Université Mohamed Khider Biskra

Faculté des Sciences
Exactes et Sciences
de la Vie

2ème Année Physique
Module: Séries et Equation Différentielles

SERIE 2

Intégrales doubles et triples

Exercice 01:

Calculer les intégrales suivantes:

$$\int_3^4\int_1^2 \frac{dx\,dy}{(x+y)^2}, \quad \int_1^2\int_x^{x\sqrt{3}} xy\,dx\,dy, \quad \int_0^{2\pi}\int_{2\sin\theta}^{2} r\,dr\,d\theta, \quad \int_0^1\int_{y-1}^{2y} xy\,dx\,dy,$$

Exercice 02:

Définir les bornes d'intégrations pour $\iint_D f(x,y)\,dx\,dy$, D étant délimités par:

a) $x=2, x=3, y=-1, y=5$,

b) $y=0, y=1-x^2$

c) $x^2+y^2=4$,

d) $y=\frac{2}{1+x^2}, y=x^2$

Exercice 03:

Calculer:

$$\iint_D |x+y|\,dx\,dy, \qquad \text{où } D=\{(x,y)\in\mathbb{R}^2/\,|x|<1, |y|<1\}$$
$$\iint_D \frac{xy}{x^2+y^2}\,dx\,dy, \qquad \text{où } D=\{(x,y)\in\mathbb{R}^2/x>0, y>0, x+y<1\}$$
$$\iint_D \frac{1}{1+x^2+y^2}\,dx\,dy, \qquad \text{où } D=\{(x,y)\in\mathbb{R}^2/x^2+y^2<1\}$$
$$\iint_D \sqrt{x^2+y^2}\,dx\,dy, \qquad \text{où } D=\{(x,y)\in\mathbb{R}^2/0<y<x<1\}$$

Exercice 04:

Calculer l'aire de la figure délimitée par les courbes:

$y^2=2x$, $y^2=4x-x^2$

$y^2=2x$, $y=x$; $y^2=4x$, $x+y=3$, $y=0$

$y=\sin x$, $y=\cos x$, $x=0$; $y^2=4x+4$, $y^2=-4x+4$

Exercice 05:
Calculer le volume délimité par les surfaces:
$\frac{x^2}{9}+\frac{y^2}{16}+\frac{z^2}{25}=1$
$x^2+z^2=R^2,\ y^2+z^2=R^2$
$x^2+y^2+z^2=1,\ x^2+y^2=z^2$

Exercice 06:
Calculer:
$\iiint\limits_V z\ dxdydz$ où $V=\{(x,y,z)\in\mathbb{R}^3/x\geqslant 0, y\geqslant 0, z\geqslant 0, z\leqslant 1-y^2 \text{ et } x+y\leqslant 1\}$
$\iiint\limits_V xyz\ dxdydz$ où $V=\{(x,y,z)\in\mathbb{R}^3/0<z<1,\ x^2+y^2<z^2\}$
$\iiint\limits_V \left(\frac{1}{\sqrt{x^2+y^2}}+\frac{1}{z}\right) dxdydz$ où $V=\{(x,y,z)\in\mathbb{R}^3/0<x^2+y^2+z^2<1,\ 0<x^2+y^2<z^2,\ z>0\}$

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Serie III

Exercice 01:

* $$\int_3^4\int_1^2 \frac{dx\,dy}{(x+y)^2} = \int_3^4\left(\int_1^2 \frac{1}{(x+y)^2}\,dy\right)dx = \int_3^4\left(\frac{-1}{x+y}\right)\Big|_1^2 dx$$

$$= \int_3^4\left(\frac{-1}{x+2}+\frac{1}{x+1}\right)dx = -\ln(x+2)+\ln(x+1)\Big|_3^4$$

$$= \ln\left(\frac{x+1}{x+2}\right)\Big|_3^4 = \ln\frac{5}{6} - \ln\frac{4}{5} = \ln\frac{5}{6}\times\frac{5}{4} = \ln\frac{25}{24}$$

* $$\int_1^2\int_x^{x\sqrt{3}} xy\,dx\,dy = \int_1^2\left(\int_x^{x\sqrt{3}} xy\,dy\right)dx = \int_1^2 x\left(\int_x^{x\sqrt{3}} y\,dy\right)dx$$

$$= \int_1^2 x\left[\frac{3}{2}x^2 - \frac{1}{2}x^2\right]dx = \int_1^2 x^3\,dx = \frac{x^4}{4}\Big|_1^2 = 4-\frac{1}{4} = \frac{15}{4}$$

* $$\int_0^{2\pi}\int_{2\sin\theta}^{2} r\,dr\,d\theta = \int_0^{2\pi}\left(\int_{2\sin\theta}^{2} r\,dr\right)d\theta = \int_0^{2\pi}\frac{r^2}{2}\Big|_{2\sin\theta}^{2}d\theta$$

$$= \int_0^{2\pi}\left[2-2\sin^2\theta\right]d\theta = \int_0 2\cos^2\theta\,d\theta$$

$$= \int_0^{2\pi}(1+\cos 2\theta)\,d\theta = \left[\theta+\frac{1}{2}\sin 2\theta\right]_0^{2\pi} = 2\pi$$

* $$\int_0^1\int_{y-1}^{2y} xy\,dx\,dy = \int_0^1 y\left(\int_{y-1}^{2y} x\,dx\right)dy$$

$$= \int_0^1 y\left(\frac{x^2}{2}\Big|_{y-1}^{2y}\right)dy = \int_0^1 y\left(2y^2 - \frac{1}{2}(y-1)^2\right]dy$$

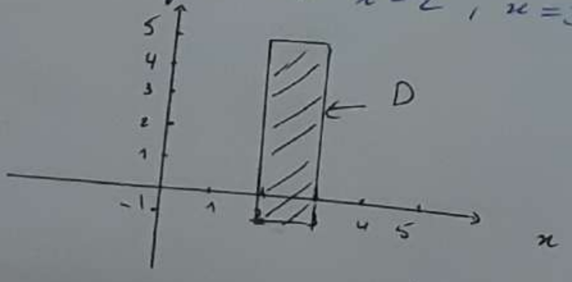
$$= \int_0^1\left(\frac{3}{2}y^3 + y^2 - \frac{1}{2}y\right)dy = \frac{3}{6}y^4 + \frac{1}{3}y^3 - \frac{1}{4}y^2\Big|_0^1$$

$$= \frac{11}{24}$$

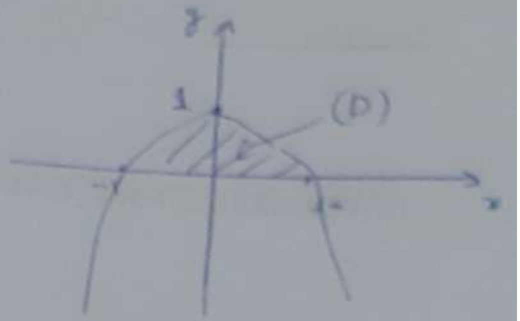
Exercice 02:

* $\iint_{①} f(x,y)\,dx\,dy$

a) $x=2,\ x=3,\ y=-1,\ y=5$



$$\int_{-1}^{5}\int_{2}^{3} f(x,y)\,dx\,dy$$



* b) $y=0$, $y=1-x^2$, $\Rightarrow$ $x=\pm\sqrt{1-y}$

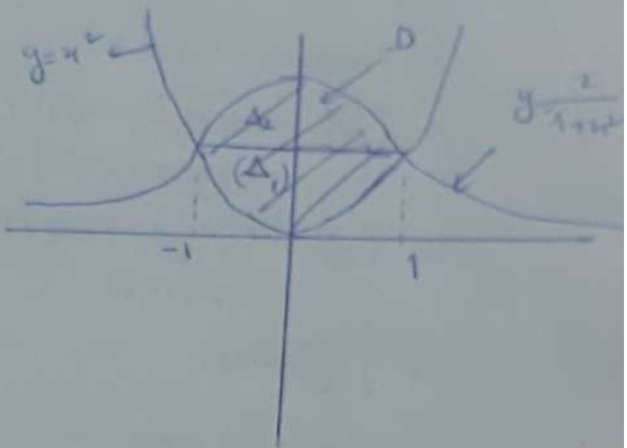
$y=1-x^2=0 \Leftrightarrow$ $x=1 \lor x=-1$

$y=0 \lor y=0$

$$\int_{0}^{1}\int_{-\sqrt{1-y}}^{+\sqrt{1-y}} f(x,y)\,dx\,dy \;\lor\; \int_{-1}^{1}\int_{0}^{1-x^2} f(x,y)\,dy\,dx$$

* c) $x^2+y^2=4 \Rightarrow x=\pm\sqrt{4-y^2}$

$$\int_{-2}^{2}\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} f(x,y)\,dx\,dy \;\lor\; \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{-2}^{2} f(x,y)\,dy\,dx$$



* d) $y=\frac{2}{1+x^2}$, $y=x^2$

$y=x^2$, $y=\frac{2}{1+x^2}$ $\Rightarrow$ $y=x^2$, $x^2+x^4-2=0$

$X=x^2$, $\Delta=9$ $X_1\to 1$ $X_2\to -1$ $x^2=1$ $x=\pm 1$

$$\int_{-1}^{1}\int_{x^2}^{\frac{2}{1+x^2}} f(x,y)\,dy\,dx \;\lor$$

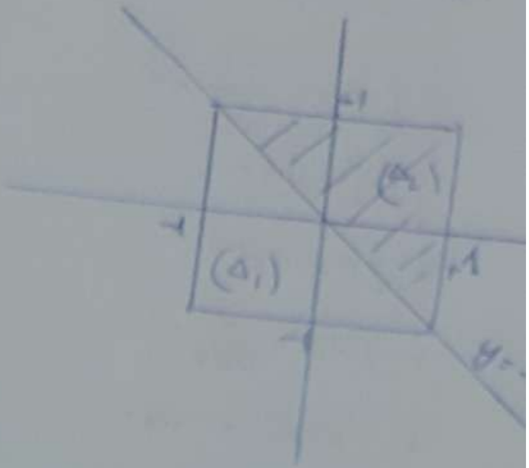
$$\iint_{\Delta} f(x,y)\,dx\,dy = \iint_{\Delta_1} + \iint_{\Delta_2}$$

$$= \int_{0}^{1}\int_{-\sqrt{y}}^{+\sqrt{y}} f(x,y)\,dx\,dy + \int_{1}^{2}\int_{-\sqrt{\frac{2-y}{y}}}^{\sqrt{\frac{2-y}{y}}} f(x,y)\,dx\,dy$$

ercice 03

$$\iint_D |x+y|\, dx\, dy \quad . \quad D = \{ (x,y) \in \mathbb{R}^2 / \ |x| < 1 \ , \ |y| < 1 \}$$

$$|x+y| = \begin{cases} x+y & \text{si } y \geq -x \\ -(x+y) & \text{si } y \leq -x \end{cases}$$



$$\iint_\Delta |x+y|\, dx\, dy = \iint_{\Delta_1} + \iint_{\Delta_2}$$

$$= \int_{-1}^{1} \int_{-1}^{-x} -(x+y)\, dy\, dx \ +$$

$$+ \int_{-1}^{1} \int_{-x}^{1} x+y \ dy\, dx$$

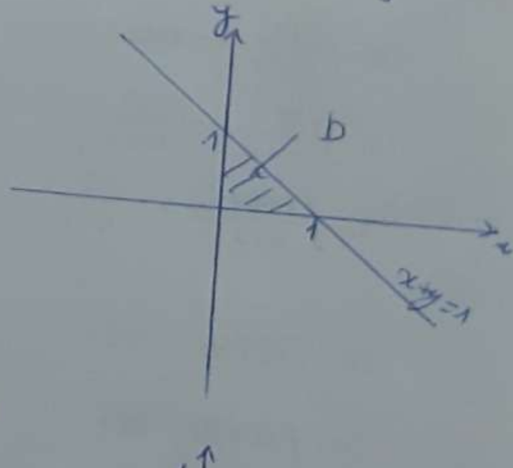
$$= -\int_{-1}^{1} (xy + \tfrac{1}{2}y^2) \Big|_{-1}^{-x} dx + \int_{-1}^{1} xy + \tfrac{1}{2}y^2 \Big|_{-x}^{1} dx$$

$$= -\int_{-1}^{1} (-x^2 + \tfrac{1}{2}x^2 + x - \tfrac{1}{2})\, dx \ + \int_{-1}^{1} (x + \tfrac{1}{2} + x^2 - \tfrac{1}{2}x^2)\, dx$$

* $$\iint_D \frac{xy}{x^2+y^2}\, dx\, dy = \int_0^1 \left(\int_0^{1-y} \frac{xy}{x^2+y^2}\, dx \right) dy$$

$$D = \{ (x,y) \in \mathbb{R}^2 / \ x > 0, \ y > 0, \ x+y < 1 \}$$

$$\iint_D - = \int_0^1 \left(\int_0^{1-x} \frac{xy}{x^2+y^2}\, dy \right) dx$$



* $$\iint \frac{1}{1+x^2+y^2}\, dx\, dy$$

$$D = \{ (x,y) \in \mathbb{R}^2 / \ x^2+y^2 < 1 \}$$

$$\iint_D - = \int_{-1} \left(\int_{-\sqrt{1-y^2}}^{\sqrt{1-y}} \frac{1}{1+x^2+y^2}\, dx \right) dy$$

$$= \int_{-1} \left(\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{1+x^2+y^2}\, dy \right) dx$$

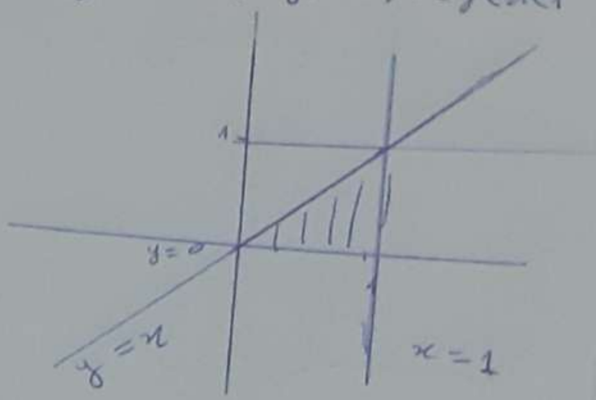
$$= \int_0^{2\pi} \int_0^1 \frac{r}{1+r^2}\, dr\, d\theta$$

$= \iint_D \sqrt{x^2+y^2}\, dx\, dy = \int_0^1 \left(\int_y^1 \sqrt{x^2+y^2}\, dx \right) dy$

$= \int_0^1 \left(\int_0^x \sqrt{x^2+y^2}\, dy \right) dx$

$= \int_0^{\pi/4} \int_0^{\frac{1}{\cos\theta}} r^2\, dr\, d\theta$

$D = \{(x,y) \in \mathbb{R}^2 / 0 < y < x < 1$



$(y = x \iff r\cos\theta = r\sin\theta$

$tg\theta = 1 \Rightarrow \theta = \pi/4$

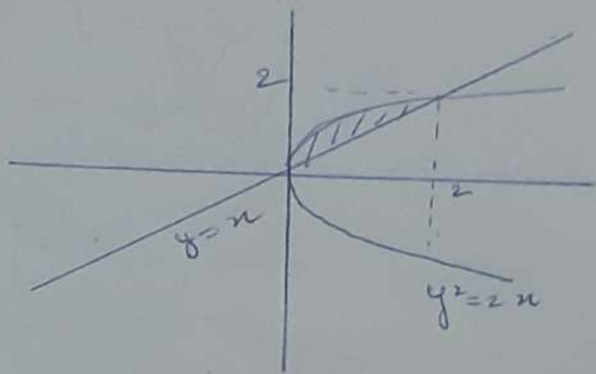
$x = 1 \iff r\cos\theta = 1 \Rightarrow r = \frac{1}{\cos\theta}$

Exercice 04 :

* $y^2 = 2x$, $y = x$

$aire(D) = \iint_D dx\, dy$

$= \int_0^2 \left(\int_{\frac{y^2}{2}}^{y} dx \right) dy$



$x = y$

$x = \frac{y^2}{2} \Rightarrow \begin{array}{l} x = y \\ y = \frac{y^2}{2} \end{array} \Rightarrow y = 2$

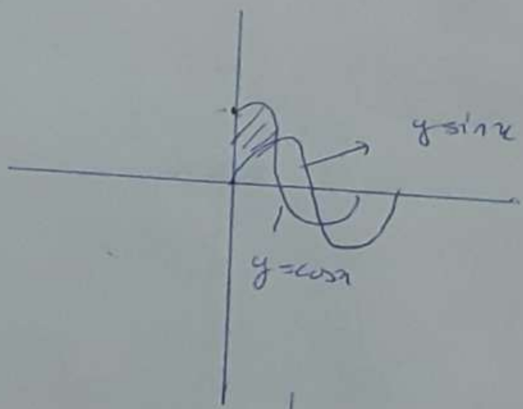
* $y = \sin x$, $y = \cos x$, $x = 0$

$y = \cos x$

$y = \sin x \iff y = \frac{\sqrt{2}}{2}$

$\cos x = \sin x \quad x = \pi/4$

$aire(D) = \int_0^{\pi/4} \left(\int_{\sin x}^{\cos x} dy \right) dx$



* $y^2 = 4x + 4$, $y^2 = -4x + 4$

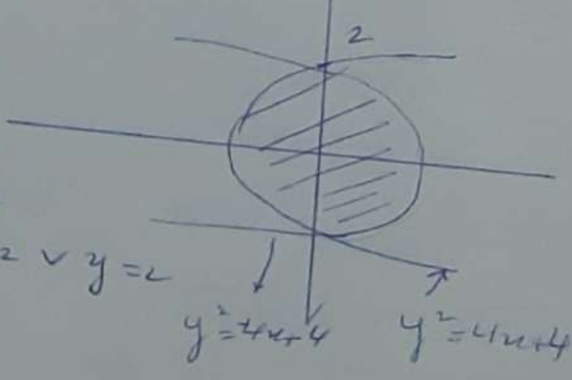
$aire(D) = \int_{-2}^{2} \int_{\frac{1}{4}(y^2-4)}^{-\frac{1}{4}(y^2-4)} dx\, dy$

$y^2 = 4x + 4$

$y^2 = -4x + 4 \Rightarrow y^2 = 4x + 4$

$4x + 4 = -4x + 4$

$\begin{cases} x = 0 \\ y = -2 \vee y = 2 \end{cases}$



* $y^2 = 2x$, $y^2 = 4x - x^2$

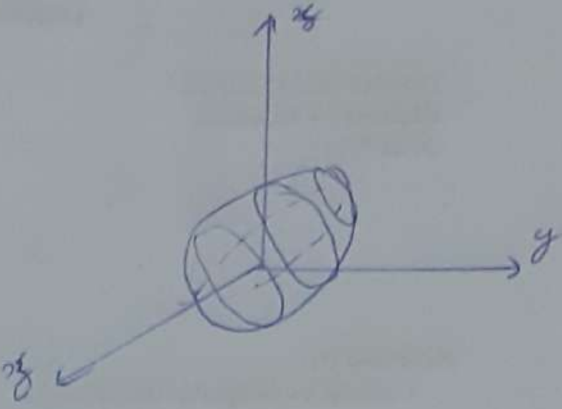
rcice 05 :

$\Omega = \{ (x,y,z) \in \mathbb{R}^3 / \quad \frac{x^2}{9} + \frac{y^2}{16} + \frac{z^2}{25} \leq 1 \}$

$V(\Omega) = \frac{4}{3}\pi R^3$

$V(\Omega) = \iiint_\Omega dx\,dy\,dz$

$= \iint_{①} \left(\int_{-5\sqrt{1-\frac{x^2}{9}-\frac{y^2}{16}}}^{+5\sqrt{1-\frac{x^2}{9}-\frac{y^2}{16}}} dz \right) dx\,dy$



$① = \{ (x,y) \in \mathbb{R}^2 / \quad \frac{x^2}{9} + \frac{y^2}{16} \leq 1 \}$
$z=0$

$2^{ème}$ méthode :

$\frac{x^2}{9} + \frac{y^2}{16} + \frac{z^2}{25} \leq 1 \iff \left(\frac{x}{3}\right)^2 + \left(\frac{y}{4}\right)^2 + \left(\frac{z}{5}\right)^2 \leq 1$

changement de variable : on pose :

$X = \frac{x}{3} \Rightarrow x = 3X \qquad \Omega' = \{ (X,Y,Z) \in \mathbb{R}^3 / \quad X^2+Y^2+Z^2 \leq 1 \}$

$Y = \frac{y}{4} \Rightarrow y = 4Y$

$Z = \frac{z}{5} \Rightarrow z = 5Z \qquad J = \begin{vmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 5 \end{vmatrix} = 60$

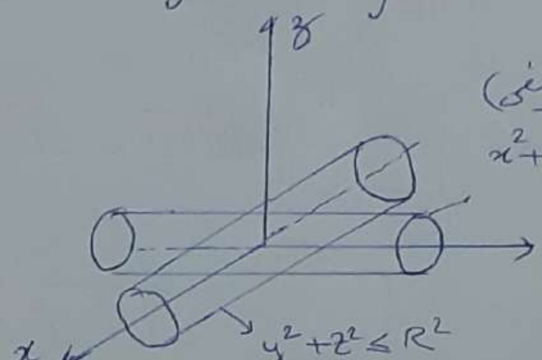
$V(\Omega') = \iiint_{\Omega'} dX\,dY\,dZ = 60 \iint_{①} \left(\int \quad dZ \right) dx\,dy$

$= 60\, V(\Omega') = 60 \cdot \frac{4}{3}\pi (1)^3 = 80\pi$

* $\Omega = \{ (x,y,z) \in \mathbb{R}^3 / \quad x^2+z^2 \leq R^2 \ \wedge \ y^2+z^2 \leq R^2 \}$

$V(\Omega) = \iiint_\Omega dx\,dy\,dz$

$= \int_{-R}^{R} S(z)\,dz$



$S(z) = \text{aire}\,(\Delta z)$
$= 4(R^2 - z^2)$

$\Delta z = \Omega \cap ($ المستوى z)

$\Delta z = \{ (x,y) \in \mathbb{R}^2 / \quad x^2 \leq R^2 - z^2 , \ y^2 \leq R^2 - z^2 \}$

$= \{ (x,y) \in \mathbb{R}^2 / \quad -\sqrt{R^2-z^2} \leq x \leq \sqrt{R^2-z^2}$
$\quad -\sqrt{R^2-z^2} \leq y \leq \sqrt{R^2-z^2}$

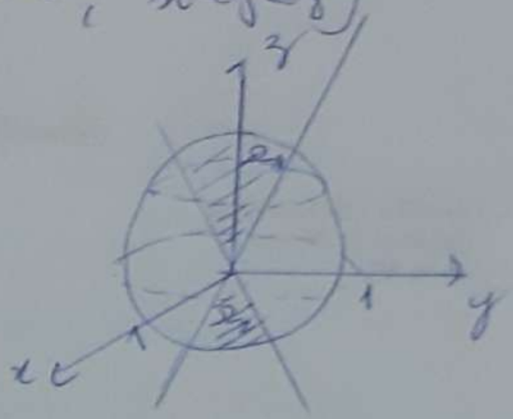
$V(\Omega) = \int_{-R}^{R} 4(R^2 - z^2)\,dz = \frac{16R^3}{3}$

$\{(x,y,z) \in \mathbb{R}^2 \,/\, x^2+y^2+z^2 \leq 1,\ x^2+y^2 \leq z^2\}$

$x^2+y^2 \leqslant z^2$

مساحة قرص

$\pi R^2 = \boxed{\pi z^2}$



$\Omega_1 \cup \Omega_2 = \Omega$

$x^2+y^2 \leq 1-z^2$

$\pi(1-z^2)$ مساحة

$$V(\Omega) = \iiint_\Omega dx\,dy\,dz = 2\iint_{\Omega_1} dx\,dy\,dz$$

$$= 2\int_0^1 S(z)\,dz = 2\left(\int_0^{\frac{\sqrt{2}}{2}} \pi z^2\,dz + \int_{\frac{\sqrt{2}}{2}}^1 \pi(1-z^2)\,dz\right)$$

$x^2+y^2+z^2=1$

$x^2+y^2=z^2 \Rightarrow z^2 = \frac{1}{2}$

$$= 2\pi(2-\sqrt{2})$$

Exercicio 06 i

$V = \{(x,y,z) \in \mathbb{R}^3 \,/\, x \geq 0,\ y \geq 0,\ z \geq 0,\ z \leq 1-y^2,\ x+y \leq 1\}$

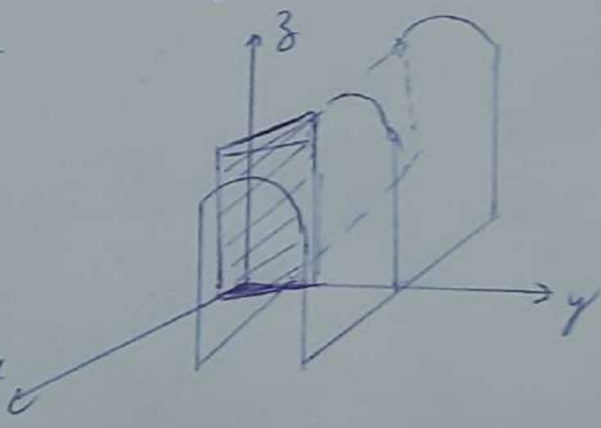
$\iiint_V z\,dx\,dy\,dz$

$z = 1-y^2$

$x \in \mathbb{R}$

$x+y=1$

$z \in \mathbb{R}$

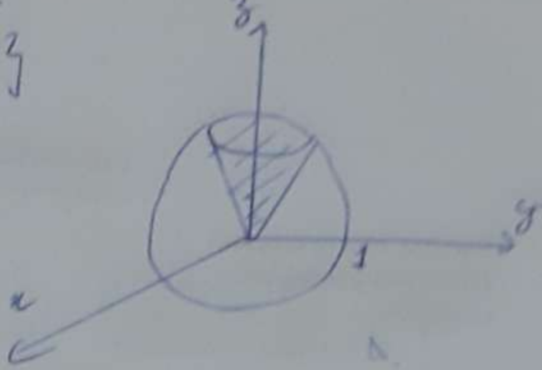


$$\iiint_V z\,dx\,dy\,dz$$

$$= \int_0^1 \left(\int_0^{1-x} \left(\int_0^{1-y^2} z\,dz \right) dy \right) dx$$

$$= \frac{11}{60}$$

$= \{(x,y,z) \in \mathbb{R}^3 / \quad 0 < x^2+y^2+z^2 < 1$
$0 \le x^2+y^2 \le z^2 , z > 0\}$



$$= \iiint_V \left(\frac{1}{\sqrt{x^2+y^2}} + \frac{1}{z}\right) dx\,dy\,dz$$

$$= \int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^1 \left(\frac{1}{r\sin\varphi} + \frac{1}{r\cos\varphi}\right) r^2 \sin\varphi \, dr\, d\varphi\, d\theta$$

$x = r\cos\theta \sin\varphi$
$y = r\sin\theta \sin\varphi$
$z = r\cos\varphi$

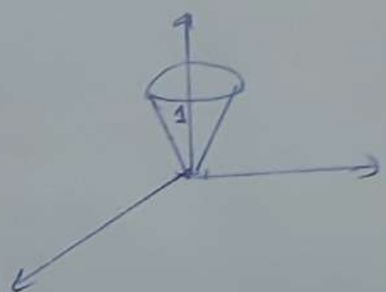
$x^2+y^2 = r^2\sin^2\varphi$
$J = r^2\sin\varphi$

$$V = \int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^1 (r\sin\varphi + r\,tg\varphi)\, d\varphi\, d\theta\, dr$$
$$= \frac{\pi}{4} - \pi \ln\left(\frac{\sqrt{2}}{2}\right)$$

* $V = \{(x,y,z) \in \mathbb{R}^3 / \quad 0 < z < 1 , \quad x^2+y^2 < z^2\}$

$\iiint_V xyz \, dx\,dy\,dz$.

$x^2+y^2 = z^2$
$r^2 = z^2 \Rightarrow r = z$



$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$ coordonnées cylindriques

$$\iiint_V \text{——} = \int_0^{2\pi}\left(\int_0^1\left(\int_0^z (r^3\cos\theta\sin\theta\, dr\right) dz\right) d\theta = 0$$