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First Year (MI+Mathematics)
Module of Algebra 2
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Series N°2

(Linear applications and matrices)

Exercise 1: Let f and g be two applications defined by:

$$f: \quad \mathbb{R}^2 \to \mathbb{R}^2 \qquad\qquad g: \quad \mathbb{R}^2 \to \mathbb{R}^2$$

$$(x,y) \mapsto f(x,y) = \left(\frac{x-y}{2}, \frac{y-x}{2}\right) \qquad\qquad (x,y) \mapsto g(x,y) = (2x-y, x-y)$$

1- Show that f and g are linear.

2- Determine $\mathrm{Ker}f, \mathrm{Ker}g, \mathrm{Im}f, \mathrm{Im}g, \mathrm{rk}f, \mathrm{rk}g$.

3- f and g are they injective ? surjective ?

4- Is $\mathbb{R}^2 = \mathrm{Ker}f \oplus \mathrm{Im}f$?

5- Show that: if $u \in \mathrm{Im}f$ then $f(u) = u$.

Exercise 2: Let be the following matrices:

$$A = \begin{pmatrix} 3 & 0 \\ -2 & 1 \end{pmatrix}, B = \begin{pmatrix} 2 & 1 \\ -1 & 0 \\ 1 & 3 \end{pmatrix}, C = \begin{pmatrix} 1 & -3 & 2 \\ 2 & 1 & -3 \\ 4 & -3 & -1 \end{pmatrix}, D = \begin{pmatrix} 2 & 1 \\ 3 & -1 \\ 5 & 4 \end{pmatrix}, , E = \begin{pmatrix} 1 & 2 & -1 \\ -1 & 2 & 3 \\ 4 & 5 & 2 \end{pmatrix}$$

1. Calculate -if possible- $A + C, B + D, 3A, A \times B, B \times A,\ A^3, C^2, B^t, A \times B^t$.
2. Calculate $\det C, \det E$ and C^{-1}, E^{-1}.

Exercise 3: Let $B = \{e_1, e_2, e_3\}$ be the canonical basis of $\mathbb{R}^3$, and let f be the linear application:

$$f: \quad \mathbb{R}^3 \to \mathbb{R}^3$$
$$(x,y,z) \mapsto (x-y, x+z, y+z)$$

1- Find the matrix of f in the canonical basis of $\mathbb{R}^3$.

2- Let $a = (1,3,-1), b = (1,3,0), c = (1,2,-1)$

a) Show that $B' = \{a,b,c\}$ is a basis of $\mathbb{R}^3$.

b) Find the passage matrix P from B to B'. Calculate P^{-1}.

c) Find the matrix of f in the basis B' using the passage matrix.

d) Find the matrix of f in the basis B' using the definition.

Exercise 4: Let $B = \{e_1, e_2, e_3\}$ the canonical basis of $\mathbb{R}^3$, and let f be the linear application:

$f: \mathbb{R}^3 \to \mathbb{R}^3$ such that

$$f(e_1) = e_3,\ f(e_2) = -e_1 + e_2 + e_3,\ f(e_3) = e_3$$

I.

1. Show that: $\forall (x, y, z) \in \mathbb{R}^3: f(x, y, z) = (-y, y, x + y + z)$
2. Find $\mathrm{Ker} f$ and $\dim \mathrm{Ker} f$, is f injective?
3. Let F a vector subspace of $\mathbb{R}^3$ defined by $F = \{(x, y, z) \in \mathbb{R}^3 /\ x = 0\}$

   show that $\mathbb{R}^3 = F \oplus \mathrm{Ker} f$.

II.

1. Find the matrix of f in the canonical basis of $\mathbb{R}^3$.
2. We set $e'_1 = e_1 - e_3,\ e'_2 = e_1 - e_2,\ e'_3 = -e_1 + e_2 + e_3$
   a) Check that $B' = \{e'_1, e'_2, e'_3\}$ is a basis of $\mathbb{R}^3$.
   b) Find the passage matrix P from the canonical basis to B'. Calculate P^{-1}.
   c) Find the matrix of f in the basis B' by using two methods .