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# Module of Algebra 2

First Year (MI+Mathematics)

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Chapter I :

Vector spaces.

## 1 Vector spaces

Consider a set $E$ endowed with two operations:

1] Addition : $\oplus$ : $E \times E \longrightarrow E$.
$(v_1, v_2) \longmapsto v_1 + v_2$.

2] Multiplication: $\otimes$ : $K \times E \longrightarrow E$
$(\lambda, v) \longmapsto \lambda \times v$.

Such that $(K, +, \cdot)$ is field (جسم)

Definition: A triple $(E, \oplus, \otimes)$ is a vector space On $(K, +, \cdot)$ if:

1- $(E, \oplus)$ is Commutative groupe

2. $\forall \alpha, \beta \in K$, $\forall v_1, v_2 \in E$ we have.

i) $(\alpha + \beta) \otimes v_1 = (\alpha \otimes v_1) \oplus (\beta \otimes v_2)$

ii) $\alpha \otimes (v_1 \oplus v_2) = (\alpha \otimes v_1) \oplus (\beta \otimes v_2)$

iii) $(\alpha \cdot \beta) \otimes v_1 = \alpha \otimes (\beta \otimes v_1)$

iv) $1_K \otimes v_1 = v_1$ (Such that $1_K$ is neutral element on $K$ with respect to $(\cdot)$)

the elements on $E$ are called vectors and the elements in $K$ scalars in general we have $K = \mathbb{R}$.

Exemple:

1) $\mathbb{R}$ itself is a vectors space on $\mathbb{R}$.
2) $\forall n \in \mathbb{N}$, $\mathbb{R}^n$ is vectors space endowed with.

i) $\oplus$ : $\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \oplus \begin{pmatrix} v_1' \\ \vdots \\ v_n' \end{pmatrix} = \begin{pmatrix} v_1 + v_1' \\ \vdots \\ v_n + v_n' \end{pmatrix}$.

ii) $\otimes$ : $\lambda \otimes \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} \lambda v_1 \\ \vdots \\ \lambda v_n \end{pmatrix}$

3) in general if $V$ is vectors space then $V^n$ is still a vector space.
4) the space of functions $f : [0,1] \longrightarrow \mathbb{R}$. is a vector space

From the properties above, one came deduce the following.

**Proposition 1:** Let $(E, \oplus, \otimes)$ be a vector space. then,

(1) $0_K \otimes v = 0_E$, for any $v \in E$.

(2) $(-c) \otimes v = c \otimes (-v)$, for any $c \in \mathbb{R}$ and $c \in v$;

3) $c \otimes 0_E = 0_E$, for any $c \in \mathbb{R}$;

4) if $c \otimes v = 0_E$, then either $c = 0$ or $v = 0$.

Proof: 1) Using the properties (iv) and (i)

$$v \oplus 0_E \otimes v = 1_K \otimes v \oplus 0_K \otimes v$$
$$= (1_K \oplus 0_K) \otimes v$$
$$= 1_K \otimes v = v$$

adding $-v$ to both sides of equality:

$$-v \oplus v \oplus 0_K \otimes v = -v \oplus v$$
$$\Rightarrow 0_K \otimes v = 0_E$$

2) By definition, $-(c \otimes v)$ is an element of $E$ using (i)

$$(-c) \otimes v \oplus c \otimes v = (-c+c) \otimes v$$
$$= 0_K \otimes v = 0_E$$

Then $(-c) \otimes v = -(c \otimes v)$ ... $(*)$

we have by using (ii)

$$c \otimes (-v) \oplus c \otimes v = c \otimes (-v+v)$$
$$= c \otimes (0_E)$$
$$= 0_E \; \ldots \quad \text{by (1)}$$

Then $c \otimes (-v) = -(c \otimes v)$ .... $(*,*)$

from $(*)$ and $(*,*)$ we have

$$(-c) \otimes v = c \otimes (-v) = -(c \otimes v)$$

4) $c \otimes v = 0_E$

if $c = 0_K$ then $0_K \otimes v = 0_E$

if $c \neq 0_K$, let $c' \in K$ such that

$$c' \cdot c = 1_K$$

we have : $v = 1_K \otimes v = (c' . c) \otimes v$

$$= c' \otimes (c \otimes v)$$

$$= c' \otimes 0_E = 0_E$$

## 1.2 Subspace:

**Definition:**

Let $(E, +, \times)$ be a vector space and $F \subseteq E$ a nonempty subspace of $E$. $F$ is said to be a vetor subspace of $E$, if it is a vector space with the induced operations;

i.e. $(F, +, \cdot)$ ($(+), (\cdot)$ are the some operations of $E$, restricted to $F$)

**Proposition 2:** Let $F$ be nonempty subset of a vector space $E$. $F$ is a vector subspace if and only if the following two condition are satisfied

a) $\forall\ u, v \in F \Rightarrow u + v \in F$

b) $\forall\ c \in K, \forall v \in F \Rightarrow c \times v \in F$

**Proof:** $(\Rightarrow)$ is obvious because $+$ is internal born in $F$ and $(\cdot)$ external born in $F$

$(\Leftarrow)$ all the conditions are satisfied because $(E, +, \cdot)$ is v.s sauf the neutral and the opposite elements we have $0 = 0 . v \in F$ and $-v = (-1) . v$

Remarque :

One Cane easily show that properties (a) and (b) above, are equivalent to the uniques Condition :

$\forall \alpha, \beta \in \mathbb{K}, \forall u, v \in F \Rightarrow \alpha \times u + \beta \times v \in F$.

The Zero vector will belong to every subspace. (that Comes from rule b: choose the scalar to be $c = 0_{\mathbb{K}}$

Example: (1) $F_1 = \{0\}$ and $F_2 = E$ are the trevial supspaces ($F_1$ Containing only the zero vector $F_2$ the whole space)

$F_1$ the smallest possible subspace and $F_2$ the biggestone

(2) (Exercice) : If $F_1$ and $F_2$ are two vectors subspaces; then intersection space $F_1 \cap F_2$ is still a subspace

(3) (Exercice) If $F_1$ and $F_2$ are two vector subspaces then the sum space

$$F_1 + F_2 := \{ v_1 + v_2 ,\ v_1 \in F_1 ,\ v_2 \in F_2 \}.$$

is a vector subspace.

observe that $F_1 \cup F_2 \subset F_1 + F_2$. In fact and in generale the union might not be a subspace One Can show that $F_1 + F_2$ is the smallest subspace Containing the union. $F_1 \cup F_2$.

i.e $F_1 + F_2 = \langle F_1 \cup F_2 \rangle$ ? (Exercice)

**Proposition** let $(E,+,\times)$ be a Vector space and $F_1, F_2$ two subspaces. The following two Conditions are equivalent:

2) for any $v \in F_1+F_2$, there exist a unique Couple $(v_1, v_2) \in F_1 \times F_2$ such that $v = v_1 + v_2$ (i.e, any Vector in the sum space Can be Written uniquely as the sum of vectors in $F_1$ and $F_2$.

**Proof**: $i \Rightarrow ii)$

Suppose that the vector $v$ can be written in two ways $\begin{cases} v = v_1 + v_2 & / v_1, v_1' \in F_1; \\ v = v_1' + v_2' & / v_2, v_2' \in F_2. \end{cases}$

$$\Rightarrow v_1 - v_1' = v_2' - v_2 \in F_1 \cap F_2.$$

$$\Rightarrow v_1 - v_1' = v_2' - v_2 = 0 \Rightarrow v_1 = v_1' \text{ and } v_2' = v_2$$

$ii) \Rightarrow (i)$ suppose by Contradiction $F_1 \cap F_2 \neq 0$

$$\Rightarrow \exists\, v \neq 0 \text{ it } v \in F_1 \cap F_2 \Rightarrow v \in F_1 \text{ and } v \in F_2.$$

$$v = 0 + v = v + 0 \in F_1 + F_2$$ Contradiction Since it implies that $v$ can be decomposed in two different ways as a vector of $F_1 + F_2$

**Définition**: Let $F_1, F_2$ two vector subspace of a Vector space $V$

- $F_1$ and $F_2$ are said Supplimentary Subspace if $F_1 + F_2 = V$ and $F_1 \cap F_2 = \{0\}$

- More generally if $F_1 \cap F_2 = \{0\}$, then $F_1 + F_2$ is called direct sum and it denoted by $w_1 \oplus w_2$.

## II] Linear independance, set of generatrice and Basis

**Définition** Let $(V, +, \cdot)$ be a vector space. $v_1, \dots, v_n \in V$ and $\alpha_1, \dots \alpha_n \in \mathbb{R}$.
The vector $\alpha_1 v_1 + \dots + \alpha_n v_n$ is called linear combination of $v_1, \dots v_n$ with coefficient $\alpha_1, \dots, \alpha_n$

**Proposition**: Let $v_1, \dots, v_k \in V$, with $k \geq 2$.
they are linearly dependent if and only if at least one of them can be written a linear combination of the remaining.

**Proof**: "$\Rightarrow$" we suppose that $v_1, \dots, v_k$ are dependent then there existe $c_1, \dots, c_k \in \mathbb{R}$ (not all zeros) such that $c_1 . v_1 + \dots c_k v_k = 0$
we can assume that $c_1 \neq 0$ therfore,

$$c_1 v_1 = -c_2 v_2 - \dots - c_k v_k$$

and multipling by $\frac{1}{c_1}$:

$$v_1 = -\frac{c_2}{c_1} v_2 - \dots - \frac{c_k}{c_1} v_k$$

"$\Leftarrow$" If, we suppose that $v_1$ is a linear combination of $v_2, \dots, v_k$, then (For suitable $a_2, \dots, a_k \in \mathbb{R}$)

$$v_1 = a_2 v_2 + \dots + a_k v_k$$

and it follows that: $1 v_1 + a_2 v_2 + \dots - a_k v_k = 0$

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Proposition: Let $v_1, \dots, v_k \in V$. They are linearly independent if and only if the following condition holds:

$$\text{if } \sum_{i=1}^{k} c_i v_i = \sum_{i=1}^{k} d_i v_i \Rightarrow c_1 = d_1, \dots, c_k = d_k$$

Proof: ($\Rightarrow$) From the hypotheses, it follows that

$$\sum_{i=1}^{k} (c_i - d_i) v_i = 0$$

therefor, since the vectors are linearly independent;

$$c_1 - d_1 = 0, \dots, c_k - d_k = 0$$

"$\Leftarrow$" suppose that $\sum_{i=1}^{k} a_i v_i = 0$

then $\sum_{i=1}^{k} a_i v_i = \sum_{i=1}^{k} 0 v_i$

$\forall i = 1, n \quad a_i = 0$

and consequently the vectors are linearly independent

Generatrice space:

Let us know introduce the concept of set of generators, or what means for a set of vectors to span a space.

Definition: Let $E$ a vector space, we define the space generated by $k$ vectors in $E$ as follow

$\langle v_1, \dots, v_k \rangle := \{ c_1 v_1 + \dots + c_k v_k, \forall c_1 \dots c_k \in \mathbb{R} \}$

This is a vector subspace and is called subspace generated by $v_1, \dots, v_k$

Observe that $\langle v_1, \dots, v_k \rangle$ is the smallest vector subspace containing $v_1, \dots, v_k$

$$\langle v_1, \dots, v_k \rangle = \bigcap_{v_1, \dots, v_k \in W} W$$

Proof; $E$ . a v.s on $K$

① we proove that $\langle v_1, \dots, v_k \rangle$ is a V.S

② $\langle v_1, \dots, v_k \rangle \neq \phi$ ;

$0_E \in \langle v_1, \dots, v_k \rangle$ because

$0_E = 0_K . v_1 + \dots + 0_K v_k$

③ $\forall x, y \in \langle v_1, \dots, v_k \rangle \Rightarrow x + y \in \langle v_1 \dots v_k \rangle$

$x \in \langle v_1 \dots v_k \rangle \Rightarrow \exists \alpha_1 \dots \alpha_k \in K$ s.t

$x = \alpha_1 v_1 + \dots \alpha_k v_k$

$y \in \langle v_1 \dots v_k \rangle \Rightarrow \exists \alpha'_1, \dots \alpha'_k \in K$ s.t

$y = \alpha'_1 v_1 + \dots \alpha'_k v_k$

$x + y = (\alpha_1 + \alpha'_1) v_1 + \dots + (\alpha_k + \alpha'_k) v_k \in \langle v_1, \dots, v_k \rangle$

④ $\forall \alpha \in K, \forall x \in \langle v_1, \dots, v_k \rangle \Rightarrow x = \alpha_1 v_1 + \dots + \alpha_k v_k$

$\Rightarrow \alpha x = \alpha \alpha_1 v_1 + \dots \alpha \alpha_k v_k$

$\Rightarrow \alpha . x \in \langle v_1, \dots, v_k \rangle$

$$\langle v_1, \dots, v_k \rangle \supseteq \bigcap_{v_1,\dots v_k \in W} W \; ?$$

" $\supset$ " ?

$\bigcap_{v_1 \dots v_k \in W} W$ the smallest v.s.s which contains $\{v_1 \dots v_k\}$ and we have

$\langle v_1, \dots v_k \rangle$ is a v.s.s s.t $\{v_1 \dots v_k\} \subset \langle v_1, \dots v_k \rangle$.

we deduce that $\langle v_1, \dots, v_k \rangle \supset \bigcap_{v_1 \dots v_k \in W} W$. (1)

" $\subset$ " ?

$$\forall x \in \langle v_1, \dots, v_k \rangle \Rightarrow \exists \alpha_1 \dots \alpha_k \in K :$$

$$x = \alpha_1 v_1 + \dots + \alpha_k v_k \in \bigcap_{v_1 \dots v_k \in W} W$$

$$\Rightarrow \langle v_1 \dots v_k \rangle \subset \bigcap_{v_1 \dots v_k \in W} W \quad (2)$$

From (1) and (2) we have.

$$\bigcap_{v_1 \dots v_k \in W} W = \langle v_1, \dots, v_k \rangle$$

<u>Definition</u> Let $E$ be a vector space and let $v_1, \dots, v_n \in E$. we will say that $\{v_1, \dots, v_n\}$ is a basis of $E$, if:

i) $v_1, \dots v_n$ are linarly independent;

ii) $\langle v_1, \dots, v_n \rangle = E$

<u>Proposition</u>: Let $E$ a v.s and $v_1, \dots, v_n \in E$ the following conditions are equivalent;

(1) $\{v_1, \dots, v_n\}$ is basis of $E$.

(2) Every vectors $v \in E$ can be written uniquely as a linear combination of $v_1, \dots, v_n$

Proof: (1) $\Rightarrow$ (2)

$v \in E = \langle v_1, \dots, v_n \rangle$

$\Rightarrow v = \alpha_1 v_1 + \dots \alpha_n v_n \;/\; \alpha_1, \dots, \alpha_n \in k$ (1)

if we suppose that there exists $\alpha_1' \dots \alpha_n'$ such that

$v = \alpha_1' v_1 + \dots + \alpha_n' v_n$ ..... (2)

(1) - (2) $\Rightarrow 0 = (\alpha_1 - \alpha_1') v_1 + \dots (\alpha_n - \alpha_n') v_n$

$\Rightarrow (\alpha_1 - \alpha_1') = 0, \dots, \alpha_n' - \alpha_n = 0$

(2) $\Rightarrow$ (1)

i) $\{v_1, \dots, v_n\}$ is independent ?

$v = \alpha_1 v_1 + \dots + \alpha_n v_n = 0 = 0v_1 + \dots + 0 v_n$

We have $v \in E$ then it can be written uniquely then $\alpha_1 = 0 = \alpha_2 = \dots = \alpha_n$

ii) $\langle v_1, \dots v_n \rangle = E$ ?

$v \in E \Rightarrow v = \alpha_1 v_1 + \dots + \alpha_n v_n \in \langle v_1, \dots v_n \rangle$

then $E \subset \langle v_1, \dots, v_n \rangle$

$\langle v_1, \dots, v_n \rangle \subset E$

In other words, the above proposition shows that there exists a bijection

$$E \longrightarrow \mathbb{R}^n$$
$$v \longmapsto (c_1, \dots, c_n) \text{ such that}$$
$$v = \sum_{i=1}^{n} c_i v_i.$$

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This bijection is clearly dependent on the choice of the basis $\{v_1, \dots, v_n\}$ of $E$.
we call $(c_1, \dots, c_n)$ the Coordinates of $v$ with respect to the basis $\{v_1, \dots, v_k\}$

### 3. Dimension of a vector space.

In this last section, we will show four importante results, ~~we will~~ about a vector space in few words;