=0}^{+\infty} f_k$ is convergent on I (or on a part $A \subset I$) if the series $\sum_{k=0}^{+\infty} f_k(x)$ is convergent at any point $x \in I$ (respectively $x \in A$), in this case we will say that the series $\sum_{k=0}^{+\infty} f_k(x)$ is simply convergent on I (respectively on A).*

We therefore have the following equivalences:

$$\sum f_n \text{ converges at } x_0 \iff \sum f_n(x_0) \text{ converges} \iff (S_n(x_0))_n \text{ converges}$$

$$\sum f_n \text{ converges on } I \iff \forall x \in I; \sum f_n(x) \text{ converges} \iff (S_n)_n \text{ converges on } I.$$

Thus: The simple convergence of a series on I is equivalent to the simple convergence of the sequence of its partial sums $(S_n)_n$ on I.

Definition 5.2.4 *Let $\sum_{n=0}^{+\infty} f_n$ be a convergent series on I and $(S_n)_n$ the sequence of its partial sums. The function $S : I \longrightarrow \mathbb{C}$ defined by $S = \lim_{n \longrightarrow +\infty} S_n$ is called the sum of the series and is denoted $S = \sum_{n=0}^{+\infty} U_n$.*

Remark 5.2.1 *The general theorems relating to numerical series remain true, with necessary modifications, for series of functions.*

5.2.2 Uniform convergence

Definition and example

Definition 5.2.5 *Let $\sum_{n=0}^{+\infty} f_n$ be a series of functions, we say that this series is uniformly convergent on I if the sequence $(S_n)_n$ of partial sums is uniformly convergent where $S_n(x) = \sum_{n=0}^{+\infty} f_n(x)$. In case of convergence, the limit S such that $S_n(x) = \sum_{n=0}^{+\infty} f_n(x)$ is called the sum of the series.*

Remark 5.2.2 *Thus the uniform convergence of $\sum f_n$ on a set E means:*

$$\forall \epsilon > 0; \exists N; /\forall n, \forall x \in E \left[n > N \implies \left| \sum_{k=n+1}^{+\infty} f_n(x) \right| < \epsilon \right].$$

Then the simple convergence is expressed by:

$$\forall x \in E; \forall \epsilon > 0; \exists N; /\forall n, \left[n > N \implies \left| \sum_{k=n+1}^{+\infty} f_n(x) \right| < \epsilon \right]$$

We will note: $\sum_{k=n+1}^{+\infty} f_n(x) = S - S_n$

Example 5.2.1 *Let $f_n(x) = x^n$. Then $\forall x \neq 1; S_n(x) = \sum_{k=0}^{n} f_k(x) = 1 + x + ... + x^n = \dfrac{x^{n+1} - 1}{x - 1}$, this series therefore simply converges in $I =]-1, +1[$ to the function $S(x) = \dfrac{1}{1-x}$. However the convergence is not uniform on $I =]-1, +1[$. Indeed, $\lim_{\substack{x \longrightarrow 1 \\ <1}} (S_n(x) - S(x)) = +\infty, \forall n$, so $\|S_n(x) - S(x)\|$ does not tend towards zero. On the other hand, in each interval $[-\delta, \delta]$ with $0 < \delta < 1$, the convergence is uniform.*

Abel's criterion for uniform convergence

Theorem 5.2.1 *We consider the series of general term functions f_n. If f_n is written in the form $\forall x \in I, f_n(x) = \varepsilon_n(x) g_n(x)$ with:*

(1) $\forall x \in I, \varepsilon_n(x)$ *is decreasing towards* 0.

(2) $\lim\limits_{n\longrightarrow+\infty}\left(\sup\limits_{x\in I}|\varepsilon_n(x)|\right)=0.$

(3) $\exists M>0$ *such that* $\forall x\in I;\forall n\in\mathbb{N}:\sum_{i=0}^{n} g_i(x)<M$.

Then the general term series f_n *converges uniformly on* I.

Example 5.2.2 *We show that this series converges uniformly on* I*, using the Abel criterion. We pose*

$$\forall n\in\mathbb{N};\forall x\in I_\delta,[\delta,2\pi-\delta] \text{ with } \delta\in]0,\pi[\text{ and } \alpha\in[0,1[, f_n(x)=\frac{e^{inx}}{(n+1)^\alpha}$$

We show that this series converges uniformly on I*, using the Abel criterion. We set*

$$\forall x\in I_\delta; f_n(x)=\varepsilon_n(x)g_n(x), \text{ with } \varepsilon_n(x)=\frac{1}{(n+1)^\alpha} \text{ and } g_n(x)=e^{inx}.$$

We have

(1) $\varepsilon_n(x)$ *decreasing towards* 0 *because* $\alpha\in]0,1[$.

(2) We have

$$\forall n\in\mathbb{N},\sup_{x\in I_\delta}|\varepsilon_n(x)|=\frac{1}{(n+1)^\alpha} \text{ and } \lim_{n\longrightarrow+\infty}\left(\sup_{x\in I_\delta}|\varepsilon_n(x)|\right)=0$$

(3) $\sum_{i=0}^{n} g_i(x)=1+r^{ix}+(e^{ix})^2+.....+(e^{ix})^n$ *which is a sum of a geometric series with first term* $g_0(x)=1$ *and reason* $q=e^{ix}$*, we deduce that,* $\sum_{i=0}^{n} g_i(x)=\dfrac{1-(e^{ix})^{n+1}}{1-e^{ix}}$ *because* $e^{ix}\neq 1,\forall x\in I_\delta$*. We must find an increase of* $|\sum_{i=0}^{n} g_i(x)|$ *which is independent of the variable* x *and* n*, since* $\left|e^{ix(n+1)}\right|=1$ *We have*

$$\left|\sum_{i=0}^{n} g_i(x)\right|\leq\frac{1+|e^{ix(n+1)}|}{|1-e^{ix}|}\leq\frac{2}{|1-e^{ix}|}.$$

We have

$$\begin{aligned}|1-e^{ix}| &=\sqrt{(1-\cos x)^2+\sin^2 x}=\sqrt{1-2\cos x+\cos^2 x+\sin^2 x}\\ &=\sqrt{2-2\cos x}=\sqrt{4\sin^2\frac{x}{2}}=2|sin^2\frac{x}{2}|\\ &=2\sin\frac{x}{2} \text{ because } 0<\frac{\delta}{2}\leq\frac{x}{2}\leq\pi-\frac{\delta}{2}<\pi\end{aligned}$$

First case: $0 < \frac{\delta}{2} \leq \frac{x}{2} \leq \frac{\pi}{2}$

The sine function is increasing on $]0, \frac{\pi}{2}]$, *which allows us to deduce:* $|1-e^{ix}| = 2\sin\frac{x}{2} \geq 2\sin\frac{\delta}{2}$.

Second case: $\frac{\pi}{2} \leq \frac{x}{2} \leq \pi - \frac{\delta}{2} < \pi$

The sine function is decreasing on $\left[\frac{\pi}{2}, \pi\right]$, *which allows us to deduce:* $\sin\frac{x}{2} \geq \sin(\pi - \frac{\delta}{2}) > \sin\pi$, *or we have* $\sin(\pi - \frac{\delta}{2}) = \sin\frac{\delta}{2}$. *Thus we obtain:* $|1-e^{ix}| = 2\sin\frac{x}{2} \geq 2\sin\frac{\delta}{2}$, *since the inequalities are the same in both cases, we deduce that:* $\sum_{i=0}^{n} g_i(x) \leq \frac{2}{2\sin\frac{\delta}{2}} = \frac{1}{\sin\frac{\delta}{2}} = M$. *Then according to Abel's criterion, we deduce that the series of functions converges uniformly on* I.

5.2.3 Normal convergence

Definition 5.2.6 *Let* $\sum_{n=0}^{+\infty} f_n, f_n(x) : I \longrightarrow \mathbb{K}$ *(*$\mathbb{K} = \mathbb{R}$ *or* $\mathbb{C}$*) be a series of functions, we say that this series is normally convergent on* I *if the numerical series:* $\sum_n \|f_n\|$ *where* $\|f_n\| = \sup_{x\in I} |f_n(x)|$, *is convergent.*

Proposition 5.2.1 *Let* $f_1, f_2 : I \longrightarrow \mathbb{K}$, *then* $\|f_1 + f_2\| \leq \|f_1\| + \|f_2\|$ *(triangular inequality).*

Theorem 5.2.2 *If the series* $\sum_{n=0}^{+\infty} f_n$ *converges normally on* I*; Then it is uniformly convergent on* I.

corollary 5.2.1 *Let* $\sum_{n=0}^{+\infty} f_n; f_n : I \longrightarrow \mathbb{K}$ *be a series of functions; if there exists a convergent positive-term numerical series* $\sum_{n=0}^{+\infty} a_n$, *such that:* $\forall n, \forall x \in I, |f_n(x)| \leq a_n$, *then the series* $\sum_{n=0}^{+\infty} f_n$ *is normally convergent on* I.

Example 5.2.3 *Let* $\sum_{n=0}^{+\infty} f_n$ *such that;* $f_n(x) = \frac{\sin(x^2+n^2)}{x^2+n^2}$, *we have* $\forall x \in \mathbb{R}; |f_n(x)| \leq \frac{1}{n^2}$. *And we have* $a_n = \frac{1}{n^2}$ *we know that* $\sum_{n=0}^{+\infty} a_n$ *is convergent, therefore the series* $\sum_{n=0}^{+\infty} f_n$ *is normally convergent, therefore uniformly convergent in* $\mathbb{R}$.

Remark 5.2.3 *Normal convergence is stronger than uniform convergence. There exist uniformly convergent series which do not converge normally. [5]*

5.2.4 Uniform convergence and properties of a series of functions

Continuity

Theorem 5.2.3 *Let* $\sum_{n=0}^{+\infty} f_n, f_n(x) : I \longrightarrow \mathbb{K}$ *be a series of functions uniformly convergent on* I. *If all functions* f_n *are continuous in* $x_0 \in I$, *the sum* S *of the series is continuous in* $x_0 \in I$.

corollary 5.2.2 *If all functions f_n are continuous on I and if $\sum_{n=0}^{+\infty} f_n$ is uniformly convergent on I, then the sum is continuous on I.*

Remark 5.2.4 *The hypotheses are those of Theorem (3.2.16), we can therefore write:*

$$\lim_{x\longrightarrow x_0}\sum_{n=0}^{+\infty} f_n(x) = \sum_{n=0}^{+\infty}\lim_{x\longrightarrow x_0} f_n(x) + \sum_{n=0}^{+\infty} f_n(x_0).$$

Generalization*:* If $\sum_{n=0}^{+\infty} f_n$ is uniformly convergent on I, and if the finite limit $\lim\limits_{x\longrightarrow x_0} f_n(x) = a_n(x_0 \in I)$ exists whatever n, the series $\sum_{n=0}^{+\infty} a_n$ is convergent and we have:

$$\lim_{x\longrightarrow x_0}\sum_{n=0}^{+\infty} f_n(x) = \sum_{n=0}^{+\infty}\lim_{x\longrightarrow x_0} f_n(x) + \sum_{n=0}^{+\infty} a_n(x_0).$$

Remark 5.2.5 *This result is often used to demonstrate that a given series is not uniformly convergent by demonstrating that the sum function $S_n(x)$ is discontinuous at a point.*

Example 5.2.4 *Study the uniform convergence of the series:*

$$x^2 + \frac{x^2}{1+x^2} + \frac{x^2}{(1+x^2)^2} + ... + \frac{x^2}{(1+x^2)^n} + ...$$

We have $\sum_{n=0}^{+\infty} \frac{x^2}{(1+x^2)^n}$, suppose $x \neq 0$, then the series is a geometric series of ratio $\frac{1}{1+x^2}$ and first term x^2. Then $S(x) = x^2\frac{1}{1+\frac{1}{1+x^2}} = 1+x^2$, If $x=0, S_n(0)=0$, hence $\lim\limits_{x\longrightarrow 0} S_n(0) = S(0)$, on the other hand we have $\lim\limits_{x\longrightarrow 0} S(x) = 1 \neq S(0)$ so S is discontinuous at the point $x = 0$. So the convergence is not uniform.

Integration

Theorem 5.2.4 *Let $\sum_{n=0}^{+\infty} f_n, f_n : I = [a,b] \longrightarrow \mathbb{K}$ be a series of functions uniformly convergent on $[a,b]$. If the functions f_n are integrable in $[a,b]$, then it is the same for the sum of the series and we have,*

$$\int_a^b S(x)dx = \int_a^b \left(\sum_{n=0}^{+\infty} f_n(x)\right)dx = \sum_{n=0}^{+\infty}\int_a^b f_n(x)dx.$$

Furthermore, the series $\sum_{n=0}^{+\infty}\int_a^b f_n(x)dx$ converges uniformly on $[a,b]$ towards $\int_a^x S(t)dt$.

Example 5.2.5 *We have $\frac{1}{1+x} = \sum_{n=0}^{+\infty}(-1)^n x^n$ and $\frac{1}{x^2} = \sum_{n=0}^{+\infty}(-1)^n x^{2n}$, converge uniformly on each interval $[a,b] \subset]-1,1[$, we can therefore integrate them term by term from 0 to*

x with $|x| < 1$. So,

$$\ln(x) = \sum_{n=0}^{+\infty} \frac{(-1)^n x^{n+1}}{n+1} = \sum_{n=0}^{+\infty} \frac{(-1)^n x^n}{n},$$
$$\arctan x = \sum_{n=0}^{+\infty} \frac{(-1)^n x^{2n+1}}{2n+1}.$$

Derivation

Theorem 5.2.5 *Let $\sum_{n=0}^{+\infty} f_n, f_n : I \longrightarrow \mathbb{K}$ be a series of functions whose general term f_n are continuously differentiable on $[a,b](f_n \in C^1([a,b],\mathbb{K})$. If*

(a) $\sum_{n=0}^{+\infty} f_n$ is convergent at a point $x_0 \subset [a,b]$,

(b) $\sum_{n=0}^{+\infty} \acute{f}_n(x)$ is uniformly convergent on $[a,b]$. Then, the series $\sum_{n=0}^{+\infty} \acute{f}_n(x)$ is uniformly convergent on $[a,b]$ and we have $\acute{S} = \left(\sum_{n=0}^{+\infty} f_n\right)' = \sum_{n=0}^{+\infty} \acute{f}_n(x)$.

Example 5.2.6 *Let $\sum_{n=0}^{+\infty} f_n$ such that $f_n(x) = \dfrac{x^n}{n^3(1+x^n)}, I = [0,1]$; we have $|f_n(x)| \leq \dfrac{1}{n^3}$ because $x^n \leq 1$ therefore $\sum \dfrac{1}{n^3}$ is convergent, We have normal convergence hence uniform convergence therefore simple convergence on $[0,1]$. Is the sum derivable?*

(i) The functions f_n are differentiable with continuous derivatives on $[0,1]$. Indeed; $\acute{f}_n(x) = \dfrac{x^{n-1}}{n^2(1+x^n)^2}$.

(ii) The series $\sum_{n=0}^{+\infty} f_n$ converges at least at a point of $[0,1]$.

(iii) The series of derivatives converges uniformly on $[0,1]$. Indeed; $\left|\acute{f}_n(x)\right| = \left|\dfrac{x^{n-1}}{n^2(1+x^n)^2}\right| \leq \dfrac{1}{n^2}$. Then the sum is therefore differentiable on $[0,1]$ and we have: For

$$x \in [0,1], \acute{S}(x) = \left(\sum_{n=0}^{+\infty} f_n(x)\right)' = \sum_{n=0}^{+\infty} \acute{f}_n(x).$$

Chapter 6

ENTIGER SERIES

The theory of integer series allows the majority of usual functions to be expressed as sums of series. We say that an analytic function is a series which can be expressed locally as a convergent integer series. This makes it possible to demonstrate properties of these functions, to calculate complicated sums and also to solve differential equations.

6.1 Definitions and properties

Definition 6.1.1 *An integer series is a series of functions with* $U_n(z) = a_n z^n$ *where* $a_n \in \mathbb{C}$ *and* $z \in \mathbb{C}$*.* a_n *is the coefficient of order* n*,* a_0 *the constant term. By convention, we set* $z^0 = 1 \forall z \in \mathbb{C}$*. If* $U_n(x) = a_n x^n$ *where* $a_n \in \mathbb{C}$ *and* $x \in \mathbb{R}$*, we speak of an integer series with a real variable.*

Proposition 6.1.1 *- If there exists* $R \in [0, +\infty[$ *such that* $|z| < R$*, the general term series* $U_n(z) = a_n z^n$ *converges.*

- If $|z| > R$*, the series diverges.*

- Moreover $0 \leq |r| < R$*, the series converges normally on the closed disk* $\overline{D_r} = \{z \in \mathbb{C}/|z| \leq r\}$*.*

Remark 6.1.1 *We consider the entire series of general term* $U_n(x) = a_n x^n$

- If there exists $R \in [0, +\infty[\cup\{+\infty\},)$ *that is to say that* R *can take the infinite value, such that* $x \in]-R, +R[$*, the general term series* $U_n(x) = a_n x^n$ *converges.*

- If $|x| > R$*, the series diverges roughly.*

- For normal convergence, it is enough to take $r \in [0, R[$ *and* $x \in [-r, +r]$*.*

Definition 6.1.2 *R is the radius of convergence of the series. $\overline{D_r} = \{z \in \mathbb{C}/|z| \leq r\}$ is the convergence disk. By convention, we have $D_0 = \emptyset$ and $D_{+\infty} = \mathbb{C}$. In the real case $]-R, R[$ is the convergence interval.*

Proposition 6.1.2 *- if $\lim_{n \longrightarrow +\infty} \sqrt[n]{|a_n|} = L \in [0, +\infty[$ then $R = \dfrac{1}{L}$.*
- if $\lim_{n \longrightarrow +\infty} \left|\frac{a_{n+1}}{a_n}\right| = L \in [0, +\infty[$ then $R = \dfrac{1}{L}$.

Example 6.1.1 *Consider the entiger series of general term $U_n(z) = n!z^n$, with $z \neq 0$. To find out if this series converges, we use the D'Alembert criterion. Thus, we have: $\left|\dfrac{a^{n+1}(z)}{a^n(z)}\right| = \left|\dfrac{(n+1)!z^{(n+1)}}{n!z^n}\right| = (n+1)|z|$. Or $\lim_{n \longrightarrow +\infty}(n+1)|z| = +\infty$ therefore for all $z \neq 0$ the series diverges, we say that the radius of convergence of this entire series is 0.*

Example 6.1.2 *Let the entiger series of general term $U_n(z) = \dfrac{z^n}{n^\alpha}$, with $\alpha \in \mathbb{R}$. We have $\left|\dfrac{a^{n+1}(z)}{a^n(z)}\right| = \dfrac{n^\alpha}{(n+1)^\alpha}|z| = \left(\frac{n}{n+1}\right)|z| \underset{n \longrightarrow +\infty}{\longrightarrow} |z|$. So the convergence radius is $R = 1, (|z| < 1)$.*

Example 6.1.3 *Let the entiger series of general term $U_n(z) = \dfrac{z^n}{n^{3n}}$. We have $\sqrt[n]{|U_n(z)|} = \dfrac{|z|}{n^3} \underset{n \longrightarrow +\infty}{\longrightarrow} 0$; Then the radius of convergence is $R = +\infty$ (convergence for all z). [5]*

6.2 Operations on entiger series

We consider the entire series of general term $U_n(z) = a_n z^n$ and $V_n(z) = b_n z^n$ with convergence radius R_a and R_b respectively.

- If we consider the entire series of general term $W_n(z) = c_n z^n = U_n(z) + V_n(z) = (a_n + b_n)z^n$. Then $R_c \geq \min(R_a, R_b)$, moreover if $R_a \neq R_b, R_c = \min(R_a, R_b)$.
- If we consider the entire series of general term $W_n(z) = d_n z^n = \sum_{n=0}^{+\infty} U_n(z) + V_{n-k}(z) = \sum_{n=0}^{+\infty} \left(a_k z^k\right)\left(b_{n-k} z^{n-k}\right) = z^n \sum_{n=0}^{+\infty}(a - kb_{n-k})$, then $R_c \geq \min(R_a, R_b)$

Example 6.2.1 *We consider the series $S_1 = \sum_{n=0}^{+\infty} z^n$ and $S_2 = \sum_{n=0}^{+\infty} -z^n$, We have $U_n(z) = z^n, V_n(z) = -z^n, (\forall n, a_n = 1, b_n = -1)$, then $R_a = 1$ and $R_b = 1$. So $S = \sum_{n=0}^{+\infty} c_n z^n = \sum_{n=0}^{+\infty}(a_n + b_n)z^n = \sum_{n=0}^{+\infty}(1 + (-1))z^n = 0$, has convergence radius $R_c = +\infty \geq \min(R_a, R_b)$.*

Example 6.2.2 *We consider the series $S_1 = \sum_{n=0}^{+\infty} z^n, R_a = 1$. We consider the entire series defined as follows: $b_0 = 1, b_1 = -1$, and $\forall n \geq 2, b_n = 0$, we deduce that: $S_2 = 1 - z$ which* *has convergence radius* $R_b = +\infty$. *If we calculate* $S = S_1 \times S_2$ *we obtain* $W_n(z) = d_n z^n = z^n \sum_{n=0}^{+\infty}(a_k b_{n-k})$, *we deduces that:* $d_0 = a_0 b_0 = 1, d_1 = a_0 b_1 + a_1 b_0 = -1 + 1 = 0$ *and therefore* $\forall n \geq 1, d_n = 0, R_d = +\infty$.

6.3 Derivation and integration of integer series

Theorem 6.3.1 *Let* $\sum_{n=0}^{+\infty} a_n z^n$ *be an integer series with sum* $f(z)$. *The function* $z \longrightarrow f(z)$ *is continuous in the convergence disk of the series.*

Proposition 6.3.1 *Let the entire series be defined by:* $S(x) = \sum_{n=0}^{+\infty} a_n x^n$ *such that* $x \in]-R, +R[$, *where* R *is the radius of convergence. If the function* S *is* $C^{+\infty}$ *then* $]-R, +R[$ *alors* $\acute{S}(x) = \sum_{n=1}^{+\infty} n a_n x^{n-1} = \sum_{n=0}^{+\infty} (n+1) a_{n+1} x^n$. *So, this new series also has a radius of convergence* R.

corollary 6.3.1 *Let the entire series be defined by:* $S(x) = \sum_{n=0}^{+\infty} a_n x^n$ *such that* $x \in]-R, +R[$, *where* R *is the radius of convergence. We set* $T(x) = \sum_{n=1}^{+\infty} a_{n-1} \frac{x^n}{n} = \sum_{n=0}^{+\infty} a_n \frac{x^{n+1}}{n+1}$. *The radius of convergence of this series is also equal to* R, *and* $\forall x \in]-R, +R[, \acute{T}(x) = S(x)$. [5]

Chapter 7

FOURIER SERIES

L Fourier series are series of functions of a particular type, which are used to study periodic functions. The idea is to express any 2π-periodic function as a linear combination of simple 2π-periodic functions, of the form $\cos(nx)$ or $\sin(nx)$, with $n \in \mathbb{N}$. This "linear combination" will, in general, be an infinite sum, that is to say a series:

7.1 Definitions et proprieties

Definition 7.1.1 (trigonometric series) *We call a trigonometric series a series of functions* $\sum f_n$ *whose general term is of the form* $f_n(x) = a_n \cos(nx) + b_n \sin(nx)$ *with* $x \in \mathbb{R}$ *and, for all* $n \in \mathbb{N}, a_n \in \mathbb{C}$ *and* $b_n \in \mathbb{C}$.

Proprietie 7.1.1 (Convergence 1) *If* $\sum a_n$ *and* $\sum b_n$ *converge absolutely, then the trigonometric series* $\sum (a_n \cos(nx) + b_n \sin(nx))$ *converges normally on* $\mathbb{R}$.

Proprietie 7.1.2 (Convergence 2) *If the sequences* $(a_n)_{n\in\mathbb{N}}$ *and* $(b_n)_{n\in\mathbb{N}}$ *are real, decreasing, and tend towards* 0 *then, for all* $x_0 \in \mathbb{R}/2\pi\mathbb{Z}$ *fixed* $\sum (a_n \cos(nx_0) + b_n \sin(nx_0))$ *converges. Moreover for all* $\varepsilon > 0$, $\sum (a_n \cos(nx) + b_n \sin(nx))$ *converges uniformly on each interval of the form* $[2n\pi + \varepsilon, 2(n+1)\pi\varepsilon]$ *with* $n \in \mathbb{Z}$.

The proof of this property is an application of the uniform Abel rule. We then have:

Proprietie 7.1.3 (Complex writing) *Any trigonometric series:*

$$\sum_{n\in\mathbb{N}} (a_n \cos(nx) + b_n \sin(nx))$$

can be rewritten in the form $\sum_{n\in\mathbb{Z}} c_n e^{inx}$ *with* $c_0 = a_0$ *and* $\forall n \in \mathbb{N}, c_n = \dfrac{a_n - ib_n}{2}$ *and* $c_{-n} = \dfrac{a_n + ib_n}{2}$. *Then,* $\forall n \in \mathbb{N}, a_n = c_n + c_{-n}$ *and* $bn = i(c_n - c_{-n})$.

When a trigonometric series converges uniformly on $[-\pi, \pi]$, we can find its coefficients according to its sum

Propretie 7.1.4 (Evaluation of the coefficients) *Let* $\sum (a_n \cos(nx) + b_n \sin(nx))$ *be a trigonometric series uniformly convergent on* $[-\pi, \pi]$. *Note,*

$$S(x) = \sum_{n\in\mathbb{N}} (a_n \cos(nx) + b_n \sin(nx)),$$

for $x \in \mathbb{R}$. *Then* $a_0 = \dfrac{1}{2\pi}\int_{-\pi}^{\pi} S(x)dx$ *and for all* $n \in \mathbb{N}^*$, $a_n = \dfrac{1}{\pi}\int_{-\pi}^{\pi} S(x)\cos(nx)dx$ *and* $b_n = \dfrac{1}{\pi}\int_{-\pi}^{\pi} S(x)\sin(nx)dx$.

Remark 7.1.1 *1. S is an* $\mathbb{R} \longrightarrow \mathbb{C}$ *function. We therefore have here integrals of functions* $\mathbb{R} \longrightarrow \mathbb{C}$ *to which we must give meaning. By definition, for* $f : \mathbb{R} \longrightarrow \mathbb{C}, \int_a^b f(x)dx = \int_a^b \mathcal{R}e(f(x))dx + i\int_a^b \mathcal{I}m(f(x))dx$.

2. We have no expression for b_0. *In fact, since* b_0 *is the coefficient of* $\sin(0x) = 0$, *it has no importance, we can choose for example* $b_0 = 0$.

If the trigonometric series is given by its complex writing, the expressions simplify:

Propretie 7.1.5 (Trigo-complexe serie) *Let* $\sum_{n\in\mathbb{Z}} c_n e^{inx}$ *be a trigonometric series written in complex form which converges uniformly on* $[-\pi; \pi]$. *Let us note, for all* $x \in \mathbb{R}, S(x) = \sum_{n=-\infty}^{+\infty} c_n e^{inx}$. *Then for all* $n \in \mathbb{Z}, c_n = \dfrac{1}{2\pi}\int_{-\pi}^{\pi} S(x)e^{-inx}dx$.

Remark 7.1.2 *Since* $\cos(nx)$ *and* $\sin(nx)$ *are* 2π*-periodic, so* $S(x)$ *is* 2π*-periodic. Because of this, we can change the integration interval: for all* $\alpha \in \mathbb{R}$, *for all* $n \in \mathbb{Z}, c_n = \dfrac{1}{2\pi}\int_{\alpha}^{\alpha+2\pi} S(x)e^{-inx}dx$. *The same is true for* a_n *and* b_n.

Now that we have studied trigonometric series, we can return to the initial program: given any 2π-periodic function, can we rewrite it as the sum of a trigonometric series? [5]

Definition 7.1.2 (Fourier series) *Let f is* 2π*-periodic, Its Fourier series is by definition the trigonometric series* $\sum_{n\in\mathbb{N}} (a_n \cos(nx) + b_n \sin(nx))$ *defined by* $a_0 = \dfrac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx$ *and for all*

$n \in \mathbb{N}, a_n = \dfrac{1}{\pi} \int_{-\pi}^{\pi} f(x)\cos(nx)dx$ and $b_n = \dfrac{1}{\pi} \int_{-\pi}^{\pi} f(x)\sin(nx)dx$, if these integrals are defined. Or, equivalently, it is the trigonometric series written in complex form $\sum_{n\in\mathbb{Z}} c_n e^{inx}$ where, for all $n \in \mathbb{Z}, c_n = \dfrac{1}{2\pi} \int_{-\pi}^{\pi} f(x)e^{-inx}dx$. The coefficients a_n and b_n (or, equivalently c_n) are called ***Fourier coefficients*** *of f.*

Propretie 7.1.6 (Parity) *1. Since f is 2π-periodic function, we can change the integration interval to $[\alpha, \alpha + 2\pi]$, for all $\alpha \in \mathbb{R}$.*

2. If f is even, for all $n \in \mathbb{N}, b_n = 0$.

3. If f is odd, for all $n \in \mathbb{N}, a_n = 0$.

Analogously to what happens when we develop a function in integer series, given a function f 2π-periodic whose Fourier coefficients are defined, two questions arise:

1. Does the Fourier series of f converge?
2. If yes, does it converge to f f?

Unfortunately, as with entire series, the answer may be no to each of these questions. There is a whole theory describing the convergence of the Fourier series under various assumptions about f. Among this theory, we will retain for this course the following result:

Theorem 7.1.1 (Dirichlet Jordan) *Let f be a 2π-periodic function continuous on $[-\pi, \pi]$ sexcept possibly at a finite number of points. We assume that at these points of discontinuity, f admits a finite right limit and a left limit. Finally, we suppose that f admits at every point of $[-\pi, \pi]$ a right derivative and a left derivative (finite). Then for all $x \in \mathbb{R}$, the Fourier series of f is convergent at x and has the sum $\dfrac{1}{2}\left(\lim\limits_{y \longrightarrow x^+} f(y) + \lim\limits_{y \longrightarrow x^-} f(y)\right)$. In particular, at any point x where f is continuous, the sum of its Fourier series is $f(x)$.*

It is convenient to reinterpret the theory of Fourier series using the notions of vector space and dot product. We can then retain certain aspects of the Fourier series by keeping in mind the analogy with the simple vector space that is $\mathbb{R}^2$, which is equipped with the scalar product $\vec{x}.\vec{y} = x_1y_1 + x_2y_2$. This analogy is written in a more natural way when we use the complex writing of Fourier series. The space which, for Fourier series, plays the role of the vector space $\mathbb{R}^2$ is the set of periodic functions $\mathcal{F} = \{f : R \longrightarrow \mathbb{C}; 2\pi - \text{and whose squareis integrable on } [-\pi, \pi]\}$.

We can define a product on $\mathcal{F}$ (a function $\mathcal{F} \times \mathcal{F} \longrightarrow \mathcal{C}$) which will play the role of the scalar product of $\mathbb{R}^2$:

Definition 7.1.3 (Scalar product) *For $f, g \in \mathcal{F}$, we call the scalar product of f and g, and we note (f, g) the complex number $(f, g) = \dfrac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\bar{g}(x)dx$ where $\bar{g}(x)$ denotes the conjugate complex number of $g(x)$.*

When we have a scalar product, we can define a norm:

Definition 7.1.4 (Norm) *Let $f \in \mathcal{F}$. We call the norm of f and we note $\|f\|$ the positive real number $\|f\| = \sqrt{(f, f)}$.*

Remark 7.1.3 *The norm of $\mathbb{R}^2$ is constructed this way from the scalar product: $\|\vec{x}\| = \sqrt{\vec{x}.\vec{x}} = \sqrt{x_1^2 + x_2^2}$.*

Propretie 7.1.7 (Orthonormal basis) *The (infinite) set of functions $\{x \mapsto e^{inx}, n \in \mathbb{Z}\}$ forms an orthonormal basis (infinite) of $\mathcal{F}$ provided with the scalar product. Indeed we have already seen that for all $n_0 \in \mathbb{Z}$,*

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{inx} e^{-in_0 x} dx = \begin{cases} 1 \text{ if } n = n_0 \\ 0 \text{ otherwise} \end{cases},$$

which translates to:

$$\left(e^{in_0 x}, e^{inx}\right) = \begin{cases} 1 \text{ if } n = n_0 \\ 0 \text{ otherwise} \end{cases},$$

which is the definition of an orthonormal family. The fact that this family contains enough elements to be considered a base requires further development:

The difference between $\mathbb{R}^2$ and $\mathcal{F}$ is that an orthonormal basis of $\mathbb{R}^2$ contains only 2 elements while an orthonormal basis of F contains infinitely many elements. We say that $\mathcal{F}$ is of infinite dimension. By analogy with $\mathbb{R}^2$, we say that we have decomposed $f \in \mathcal{F}$ according to the orthonormal basis $\{x \mapsto e^{inx}, n \in \mathbb{Z}\}$ if we found coefficients $c_n \in \mathbb{Z}$ such that $\lim_{N \longrightarrow +\infty} \left\| f(x) - \sum_{n=-N}^{+N} c_n e^{inx} \right\| = 0$. The previous proposition asserts that this decomposition is possible for all $f \in \mathcal{F}$. Then we get the following interpretation. [5]

7.2 Geometric interpretation of Fourier series

Let $f \in \mathcal{F}$. Its Fourier series is nothing other than its decomposition according to the orthonormal basis $\{x \mapsto e^{inx}, n \in \mathbb{Z}\}$. This interpretation allows us to retain the expression of the Fourier coefficients of f:

Propretie 7.2.1 (Orthogonal projection) *Let $f \in \mathcal{F}$. For all $n \in \mathbb{Z}$ its Fourier coefficient c_n is the orthogonal projection of f on e^{inx}, i.e. $c_n = (f(x), e^{inx}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)e^{-inx}dx$.*

Finally, this interpretation makes it possible to connect the norm of f with its Fourier coefficients:

Theorem 7.2.1 (Parseval-Bessel) *Let $f \in \mathcal{F}$ and $\{c_n, n \in \mathbb{Z}\}$ be its Fourier coefficients in complex writing, $\{(a_n, b_n), n \in \mathbb{N}\}$ be its Fourier coefficients in real writing. Then the norm of f verifies:*

1. ***Bessel inegality****: for all $N \in \mathbb{N}$,*

$$\begin{aligned}\|f\|^2 = (f, f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\bar{f}(x)dx &\geq \sum_{n=-N}^{N} |c_n|^2 \\ &= |a_0|^2 + \frac{1}{2}\sum_{n=1}^{N}(|a_n|^2 + |b_n|^2).\end{aligned}$$

2. ***Parseval egality:***

$$\|f\|^2 = (f, f) = \sum_{n=-\infty}^{+\infty} |c_n|^2 = |a_0|^2 + \frac{1}{2}\sum_{n=1}^{+\infty}(|a_n|^2 + |b_n|^2).$$

Chapter 8

LAPLACE TRANSFORMATION

The Laplace transformation is, along with the Fourier transformation, one of the most important integral transformations. It intervenes in many questions of mathematical physics, probability calculation, automation, etc., but it also plays a major role in classical analysis. It very legitimately bears the name of Pierre-Simon Laplace (1749-1827).

8.1 Definition, convergence abscissa

Definition 8.1.1 *Let $f : [0, +\infty[$ or $]0, +\infty[\longrightarrow \mathbb{R}$ or $\mathbb{C}$ be a piecewise continuous function on any segment. We call the Laplace transform of f the function of a real or complex variable:*

$$F(p) = \mathcal{L}(p) = \int_0^{+\infty} e^{-pt} f(t)dt.$$

Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ or $\mathbb{C}$ be a piecewise continuous function on any segment. We call the Laplace transform of f the function of a real or complex variable:

$$F(p) = (p) = \int_{-\infty}^{+\infty} e^{-pt} f(t)H(t)dt = \int_0^{+\infty} e^{-pt} f(t)dt.$$

where $H(t)$ is the Heaviside function defined by $H(t) = 0$ for $t < 0, 1$ for $t > 0$.

The function $f(t)$ is called original, object function, or causal function. The function $F(p)$ is called the image of $f(t)$. We note $f(t)]F(p)$ for this correspondence.

The following problems naturally arise:

- At what points is the function F defined?

- What are its properties within its domain of definition?
- What are its properties at the edge of this domain?
- What are the algebraic properties, differential and integral, of the Laplace transformation $\mathcal{L} : f \to F$?
- Can we go back from F to f? That is, is there an inverse Laplace transform?

Let us denote by $D(f)$ the set of complexes $p = a + ib$ such that the function $t \longrightarrow e^{-pt}f(t)$ is integrable on $]0, +\infty[$, that is to say $\int_0^{+\infty} e^{-pt}f(t)dt$ is absolutely convergent. $D(f)$ is called the domain of absolute convergence of the Laplace transform. Like $|e^{-pt}f(t)| = e^{-at}|f(t)|, p \in D(f) \iff a = \mathcal{R}e(p) \in D(f)$. Moreover, if $p \in D(f)$, then for all $\acute{a} > a$, $e^{-\acute{a}t}f(t)$ is integrable. We deduce from this that the set $D(f)$ is of one of the following four forms: [8]

$$\varnothing\ , \mathbb{C}, \{p; \mathcal{R}e\,(p) \in]A, +\infty[\} \text{ or } \{p; \mathcal{R}e\,(p) \in [A, +\infty[\}.$$

The real $A = a(f)$ is called the abscissa of absolute convergence of the Laplace transform. We agree that $A = +\infty$ if $D(f) = \varnothing$, $A = -\infty$ if $D(f) = \mathbb{C}$.

Example 8.1.1 *1) If $f(t) = \exp(t^2), D(f) = \varnothing$, because $t \longrightarrow e^{-pt}e^{t^2}$ is never integrable.*

2) If $f(t) = 0$ or if $f(t) = \exp(-t^2)$, $D(f) = \mathbb{C}$, because $t \longrightarrow e^{-pt}f(t)$ is always integrable.

3) If $f(t) = 1$ or $H(t), D(f) = \{p; \mathcal{R}e\,(p) > 0\}$ and $\mathcal{L}(1)(p) = \mathcal{L}(H)(p) = \int_0^{+\infty} e^{-pt}dt = \dfrac{1}{p}$.

4) If $f(t) = e^{at}$ or $e^{at}H(t), D(f) = \{p; \mathcal{R}e\,(p) > a\}$ and $\mathcal{L}(e^{at})(p) = \mathcal{L}(e^{at}H(t))(p) = \int_0^{+\infty} e^{(a-p)t}dt = \dfrac{1}{p-a}$.

5) If $f(t) = \dfrac{1}{1+t^2}, D(f) = \{p; \mathcal{R}e\,(p) \geq 0\}$.

6) If $f(t) = \dfrac{1}{\sqrt{t}}, D(f) = \{p; \mathcal{R}e\,(p) > 0\}$.

The following proposition gives a sufficient condition for a function f to have a Laplace transform:

Proposition 8.1.1 *Let $f :]0, +\infty[\longrightarrow \mathbb{R}$ or $\mathbb{C}$ continue piecewise on any segment. If the integral $\int_0^1 |f(t)|\,dt$ converges, and if $\exists(M, \gamma, A)\forall t \geq A|f(t)| \leq Me^{\gamma t}, D(f)$ is non-empty. The function f is said to be of exponential order if it satisfies this last condition.*

8.2 General properties

In the following, we freely use the abusive notation $F(p) = \mathcal{L}(f(t))(p)$ for $f(t)]F(p)$. The variable p is assumed to be real.

Proposition 8.2.1 (linearity) *If $D(f)$ and $D(g)$ are non-empty„ $D(\alpha f + \beta g)$ is non-empty and, on $D(f) \cap D(g)$:*

$$\mathcal{L}(\alpha f + \beta g)(p) = \alpha\mathcal{L}(f)(p) + \beta\mathcal{L}(g)(p).$$

Proposition 8.2.2 (translation) *If $D(f)$ is non-empty, for all α, $D(e^{-at}f(t))$ is non-empty and $\mathcal{L}(e^{-at}f(t))(p) = ()\,(p+\alpha)$.*

Proof. $\mathcal{L}(e^{-\alpha t}f(t))(p) = \int_0^{+\infty} e^{-pt}e^{-\alpha t}f(t)dt = \int_0^{+\infty} e^{-(p+\alpha)t}f(t)dt = (())(p+\alpha)$. ■

Proposition 8.2.3 (delay) *If $D(f)$ is non-empty, $a > 0, g(t) = f(t-a)$ for $t > a$ for $t < a$, and $\mathcal{L}(f(t-a))(p) = e^{-ap}\,()\,(p)$.*

Proof. $\mathcal{L}(g)(p) = \int_0^{+\infty} e^{-pt}g(t)dt = \int_0^{a} e^{-pt}g\,(t)\,dt + \int_a^{+\infty} e^{-pt}g\,(t)\,dt = \int_a^{+\infty} e^{-pt}f(t-a)dt = \int_0^{+\infty} e^{-p(u+a)}f(u)du = e^{-ap}\,()\,(p)$. ■

Proposition 8.2.4 (change of scale) *Si $D(f)$ is non-empty, $D(f(at))$ is non-empty for all $a > 0$, and $\mathcal{L}(f(at))(p) = \frac{1}{a}\,()\,(\frac{p}{a})$.*

Proof. $\mathcal{L}(f(at))(p) = \int_0^{+\infty} e^{-pt}f(at)dt = \frac{1}{a}\int_0^{+\infty} e^{\frac{pu}{a}}f(u)du = \frac{1}{a}(\frac{p}{a})$. ■

Proposition 8.2.5 (derived from the image) *If $D(f)$ is non-empty, the function $= F$ is of class $\mathcal{C}^{\infty}$ on the interval $]a(f), +\infty[$, and $\mathcal{L}(t^n f(t))(p) = (-1)^n F^{(n)}(p)$.*

Proof. Here, the variable p is assumed to be real. Let $p > a(f)$. Let us choose b such that $a(f) < b < p$. The function $e^{-bt}f(t)$ is integrable on $]0, +\infty[$. As $t^n e^{-pt}\,|f(t)| = O(e^{-bt}f(t))$ at $V(+\infty)$, each of the functions $t^n e^{-pt}f(t)$ is integrable. The parameter integral differentiation theorem applies:

- Each function $t \longrightarrow t^n e^{-pt}f(t)$ is piecewise continuous and integrable;

- Each function $p \longrightarrow t^n e^{-pt}f(t)$ is continuous;

- For $p \geq b > a(f), t^n e^{-pt}f(t) \leq Me^{-bt}\,|f(t)|$, integrable upper bound. [6] ■

corollary 8.2.1 *If $f(t)$ has positive real values, $F(p)$ is positive, decreasing, convex, and completely monotonic, in the sense that its n^{th} derivative has the sign of $(-1)^n$.*

Proposition 8.2.6 (Image de la dérivée) *If f is $\mathcal{C}^1$ over $\mathbb{R}_+$, then $\mathcal{L}(\acute{f})(p) = pF(p) - f(0)$.*

If f is $\mathcal{C}^2$ over $\mathbb{R}_+$, then $\mathcal{L}(\acute{\acute{f}})(p) = p^2F(p) - pf(0) - \acute{f}(0)$.

If f is $\mathcal{C}^n$ over $\mathbb{R}_+$, then $\mathcal{L}(f^{(n)})(p) = p^nF(p) - (p^{n-1}f(0) + p^{n-2}\acute{f}(0) + \ldots + pf^{(n-2)}(0) + f^{(n-1)}(0))$.

Proof. Just integrated by parts. ■

Proposition 8.2.7 (Image of the integral) *If $D(f)$ is non-empty and if f is piecewise continuous $\mathcal{L}\left(\int_0^t f(u)\,du\right)(p) = \dfrac{F(p)}{p}$.*

Proposition 8.2.8 (Convolution) *Let f and g be two continuous functions $[0, +\infty[\longrightarrow \mathbb{C}$, exponential order, their convolution product $f*g$, defined by $\forall x \geq 0, (f*g)(x) = \int_0^x f(x-t)\,g(t)\,dt$ is continuous, of exponential order, and $\mathcal{L}(f*g)(x)(p) = \mathcal{L}(f)(p).\mathcal{L}(g)(p)$.*

Proof. The proof scheme, based on double integrals, is as follows:

$$\begin{aligned}
\mathcal{L}(f*g)(x)(p) &= \int_0^{+\infty} e^{px}(f*g)(x)dx \\
&= \int_0^{+\infty} e^{px}(\int_0^x f(x-t)g(t)dt)dx = \iint_\Delta f(x-t)g(t)e^{-px}dtdx \\
&= \iint_\Delta f(x-t)g(t)e^{-p(x-t)}e^{-pt}dtdx = \iint_\Delta f(x-t)g(t)e^{-p(x-t)}e^{-pt}dxdt \\
&= \int_0^{+\infty}\left(\int_t^{+\infty} f(x-t)\,g(t)\,e^{-p(x-t)}e^{-pt}dx\right)dt \\
&= \int_0^{+\infty}\left(\int_t^{+\infty} f(x-t)\,e^{-p(x-t)}dx\right)g(t)\,e^{-pt}dt \\
&= \int_0^{+\infty}\left(\int_0^{+\infty} f(u)\,e^{-pu}du\right)g(t)\,e^{-pt}dt = \int_0^{+\infty} F(p)g(t)e^{-pt}dt \\
&= F(p)G(p) = \mathcal{L}(f)(p)\mathcal{L}(g)(p).
\end{aligned}$$

■

8.3 Table of usual Laplace transforms

Just as there are tables of usual primitives, tables of usual limited expansions, there exist tables of Fourier transforms and tables of Laplace transforms of usual functions. In the table below, it is strictly necessary to indicate the convergence abscissa. From this table and the calculation rules above, we deduce that the Laplace transformation induces an isomorphism of the vector space of exponential-polynomials, that is to say the linear combinations of the functions $t^n e^{at}$ (a réel ou complexe), on the vector space of rational fractions of degree < 0. (See appendices).

8.4 Inverse Laplace transform

If $f(t)$ has Laplace transform $F(p), F =$, we symbolically write $f = \mathcal{L}^{-1}F$ and we say that f is a Laplace transform inverse of F.

Warning: the Laplace transformation is not injective!

- On the one hand, only the values taken by $f(t)$ on $t > 0.$ come into play. The functions 1 and $H(t)$ even have a Laplace transform.
- On the other hand, two functions which differ on $\mathbb{R}_+^*$ can have the same Laplace image. A zero function almost everywhere has a zero Laplace transform.

The functions $f(t) = e^{-2t}$ and $g(t) = 0$ for $t = 5$, e^{-2t} for $t \neq 5$, even have a Laplace transform: $()(p) = ()(p) = \dfrac{1}{p+2}$.

However, the Laplace transformation is injective if we restrict it to certain classes of functions: exponential-polynomials, Lerch's theorem...

8.5 Introduction to symbolic calculus

Symbolic calculus, or operational calculus, was invented by Heaviside to solve in particular linear differential equations and systems, but also certain integral equations. It bridges the gap between analysis and algebra. We will develop it using a few examples.

Example 8.5.1 (Solve the differential equation) $\H{y}+3\acute{y}+2y = t, y(0) = \acute{y}(0) = 0$. *It is a linear differential equation with constant coefficients. Let us denote* $F(p) = (Lf)(p)$ *as the Laplace transform of* $y(t)$*.*

$$L(\H{y}+3\acute{y}+2y)(p) = \mathcal{L}(t)(p)$$

$$p(pF(p) - y(0)) - \acute{y}(0) + 3p(F(p) - y(0)) + 2F(p) = \frac{1}{p^2}$$

$$(p^2+3p+2)F(p)-4py(0)-\acute{y}(0)=\frac{1}{p^2}$$

$$F(p)=\frac{1}{p^2\left(p^2+3p+2\right)}=\frac{1}{p^2\left(p+1\right)\left(p+2\right)}=\frac{1}{2}\frac{1}{p^2}-\frac{3}{4}\frac{1}{p}+\frac{1}{p+1}-\frac{1}{4}\frac{1}{p+2}.$$

The decomposition into simple elements of the fraction allows us to go back to the causal function. $F(p)$ is Laplace transform of:

$$y(t)=\frac{1}{2}t-\frac{3}{4}+e^{-t}-\frac{1}{4}e^{-2t}.$$

This method provides the correct result, but it poses problems of rigor. [7]

1^{st} problem: does the solution $y(t)$ have a Laplace transform? It would be necessary to show that the solutions of linear differential equations with constant coefficients and with an exponential-polynomial second member are all dominated by $O(e^{Mt})$ for a suitable M. This is indeed the case.

2^{nd} problem: a uniqueness argument is missing to go back from $F(p)$ to the source $y(t)$. It would be necessary to demonstrate that the Laplace transformation $y(t)\longrightarrow F(p)$ is injective on a sufficiently large class of functions (exponential-polynomials in particular).

Example 8.5.2 (Find the continuous function f from $\mathbb{R}$ in $\mathbb{R}$) *checking:*

$$\forall x\in\mathbb{R}\qquad f(x)=x^2+\int_0^x\sin(x-t)f(t)dt. \tag{8.1}$$

It is a functional convolution equation, which is written:: $f(x)=x^2+(\sin*f)(x)$. Let us denote $F(p)=()(p)$ as the Laplace transform of $f(x)$. It comes $F(p)=\frac{2}{p^3}+\frac{F\left(p\right)}{p^2+1}$, so $F(p)=\frac{2}{p^3}+\frac{2}{p^5}.F(p)$ is the Laplace transform of $f(x)=x^2+\frac{1}{12}x^4$. The converse is easy.

NB : We could give a more rigorous and more basic direct solution. Indeed, (8.1) is written: $\forall x\in\mathbb{R}\qquad f(x)=x^2+\sin\left(x\right)\int_0^x\cos\left(t\right)f(t)dt-\cos\left(x\right)\int_0^x\sin\left(t\right)f(t)dt$. We deduce that f is $\mathcal{C}^1$ and, step by step, $\mathcal{C}^{+\infty}$. If we differentiate it twice, we come across a differential equation...

Chapter 9

FOURIER TRANSFORMATION

9.1 Definitions

Let $f : \mathbb{R}^d \longrightarrow \mathbb{C}$ be a piecewise continuous function (or more generally locally integrable in the Riemann sense). We will say that f belongs to the space $L^1(\mathbb{R}^d)$, if:

$$\int_{\mathbb{R}^d} |f(x)|\, dx < \infty,$$

that is to say if the integral above is convergent. Likewise we will say that f belongs to the space $L^2(\mathbb{R}^d)$ if:

$$\int_{\mathbb{R}^d} |f(x)|^2\, dx < \infty,$$

We notice

$$\|f\|_1 := \int_{\mathbb{R}^d} |f(x)|\, dx, \qquad \text{for } f \in L^1(\mathbb{R}^d),$$

$$\|f\|_2 := \left(\int_{\mathbb{R}^d} |f(x)|^2\, dx\right)^{\frac{1}{2}}, \qquad \text{for } f \in L^2(\mathbb{R}^d)$$

The quantities $\|f\|_1$ and $\|f\|_1$ are norms, that is to say that: $\|f+g\|_i \leq \|f\|_i + \|g\|_i$, $\|\lambda f\|_i = |\lambda|\, \|f\|_i$ and $\|f\|_i = 0 \implies f = 0$. For $f \in L^1(\mathbb{R}^d)$, we set:

$$\hat{f}(k) := \int_{\mathbb{R}^d} e^{-ikx} f(x) dx, \qquad k \in \mathbb{R}^d.$$

where $k.x = \sum_{i=1}^{d} k_i x_i$. The function $\hat{f}$ is called the **Fourier transform** of the function f. We also write:

$$\hat{f} = \mathcal{F}f \qquad \text{or} \qquad \mathcal{F}(f),$$

where the transformation:

$$\mathcal{F} : f \mapsto \hat{f}$$

is called the Fourier transform. It is therefore an operator which transforms functions of the variable x into functions of the variable k. If the variable x represents a position (its dimension is therefore m), the variable k represents an impulse, its dimension is m^{-1}. In signal processing, we have $d = 1$, the variable x is denoted t and has the dimension of a time (s), the variable k is denoted and has the dimension of a frequency (s^{-1}). [3]

9.2 Properties

We use the following notations: the symbol ∂_{x_j}, designates the derivation operator with respect to x_j:

$$\partial_{x_j} f(x) := \frac{\partial}{\partial_{x_j}} f(x).$$

The symbol x_j denotes the multiplication operator by x_j:

$$x_j f(x) := x_j f(x).$$

We use the same conventions for the symbols ∂_{k_j} and k_j, which act on functions of the variable k.

Proposition 9.2.1 *(1) if $f \in L^1(\mathbb{R}^n)$, $\mathcal{F}f$ is a continuous and bounded function on $\mathbb{R}^d$,*
(2) si $f \in L^1(\mathbb{R}^d)$ and $x_j f \in L^1(\mathbb{R}^d)$, $\mathcal{F}f$ is a function of class $\mathcal{C}^1$ and:

$$\partial_{k_j} \mathcal{F}f(k) = -iF(x_j f)(k);$$

(3) if $f \in L^1(\mathbb{R}^d)$ and $\partial_{x_j} f \in L^1(\mathbb{R}^d)$, then $k_j \mathcal{F}f$ is bounded and:

$$k_j \mathcal{F}(f)(k) = -i\mathcal{F}(\partial_{x_j} f)(k).$$

The thing to remember is that the Fourier transform $\mathcal{F}$ transforms the momentum operator $D_j = i^{-1}\partial_{x_j}$; in the multiplication operator k_j :

$$\mathcal{F}(D_j f) = k_j \mathcal{F}(f).$$

Proposition 9.2.2 (Link to convolution) *Let $f, g \in L^1(\mathbb{R}^d)$. We set:*

$$h(x) := f * g(x) = \int_{\mathbb{R}^d} f(x-y)g(y)dy.$$

The function $f * g$ is called the convolution product of f and g. We have:

(1) $f * g = g * f$,

(2) $f * g \in L^1(\mathbb{R}^d)$ and $\|f * g\|_1 \leq \|f\|_1 \|g\|_1$;

(3) $\mathcal{F}(f * g) = \mathcal{F}(f)\mathcal{F}(g)$.

In other words, the Fourier transform transforms the convolution product into the ordinary product of functions.

9.3 Table of usual Fourier transforms

We now give some Fourier transforms of usual functions:

1. We start with the case $d = 1$.

$$f(x) = \mathbb{I}_{[-a,a](x)}, \qquad \mathcal{F}f(k) = \begin{cases} 2\dfrac{\sin(ak)}{a} & k \neq 0 \\ 0 & k = 0 \end{cases}.$$

We recall that $\mathbb{I}_I(x)$ designates the indicator function of the set I, equal to 1 if $x \in I$ and to 0 otherwise.

$$f(x) = e^{-a|x|}, a > 0, \qquad \mathcal{F}f(k) = 2\frac{a}{a^2 + k^2}.$$

$$f(x) = e^{\frac{-ax^2}{2}}, a > 0, \qquad \mathcal{F}f(k) = (\frac{2\pi}{a})^{\frac{1}{2}} e^{-a^{-1}k^2/2}. \tag{9.1}$$

(The Fourier transform of a Gaussian is a Gaussian).

2. In any dimension d, the last formula generalizes:

$$f(x) = e^{-\sum_1^d a_i x_i^2/2}, \qquad \mathcal{F}f(k) = \prod_1^d (\frac{2\pi}{a_i})^{\frac{1}{2}} e^{-\sum_1^d a_i^{-1} k_i^2/2}, \qquad \text{for} \qquad a_i > 0.$$

3. For more details see the annexes.

9.4 Inverse Fourier Transform

Proposition 9.4.1 *Let $f \in L^1(\mathbb{R}^d)$ be a function such that $\hat{f} \in L^1(\mathbb{R}^d)$. So we have:*

$$f(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{ik.x} \hat{f}(k) dk.$$

We can rewrite this result as:

$$\mathcal{F}^{-1} g(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{ik.x} g(k) dk.$$

where $\mathcal{F}^{-1}$ denotes the inverse Fourier transform, which transforms functions of the variable k into functions of the variable x.

9.5 Application of Fourier transform to solve differential equations

A powerful application of Fourier methods is in the solution of differential equations. This is because of the following identity for the FT of a derivative:

$$FT\left[f^{(p)}(x)\right] = FT\left[\frac{d^p f}{dx^p}\right] = (ik)^p \tilde{f}(k)$$

Thus applying a FT to terms involving derivatives replaces the differential equation with an algebraic equation for $\tilde{f}$, which may be easier to solve. Let's remind ourselves of the origin of this fundamental result. The simplest approach is to write a function $f(x)$ as a Fourier integral: $f(x) = \int \tilde{f}(k) exp(ikx) dk/2\pi$. Differentiation with respect to x can be taken inside the integral, so that $df/dx = \int \tilde{f}(k) exp(ikx) dk/2\pi$. From this we can immediately recognise ik $\tilde{f}(k)$ as the FT of df/dx. The same argument can be made with a Fourier series.

Fourier Transforms can also be applied to the solution of differential equations. To introduce this idea, we will run through an Ordinary Differential Equation (ODE) and look at how we can use the Fourier Transform to solve a differential equation.

Consider the ODE in Equation:

$$\frac{d^2y(t)}{dt^2} - y(t) = -g(t) \tag{9.2}$$

We are looking for the function $y(t)$ that satisfies Equation 9.2 above. We know that we can take the Fourier Transform of a function, so why not take the fourier transform of an equation? It turns out there is no reason we can't. And since the Fourier Transform is a linear operation, the time domain will produce an equation where each term corresponds to the a term in the frequency domain. Taking the Fourier Transform of Equation 9.2 , we get Equation 9.3:

$$F\left(\frac{d^2y(t)}{dt^2}\right) - F\left(y(t)\right) = F\left(-g(t)\right) \Longleftrightarrow F\left(\frac{d^2y(t)}{dt^2}\right) - Y(f) = -G(f) \tag{9.3}$$

Hence, Equation 9.3 becomes:

$$(2\pi i f)^2 Y(f) - Y(f) = -G(f) \tag{9.4}$$

Equation 9.4 is a simple algebraic equation for $Y(f)$! This can be easily solved. This is the utility of Fourier Transforms applied to Differential Equations: They can convert differential equations into algebraic equations. Equation 9.4 can be easiliy solved for $Y(f)$:

$$Y(f) = \frac{-G(f)}{(2\pi i f)^2 - 1} = \frac{G(f)}{1 + 4\pi^2 f^2} \tag{9.5}$$

In general, the solution is the inverse Fourier Transform of the result in Equation 9.5. For this case though, we can take the solution farther. Recall that the multiplication of two functions in the time domain produces a convolution in the Fourier domain, and correspondingly, the multiplication of two functions in the Fourier (frequency) domain will give the convolution in the time domain. Hence, Equation 9.5 becomes:

$$\begin{aligned} y(t) = F^{-1}\left(Y(f)\right) &= F^{-1}\left(\frac{1}{1+4\pi^2 f^2}.G(f)\right) \\ &= F^{-1}\left(\frac{1}{1+4\pi^2 f^2}\right) * F^{-1}\left(G(f)\right) \end{aligned} \tag{9.6}$$

Equation 9.6 might not look helpful, but note that we already know the inverse Fourier Transform for the left-most inverse Fourier transform in the second line of 9.6: it's one half of the two-sided decaying exponential function. Hence, we can start to simplify equation 9.6:

$$\begin{aligned} y(t) = F^{-1}\left(\frac{1}{1+4\pi^2 f^2}\right) * F^{-1}\left(G(f)\right) &= \frac{e^{-|t|}}{2} * g(t) \\ = \frac{1}{2}\int_{-\infty}^{+\infty} e^{-|t-\tau|} g(\tau) d\tau \end{aligned}$$

Now for the fine print. When we went from Step 1 to Step 2, we assumed the Fourier Transform for $y(t)$ existed. This is a non-trivial assumption. You may recall from your differential equations class that the solution should also contain the so-called homogeneous solution, when $g(t) = 0$:

$$\frac{d^2 y_h(t)}{dt^2} - y_h(t) = 0 \Longrightarrow y_h(t) = c_1 e^t + c_2 e^{-t} \tag{9.7}$$

The "total" solution is the sum of the solution we obtained in equation 9.6 and the homogeneous solution y_h of equation 9.7. So why does the homogeneous solution not come out of our method? The answer is simple: the non-decaying exponentials of equation 9.7 do not have Fourier Transforms. That is, if you try to take the Fourier Transform of $exp(t)$ or $exp(-t)$, you will find the integral diverges, and hence there is no Fourier Transform. This is a very important caveat to keep in mind. [5]

Chapter 10

Dirigated Works

10.1 Dirigated Work $N°1$: (SINGLE, DOUBLE AND TRIPLE INTEGRALS)

Exercise 10.1.1 *Calculate the following integrals:*

$$\int \frac{dx}{1-x}, \qquad \int \frac{dx}{x^2-3x-4}, \qquad \int \sin^2 x \cos x dx, \qquad \int \sin^2 x dx.$$

1. *Calculate the following integrals (Using a primitive):*

$$\int_{-1}^{2} x^2 dx, \qquad \int_{0}^{2} x\left(x^3+1\right) dx, \qquad \int_{0}^{1} \frac{e^x}{1+e^{2x}} dx.$$

2. *Calculate the following integrals (Using integration by parts):*

$$\int_{1}^{e} x^2 \ln x dx, \qquad \int_{0}^{\frac{\pi}{2}} x \cos x dx, \qquad \int_{0}^{1} x e^{3x} dx.$$

3. *Calculate the following integrals (Using variable change):*

$$\int_{e}^{e^3} \frac{dx}{x \ln x} \quad (x>0), \qquad \int_{0}^{1} x^2 \sqrt{a^2-x^2} dx \quad (a>0), \qquad \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin(2x-\frac{\pi}{6}) dx.$$

Exercise 10.1.2 *Calculate the following integrals:*

$$\int_3^4\int_1^2 \frac{dxdy}{(x+y)^2},\qquad \int_1^2\int_x^{x\sqrt{3}} xydxdy,\qquad \int_0^{2\pi}\int_{2\sin\theta}^{2} rdrd\theta,\qquad \int_0^1\int_{y-1}^{2y} xydxdy.$$

Exercise 10.1.3 *Define the integration limits for* $\iint_D f(x,y)dxdy, D$ *being delimited by:*

a) $x=2, x=3, y=-1, y=5$ *b)* $y=0, y=1-x^2$

c) $x^2+y^2=4$ *d)* $y=\frac{2}{1+x^2}, y=x^2$

Exercise 10.1.4 *Calculate the following integrals:*

a) $\iint\limits_D |x+y|\,dxdy,$ *où* $D=\left\{(x,y)\in\mathbb{R}^2/\,|x|<1,|y|<1\right\}.$

b) $\iint\limits_D \frac{1}{1+x^2+y^2}dxdy,$ *où* $D=\left\{(x,y)\in\mathbb{R}^2/x^2+y^2<1\right\}.$

c) $\iint\limits_D \frac{xy}{x^2+y^2}dxdy,$ *où* $D=\left\{(x,y)\in\mathbb{R}^2/x>0,y>0,x+y<1\right\}.$

d) $\iint\limits_D \sqrt{x^2+y^2}dxdy,$ *où* $D=\left\{(x,y)\in\mathbb{R}^2/0<y<x<1\right\}.$

Exercise 10.1.5 *Calculate the area of the figure bounded by the curves:*

a) $D=\left\{(x,y)\in\mathbb{R}^2/y^2=2x, y=x\right\};$ *c)* $D=\left\{(x,y)\in\mathbb{R}^2/y^2=4x, x+y=3, y\geq 0\right\}$

b) $D=\left\{(x,y)\in\mathbb{R}^2/y=\sin x, y=\cos x, x=0\right\};$ *d)* $D=\left\{(x,y)\in\mathbb{R}^2/y^2=4x+4, y^2=-4x+4\right\}$

Exercise 10.1.6 *Calculate the volume bounded by the surfaces:*

a) $V=\left\{(x,y,z)\in\mathbb{R}^3/\frac{x^2}{9}+\frac{y^2}{16}+\frac{z^2}{25}=1\right\}$ *b)* $V=\left\{(x,y,z)\in\mathbb{R}^3/x^2+z^2=R^2, y^2+z^2=R^2\right\}$ *c) V*

Exercise 10.1.7 *Calculate the following integrals:*

$\iiint\limits_V zdxdydz$ *où* $V=\left\{(x,y,z)\in\mathbb{R}^3/x\geq 0, y\geq 0, z\geq 0, z\leq 1-y^2 \text{ et } x+y\leq 1\right\}$

$\iiint\limits_V xyzdxdydz$ *où* $V=\left\{(x,y,z)\in\mathbb{R}^3/0<z<1, x^2+y^2<z^2\right\}$

$\iiint\limits_V \left(\frac{1}{\sqrt{x^2+y^2}}+\frac{1}{z}\right)dxdydz$ *où* $V=\left\{(x,y,z)\in\mathbb{R}^3/0<x^2+y^2+z^2<1, 0<x^2+y^2<z^2, z>0\right\}$

10.2 Dirigated Work $N°$2: (IMPROPER INTEGRALS)

Exercise 10.2.1 *Calculate the following integrals:*

$$\int_0^{+\infty} \frac{1}{x^2+4x+9}dx, \quad \int_0^1 \ln x dx, \quad \int_1^{+\infty} \frac{\ln x}{x^2}dx, \quad \int_0^{+\infty} e^{-2x}\sin x dx, \quad \int_0^1 \frac{\ln x}{\sqrt{1-x}}dx,$$

$$\int_0^{+\infty} \frac{\ln x}{\sqrt{x}\,(1-x)^{3/2}}dx, \quad \int_0^{+\infty} \frac{x\ln x}{(1+x^2)^2}dx, \quad \int_0^{+\infty} \frac{\arctan x}{(1+x^2)^{3/2}}dx, \quad \int_0^{a} \frac{x^2}{\sqrt{a^2-x^2}}dx,$$

Exercise 10.2.2 *Study the nature of the following integrals:*

$$\int_1^{+\infty} \frac{dx}{x^\alpha}, \quad \int_0^1 \frac{dx}{x^\alpha}, \quad \int_1^{+\infty} \frac{\sqrt{x}}{(1+x)^\alpha}dx, \quad \int_0^{\pi} \frac{dx}{(1-\cos x)^\alpha},$$

$$\int_1^{+\infty} \frac{\ln x}{x+e^{-x}}dx, \quad \int_0^{+\infty} \frac{e^{\sin x}}{\sqrt{x}}dx. \quad \int_0^{+\infty} \frac{\arctan x}{x^\alpha}dx.$$

Exercise 10.2.3 *Using the variable change, calculate the following integrals:*

$$\int_0^{\frac{\pi}{2}} \sqrt{\tan x}dx, \quad \int_0^{+\infty} \cos\left(e^x\right)dx, \quad \int_0^{+\infty} \sin\left(x^2\right)dx,$$

Exercise 10.2.4 *Study the absolute convergence and the semi-convergence of the following integrals:*

$$\int_1^{+\infty} \frac{\sin x}{x^2}dx, \quad \int_0^1 \frac{\sqrt{x}\sin\left(\frac{1}{x^2}\right)}{\ln(1+x)}dx, \quad \int_1^{+\infty} \frac{\sin x}{x}dx, \quad \int_0^{+\infty} \frac{\sqrt{x}\sin x}{x+1}dx.$$

Exercise 10.2.5 *Determine the set of pairs (α, β) for which the generalized integral is convergent:*

$$\int_1^{+\infty} \frac{dx}{x^\alpha\left(1+x^\beta\right)}$$

10.3 Dirigated Work N°3 : (DIFFERENTIAL EQUATIONS)

Exercise 10.3.1 (Homogeneous linear equations of the 1st order) *Solve the following differential equations:*

$$1).\begin{cases} \acute{y} + 4y = 0 \\ y(0) = 2 \end{cases}$$

$$2).\begin{cases} x\acute{y} + (1+x)\,y = 0 \\ y(1) = 1 \end{cases}$$

$$3).\begin{cases} \left(1+x^2\right)\acute{y} - xy = 0 \\ y(0) = 1 \end{cases}$$

Exercise 10.3.2 (No homogeneous linear equations of the 1st order) *Solve the following differential equations:*

$$1).\begin{cases} x\acute{y} + y = x \\ y(2) = 0 \end{cases}$$

$$2).\begin{cases} x\acute{y} - 2y = x^4 \\ y(1) = 1 \end{cases}$$

$$3).\begin{cases} \acute{y} - 2y = \dfrac{-2}{1+\exp(-2x)} \\ y(0) = 2 \end{cases}$$

$$4).\begin{cases} \acute{y} + y = x\exp(-x) \\ y(0) = 1 \end{cases}$$

$$5).\quad \acute{y} + 2y = x^2$$

Exercise 10.3.3 (Linear equations with separate variables) *Solve the following differential equations:*

$$1).\begin{cases} 2x + y\acute{y} = 0 \\ y(1) = 1 \end{cases} \qquad 2).\begin{cases} \acute{y} = \dfrac{1-y}{1-2x} \\ y(0) = 0 \end{cases} \qquad 3).\begin{cases} \left(4-x^2\right)y\acute{y} = 2(1+y^2) \\ y(1) = 0 \end{cases}$$

Exercise 10.3.4 *Solve the following differential equation on* $\mathbb{R}$*:*

$$\left(1+x^2\right)\acute{y}+2xy=e^x+x.$$

Exercise 10.3.5 (Homogeneous linear equations of the 2nd order) *Solve the following differential equations:*

1). $\acute{\acute{y}}+y=0$ 2). $\acute{\acute{y}}$-$4y=0$ 3). $2\acute{\acute{y}}+\acute{y}-y=0$ 4). $\acute{\acute{y}}-6\acute{y}+9y=0.$

Exercise 10.3.6 (No homogeneous linear equations of the 2nd order) *Solve the following differential equations:*

1). $\acute{\acute{y}}+y=x^2-1.$ 2). $\acute{\acute{y}}$-$4y=13\cos(3x).$

3). $2\acute{\acute{y}}+\acute{y}-y=3\cos(2x)-\sin(2x).$ 4). $\acute{\acute{y}}-6\acute{y}+9y=e^{3x}.$

10.4 Dirigated Work N°4: (Numerical series, Function sequences and series, Integer series)

Exercise 10.4.1 *Study the convergence of the following numerical series:*

$$\sum_{n=1}^{+\infty}\frac{1}{n(n+1)},\quad \sum_{n=2}^{+\infty}\ln(1+\frac{1}{n}),\quad \sum_{n=0}^{+\infty}\frac{4n+3}{n+1},\quad \sum_{n=1}^{+\infty}\frac{2^n n!}{n^n},\quad \sum_{n=2}^{+\infty}\frac{(-1)^n}{\ln(n)},$$

$$\sum_{n=0}^{+\infty}\frac{(-1)^n}{(2n-1)^2},\quad \sum_{n=1}^{+\infty}\frac{2n}{n+2^n},\quad \sum_{n=1}^{+\infty}\left(\frac{n}{n+1}\right)^{n^2},\quad \sum_{n=1}^{+\infty}\frac{2^{n+1}}{3^n},$$

Exercise 10.4.2 *Consider the following sequence of functions:*

$$f_n(x)=\frac{\sqrt{n}x}{1+\sqrt{n}x^2}, x\in\left]0,+\infty\right].$$

1) - Study its simple convergence.

2) - Study its uniform convergence on $[b,+\infty[\,, b>0.$

Exercise 10.4.3 *Let the sequence of functions be defined by:*

$$f_n(x) = \frac{e^{-x}}{1+n^2x},\ n \geq 1.$$

1) - Study its simple convergence and its uniform convergence on $[1, +\infty[$.

2) - Study the simple convergence of the series $\sum_{n\geq 1} f_n(x)$ *on* $[1, +\infty[$.

3) - Show that $\sum_{n\geq 1} f_n(x)$ *converges normally on* $[1, +\infty[$.

Exercise 10.4.4 *1) - Determine the domain of convergence of the following integer series:*

$$\sum_{n>1} \frac{(-1)^n}{n^3} x^n.$$

2) - We consider the entiger series $f(x) = \sum_{n>0} n\,(-2)^n\,x^n$.

a) - What is the radius of convergence of $f(x)$.

b) - What is the radius of convergence of $g(x) = \sum_{n>0} (-2)^n\,x^n$.

c) - Deduce that $f(x) = xg'(x)$.

10.5 Dirigated Work N°5: (Fourier series, Fourier Transform, Laplace Transform)

Exercise 10.5.1 *(Explicit calculations of Laplace transforms) Calculate the Laplace transforms of*

the following functions:

$$f(t) = 1, \qquad f(t) = t^n, \qquad f(t) = e^{-at},$$
$$f(t) = \sin(\omega t), \qquad f(t) = \cos(\omega t), \qquad f(t) = t\sin(\omega t),$$

Exercise 10.5.2 *(Explicit calculations of inverse Laplace transforms) For each of the following functions,*

find a function $f(t)$ *such that* $\mathcal{L}\left[f(t)\right] = F(p)$*:*

$$F(p) = \frac{1}{(p+2)(p-1)}, \qquad F(p) = \frac{p}{(p+1)(p^2+1)},$$

Exercise 10.5.3 *(Application of Laplace transforms in the resolution of Diff Eqs)*

We consider the differential equation:

$$\begin{cases} \H{y} + 2\acute{y} + y = e^{-t} \\ y(0) = 0 \\ \acute{y}(0) = 2 \end{cases} \quad , \qquad t \geq 0.$$

Exercise 10.5.4 *(Trigonometric form of Fourier series)*

Calculate the Fourier series, in trigonometric form, of the 2π*-periodic function* $f : \mathbb{R} \longrightarrow \mathbb{R}$ *such that:* $f(x) = x^2$ *on* $[0, 2\pi[$*. Does the series converge to* f*?*

Exercise 10.5.5 *(Complex form of Fourier series)*

Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ *be the* 2π*-periodic function such that* $f(x) = e^x$ *for all* $x \in]-\pi, \pi]$*.*

1. *Calculate the complex Fourier coefficients of the function* f*.*

2. *Study the (simple, uniform) convergence of the Fourier series of* f*.*

Exercise 10.5.6 *(Fourier transform)*

Let $a > 0$ *and* f *be the function defined on* $\mathbb{R}$ *by* $f(x) = e^{-a|x|}$*.*

1. *We consider a function* $g : \mathbb{R} \longrightarrow \mathbb{R}$ *which is integrable and even. Show that:*

$$\hat{g}(\omega) = 2\int_0^{\infty} g(x)\cos(\omega x)dx.$$

2. *Use the previous question to calculate the Fourier transform of* f*.*

3. *With a particular value of* a*, deduce the value of the integral:* $\int_0^{+\infty} \frac{cos(\omega x)}{1+\omega^2} d\omega.$

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Annexs

MODULE CONTENT

UEF12 / F121
Series & Differential Equations

- Chapter 1: Simple and multiple integrals: Reminders on the Riemann integral and on the calculation of primitives. Double and triple integrals. Application to the calculation of areas, volumes, etc.
- Chapter 2: Improper integral: Integrals of functions defined on an unbounded interval. Integrals of functions defined on a bounded interval, infinite at one of the ends.
- Chapter 3: Differential equations: Ordinary differential equations of the 1st and 2nd order. Elements of partial differential equations.
- Chapter 4: Series: Numerical series. Sequences and series of functions. Whole series, Fourrier series.
- Chapter 5: Laplace transformation: Definition and properties. Application to solving differential equations.
- Chapter 6: Fourier Transform: Definition and properties. Application to solving differential equations.

PRIMITIVES USUELLES

Fonction	Primitive	Domaine de validité
$x \longmapsto x^n \quad (n \in \mathbb{N})$	$x \longmapsto \dfrac{x^{n+1}}{n+1}$	$\mathbb{R}$
$x \longmapsto x^p \quad (p \in \mathbb{Z}\backslash\{-1\})$	$x \longmapsto \dfrac{x^{p+1}}{p+1}$	$]-\infty, 0[$ ou $]0, +\infty[$
$x \longmapsto x^q \quad (q \in \mathbb{R}\backslash\{-1\})$	$x \longmapsto \dfrac{x^{q+1}}{q+1}$	$]0, +\infty[$
$x \longmapsto u'(x)\,[u(x)]^n$	$x \longmapsto \dfrac{1}{n+1}[u(x)]^{n+1}$	selon D_u
$x \longmapsto \dfrac{u'(x)}{u(x)^n}$	$x \longmapsto \dfrac{-1}{n-1}\,\dfrac{1}{[u(x)]^{n-1}}$	selon D_u
$x \longmapsto \dfrac{1}{x^2}$	$x \longmapsto \dfrac{-1}{x}$	$]-\infty, 0[$ ou $]0, +\infty[$
$x \longmapsto \dfrac{u'(x)}{u(x)^2}$	$x \longmapsto \dfrac{-1}{u(x)}$	$\{x \in D_u\ ;\ u(x) \neq 0\}$
$x \longmapsto \dfrac{1}{\sqrt{x}}$	$x \longmapsto 2\sqrt{x}$	$]0, +\infty[$
$x \longmapsto \dfrac{u'(x)}{\sqrt{u(x)}}$	$x \longmapsto 2\sqrt{u(x)}$	$]0, +\infty[$
$x \longmapsto \dfrac{1}{x}$	$x \longmapsto \ln\|x\|$	$]-\infty, 0[$ ou $]0, +\infty[$
$x \longmapsto \dfrac{u'(x)}{u(x)}$	$x \longmapsto \ln\|u(x)\|$	$\{x \in D_u\ ;\ u(x) \neq 0\}$
$x \longmapsto e^x$	$x \longmapsto e^x$	$\mathbb{R}$
$x \longmapsto a^x$	$x \longmapsto \dfrac{a^x}{\ln a}$	$\mathbb{R}$
$x \longmapsto u'(x)\, e^{u(x)}$	$x \longmapsto e^{u(x)}$	D_u
$x \longmapsto \sin x$	$x \longmapsto -\cos x$	$\mathbb{R}$
$x \longmapsto \cos x$	$x \longmapsto \sin x$	$\mathbb{R}$
$x \longmapsto \tan x$	$x \longmapsto -\ln\|\cos x\|$	$\left]-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right[\ +\ \pi\mathbb{Z}$
$x \longmapsto \cot x$	$x \longmapsto \ln\|\sin x\|$	$]0, \pi[+\pi\mathbb{Z}$
$x \longmapsto \sin^2 x$	$x \longmapsto \dfrac{2x - \sin 2x}{4}$	$\mathbb{R}$
$x \longmapsto \cos^2 x$	$x \longmapsto \dfrac{2x + \sin 2x}{4}$	$\mathbb{R}$
$x \longmapsto \dfrac{1}{\sin^2 x}$	$x \longmapsto -\cot x$	$]0, \pi[+\pi\mathbb{Z}$
$x \longmapsto \dfrac{1}{\cos^2 x}$	$x \longmapsto \tan x$	$\left]-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right[\ +\ \pi\mathbb{Z}$

Fonction		Primitive	Domaine de validité
$x \longmapsto \operatorname{sh} x$		$x \longmapsto \operatorname{ch} x$	$\mathbb{R}$
$x \longmapsto \operatorname{ch} x$		$x \longmapsto \operatorname{sh} x$	$\mathbb{R}$
$x \longmapsto \operatorname{th} x$		$x \longmapsto \ln(\operatorname{ch} x)$	$\mathbb{R}$
$x \longmapsto \coth x$		$x \longmapsto \ln\lvert\operatorname{sh} x\rvert$	$]-\infty,0[$ ou $]0,+\infty[$
$x \longmapsto \dfrac{1}{\operatorname{sh}^2 x}$		$x \longmapsto -\coth x$	$]-\infty,0[$ ou $]0,+\infty[$
$x \longmapsto \dfrac{1}{\operatorname{ch}^2 x}$		$x \longmapsto \operatorname{th} x$	$\mathbb{R}$
$x \longmapsto \dfrac{1}{a^2-x^2}$		$x \longmapsto \dfrac{1}{2a}\ln\left\lvert\dfrac{a+x}{a-x}\right\rvert$	$]-\infty,-a[$ ou $]-a,a[$ ou $]a,+\infty[$
$x \longmapsto \dfrac{1}{1+x^2}$		$x \longmapsto \arctan x$	$\mathbb{R}$
$x \longmapsto \dfrac{1}{a^2+x^2}$		$x \longmapsto \dfrac{1}{a}\arctan\left(\dfrac{x}{a}\right)$	$\mathbb{R}$
$x \longmapsto \dfrac{1}{(x-a)^n}$	$(n \neq 1)$	$x \longmapsto \dfrac{-1}{n-1}\,\dfrac{1}{(x-a)^{n-1}}$	$]a,+\infty[$
$x \longmapsto \dfrac{1}{x-a}$		$x \longmapsto \ln\lvert x-a\rvert$	$]-\infty,a[$ ou $]-a,a[$ ou $]a,+\infty[$
$x \longmapsto \dfrac{1}{\sqrt{a^2-x^2}}$		$x \longmapsto \arcsin\left(\dfrac{x}{a}\right)$	$]-a,a[$
$x \longmapsto \dfrac{x}{\sqrt{x^2-1}}$		$x \longmapsto \sqrt{x^2-1}$	$]-\infty,-1[$ ou $]1,+\infty[$
$x \longmapsto \dfrac{1}{\sqrt{a^2+x^2}}$		$x \longmapsto \ln\left(x+\sqrt{a^2+x^2}\right)$	$\mathbb{R}$
$x \longmapsto \dfrac{1}{\sqrt{x^2-a^2}}$		$x \longmapsto \ln\left\lvert x+\sqrt{x^2-a^2}\right\rvert$	$]-\infty,-a[$ ou $]a,+\infty[$

Primitives complexes Dans ce tableau, $\alpha \in \mathbb{C}\backslash\mathbb{R}$ et $p \in \mathbb{Z}\backslash\{0,-1\}$.
Les fonctions complexes suivantes sont définies sur $\mathbb{R}$ et leurs primitives sont valables sur cet intervalle.

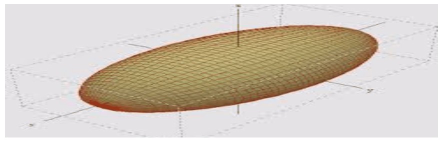
Fonction	Primitive
$x \longmapsto e^{\alpha x}$	$x \longmapsto \dfrac{1}{\alpha}\,e^{\alpha x}$
$x \longmapsto \dfrac{1}{x-\alpha}$	$x \longmapsto \ln\lvert x-\alpha\rvert + i\cdot\arctan\left(\dfrac{x-\Re(\alpha)}{\Im(\alpha)}\right)$
$x \longmapsto (x-\alpha)^p$	$x \longmapsto \dfrac{1}{p+1}\,(x-\alpha)^{p+1}$

Complément : « Surfaces dans l'espace »

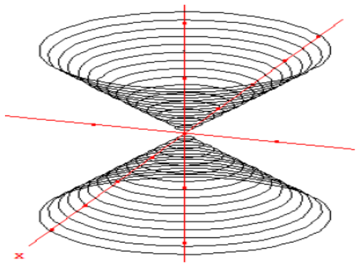
1) Sphère : L'équation cartésienne d'une sphère centrée en (x_0, y_0, z_0) et de rayon R est :

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = R^2.$$

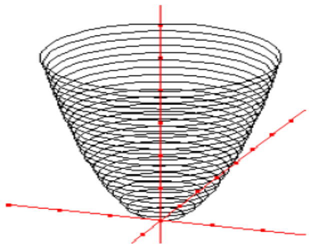
2) Ellipsoïde : est une surface d'équation de la forme : $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$



3) Cône : C'est une surface de l'espace d'équation de la forme : $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}$

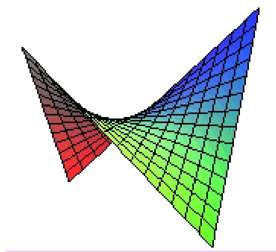


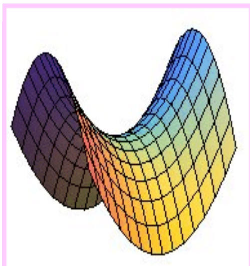
4) Paraboloïde elliptique (bol) : est d'équation de la forme : $z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$



5) Paraboloïde hyperbolique : (à selle) est d'équation de la forme :

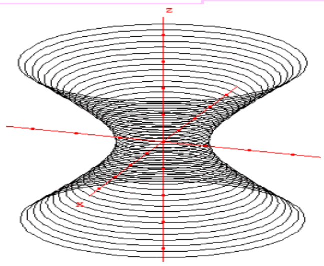
$z = \frac{x^2}{a^2} - \frac{y^2}{b^2}$; par un changement de variable l'équation se transforme en $z = x\,y$





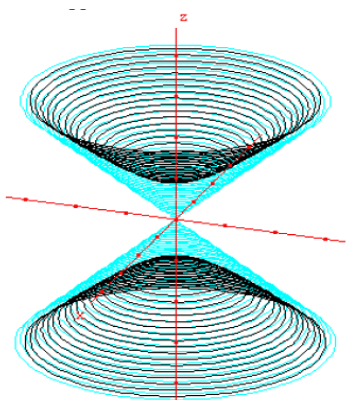
6) Hyperboloïde à une nappe : d'équation de la forme :

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$



7) Hyperboloïde à deux nappes : est d'équation :

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} + 1 = 0$$



Fonction	Transformée de Laplace et inverse
$\delta(t)$	1
1	$\frac{1}{p}$
t	$\frac{1}{p^2}$
t^n	$\frac{n!}{p^{n+1}}$
$\sqrt{t}$	$\frac{1}{2}\sqrt{\frac{\pi}{p^3}}$
$\frac{1}{\sqrt{t}}$	$\sqrt{\frac{\pi}{p}}$
$e^{-c.t}$	$\frac{1}{p+c}$
$t.e^{-c.t}$	$\frac{1}{(p+c)^2}$
$t^2.e^{-c.t}$	$\frac{2}{(p+c)^3}$
$t^n.e^{-c.t}$	$\frac{n!}{(p+c)^{n+1}}$
a^t	$\frac{1}{p-\ln a}$
$\sin(a.t)$	$\frac{a}{p^2+a^2}$
$t.\sin(a.t)$	$\frac{2a.p}{(p^2+a^2)^2}$
$t^2.\sin(a.t)$	$\frac{2a(3p^2-a^2)}{(p^2+a^2)^3}$
$\cos(a.t)$	$\frac{p}{p^2+a^2}$
$t.\cos(a.t)$	$\frac{p^2-a^2}{(p^2+a^2)^2}$
$t^2.\cos(a.t)$	$\frac{2p(p^2-3a^2)}{(p^2+a^2)^3}$
$\sin(a.t+b)$	$\frac{a\cos b+p\sin b}{p^2+a^2}$
$\cos(a.t+b)$	$\frac{p.\cos b-a\sin b}{p^2+a^2}$
$\sinh(a.t)$	$\frac{a}{p^2-a^2}$
$t.\sinh(a.t)$	$\frac{2a.p}{(p^2-a^2)^2}$
$\cosh(a.t)$	$\frac{p}{p^2-a^2}$
$t.\cosh(a.t)$	$\frac{p^2+a^2}{(p^2-a^2)^2}$

Transformée de Laplace	Fonction
1	$\delta(t)$
$\frac{1}{p}$	1
$\frac{1}{p^2}$	t
$\frac{1}{p^n}$	$\frac{t^{n-1}}{(n-1)!}$
$\frac{1}{\sqrt{p}}$	$\frac{1}{\sqrt{\pi t}}$
$\frac{1}{\sqrt{p^3}}$	$2\sqrt{\frac{t}{\pi}}$
$\frac{1}{p+a}$	$e^{-a.t}$
$\frac{1}{p(p+a)}$	$\frac{1}{a}\left(1-e^{-a.t}\right)$
$\frac{1}{p^2(p+a)}$	$\frac{e^{-a.t}}{a^2}+\frac{t}{a}-\frac{1}{a^2}$
$\frac{1}{p(p+a)^2}$	$\frac{1}{a^2}\left(1-e^{-a.t}-a.t.e^{-a.t}\right)$
$\frac{1}{(p+a)(p+b)}$	$\frac{e^{-b.t}-e^{-a.t}}{a-b}$
$\frac{p}{(p+a)(p+b)}$	$\frac{ae^{-a.t}-be^{-b.t}}{a-b}$
$\frac{1}{(p+a)(p+b)(p+c)}$	$\frac{e^{-a.t}}{(b-a)(c-a)}+\frac{e^{-b.t}}{(a-b)(c-b)}+\frac{e^{-c.t}}{(a-c)(b-c)}$
$\frac{1}{(p+a)^2}$	$t.e^{-a.t}$
$\frac{p}{(p+a)^2}$	$e^{-a.t}(1-a.t)$
$\frac{1}{(p+a)(p+b)^2}$	$\frac{e^{-a.t}-[1+(b-a)t]\,e^{-b.t}}{(b-a)^2}$
$\frac{1}{p(p+a)(p+b)}$	$\frac{1}{a.b}\left(1+\frac{b.e^{-a.t}-a.e^{-b.t}}{a-b}\right)$
$\frac{p+c}{p(p+a)(p+b)}$	$\frac{c}{a.b}+\frac{c-a}{a(a-b)}.e^{-a.t}+\frac{c-b}{b(b-a)}.e^{-b.t}$
$\frac{p^2+c.p+d}{p(p+a)(p+b)}$	$\frac{d}{a.b}+\frac{a^2-a.c+d}{a(a-b)}.e^{-a.t}+\frac{b^2-b.c+d}{b(b-a)}.e^{-b.t}$
$\frac{1}{(p+a)^3}$	$\frac{t^2.e^{-a.t}}{2}$
$\ln\left(\frac{p+a}{p+b}\right)$	$\frac{e^{-b.t}-e^{-a.t}}{t}$
$\frac{1}{p^2+a^2}$	$\frac{1}{a}\sin(a.t)$

$e^{-c.t}.\sin(a.t)$	$\frac{a}{(p+c)^2+a^2}$
$e^{-c.t}.\cos(a.t)$	$\frac{p+c}{(p+c)^2+a^2}$
$e^{-c.t}.\sin(a.t+b)$	$\frac{a\cos b+(p+c)\sin b}{(p+c)^2+a^2}$
$e^{-c.t}.\cos(a.t+b)$	$\frac{(p+c)\cos b+a\sin b}{(p+c)^2+a^2}$
$e^{-c.t}.\sinh(a.t)$	$\frac{a}{(p+c)^2-a^2}$
$e^{-c.t}.\cosh(a.t)$	$\frac{p+c}{(p+c)^2-a^2}$
$\sin^2(a.t)$	$\frac{2a^2}{p(p^2+4a^2)}$
$\sin^3(a.t)$	$\frac{6a^3}{(p^2+a^2)(p^2+9a^2)}$
$\cos^2(a.t)$	$\frac{p^2+2a^2}{p(p^2+4a^2)}$
$\cos^3(a.t)$	$\frac{p(p^2+7a^2)}{(p^2+a^2)(p^2+9a^2)}$
$\sinh^2 t$	$\frac{2}{p(p^2-4)}$
$\cosh^2 t$	$\frac{p^2-2}{p(p^2-4)}$
$\sin(a.t).\sin(b.t)$	$\frac{2a.b.p}{[(p^2+(a-b)^2].[(p^2+(a+b)^2]}$
$\cos(a.t).\cos(b.t)$	$\frac{p^2(p^2+a^2+b^2)}{[(p^2+(a-b)^2].[(p^2+(a+b)^2]}$
$\sin(a.t).\cos(b.t)$	$\frac{a(p^2+a^2-b^2)}{[(p^2+(a-b)^2].[(p^2+(a+b)^2]}$

$\frac{1}{p(p^2+a^2)}$	$\frac{1}{a^2}(1-\cos(a.t))$
$\frac{p}{p^2+a^2}$	$\cos(a.t)$
$\frac{p+a}{p(p^2+b^2)}$	$\frac{a}{b^2}-\frac{\sqrt{a^2+b^2}}{b^2}\cos\left(b.t+\arctan\frac{b}{a}\right)$
$\frac{p^2+c.p+d}{p(p^2+b^2)}$	$\frac{d}{b^2}-\frac{\sqrt{(d-b^2)^2+c^2b^2}}{b^2}\cos\left(b.t+\arctan\frac{bc}{d-b^2}\right)$
$\frac{1}{p^2-a^2}$	$\frac{1}{a}\sinh(a.t)$
$\frac{p}{p^2-a^2}$	$\cosh(a.t)$
$\frac{1}{(p+b)^2+a^2}$	$\frac{1}{a}e^{-b.t}.\sin(a.t)$
$\frac{p+b}{(p+b)^2+a^2}$	$e^{-bt}.\cos(a.t)$
$\frac{1}{(p^2+a^2)^2}$	$\frac{\sin(a.t)}{2a^3}-\frac{t.\cos(a.t)}{2a^2}$
$\frac{p}{(p^2+a^2)^2}$	$\frac{t}{2a}\sin(a.t)$
$\frac{p^2}{(p^2+a^2)^2}$	$\frac{1}{2a}(\sin(a.t)+a.t.\cos(a.t))$
$\frac{1}{p^3+a^3}$	$\frac{1}{3a^2}\left[e^{-at}-e^{\frac{at}{2}}\left(\cos\left(\frac{\sqrt{3}}{2}at\right)-\sqrt{3}\sin\left(\frac{\sqrt{3}}{2}at\right)\right)\right]$
$\frac{p}{p^3+a^3}$	$\frac{1}{3a}\left[-e^{-at}-e^{\frac{at}{2}}\left(\cos\left(\frac{\sqrt{3}}{2}at\right)+\sqrt{3}\sin\left(\frac{\sqrt{3}}{2}at\right)\right)\right]$
$\frac{p^2}{p^3+a^3}$	$\frac{1}{3}\left[e^{-at}+2e^{\frac{at}{2}}\cos\left(\frac{\sqrt{3}}{2}at\right)\right]$
$\frac{1}{(\tau 1p+1)(\tau 2p+1)p^2}$	$t-(\tau 1+\tau 2)+\frac{1}{(\tau 1-\tau 2)}.(\tau 1^2.e^{-t/\tau 1}-\tau 2^2.e^{-t/\tau 2})$