Moment Calculation about a Specified Axis

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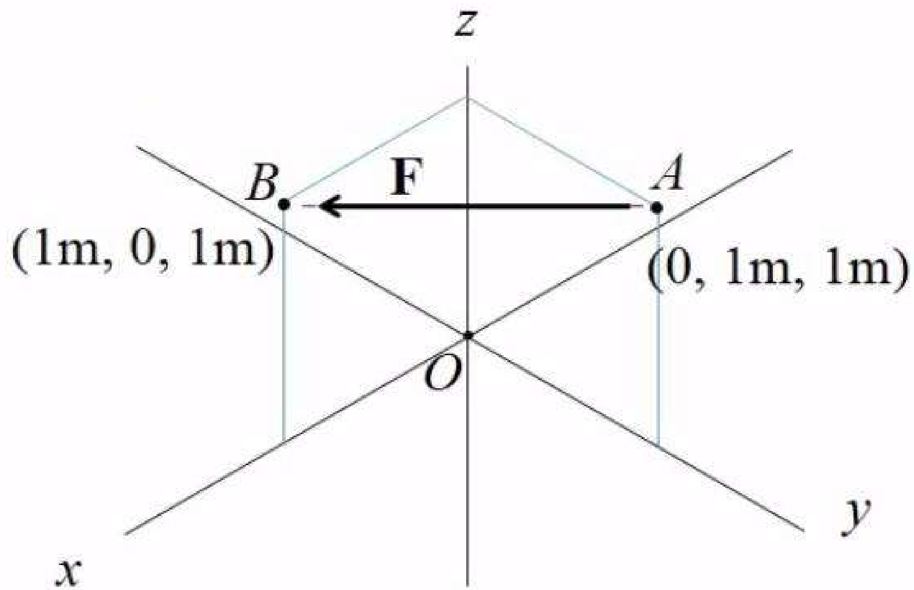
Objectives :

- To determine the moment caused by a force about a specified axis.
- To compare the moment about a specified axis to the projection of a force.

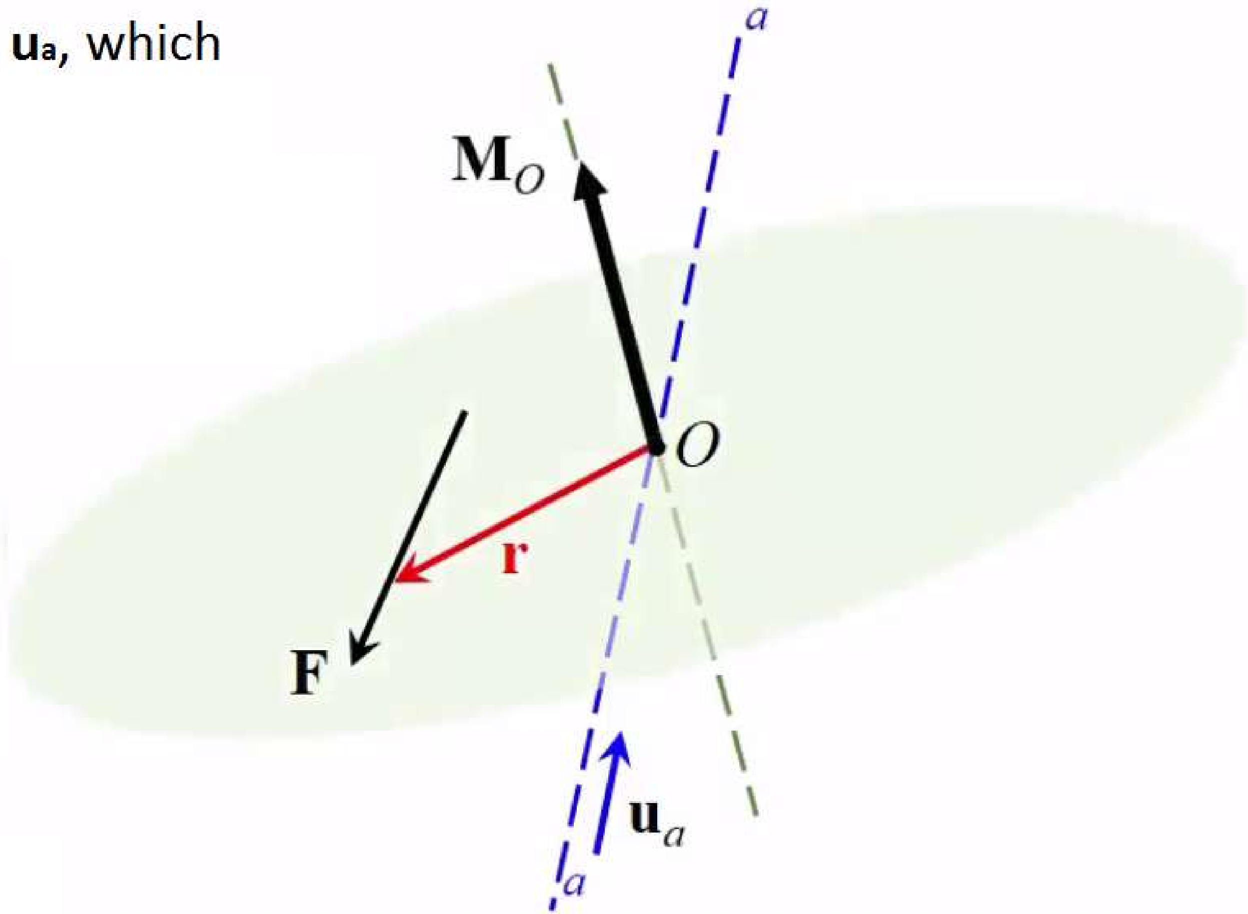
Question 1: If the 100-N force **F** is directed from point *A* to point *B* as shown, what are the moments it causes about the *x*, *y* and *z* axes respectively? Try work it out yourself first. But if you need a hint: what's the Cartesian vector moment of **F** about the origin point *O* and what do the three components (including +/− sign) physically mean?



In a specified axis (aa), the unit vector is $\mathbf{u}_a$, which specifies the direction of the axis.

To determine **a moment** caused by force **F** about this particular axis, we can draw **an arbitrary position vector r**, as long as **it starts** from **an arbitrary point** *O* an the axis and ends **anywhere** on the **line of action** of force **F**.

$$\mathbf{M}_O = \mathbf{r} \times \mathbf{F}$$



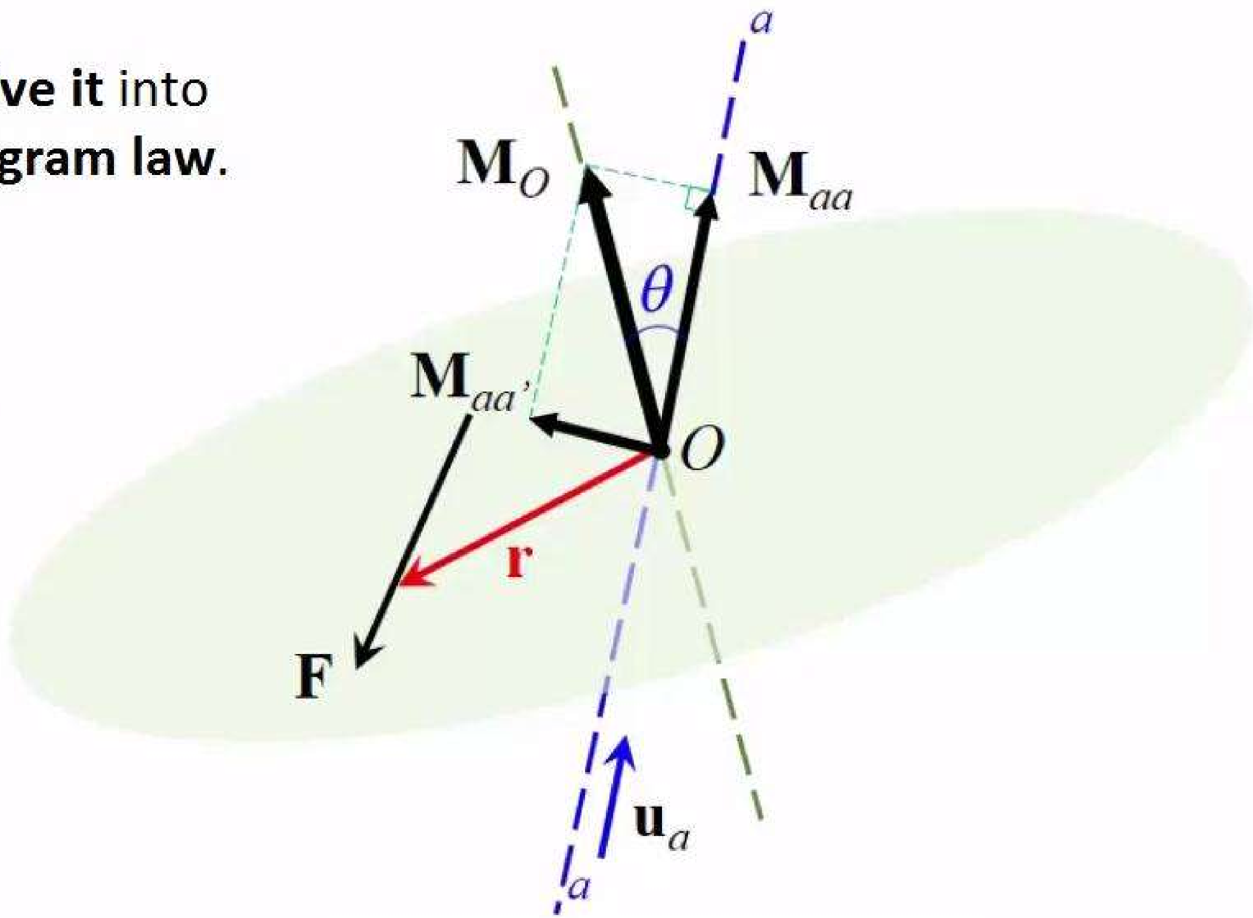
The direction of this moment is not necessarily along the (aa) axis as we want it

Since moment is a vector, we can **resolve it** into components according to the **parallelogram law**.

$\mathbf{M}_{aa}$: is along the axis.

$\mathbf{M}_{aa}$: is the projection of the moment along the axis.

$$M_{aa} = M_{o} \cdot \cos\theta$$



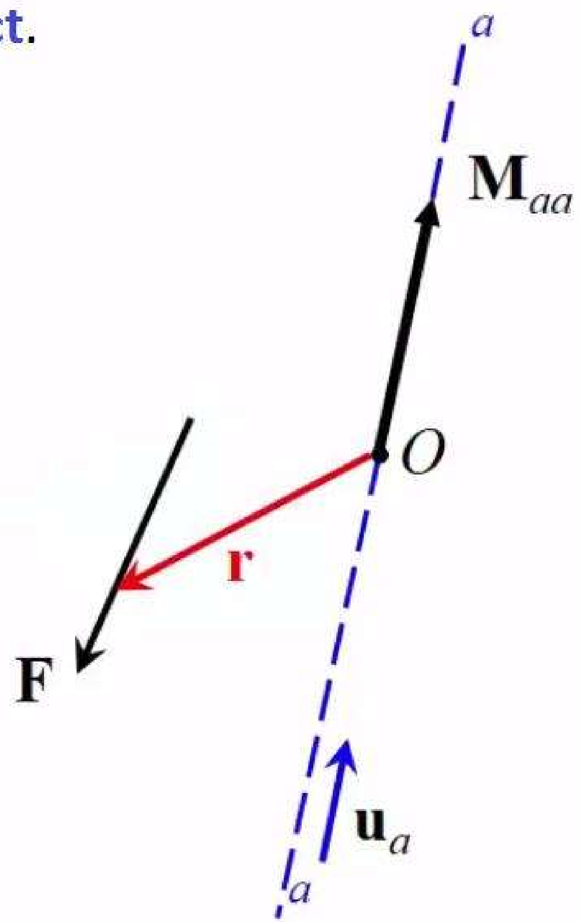
Just like finding **projection** of a force, we can also use **dot product**.

$$M_{aa} = \mathbf{u}_a \cdot \mathbf{M}_O$$

Or more directly:

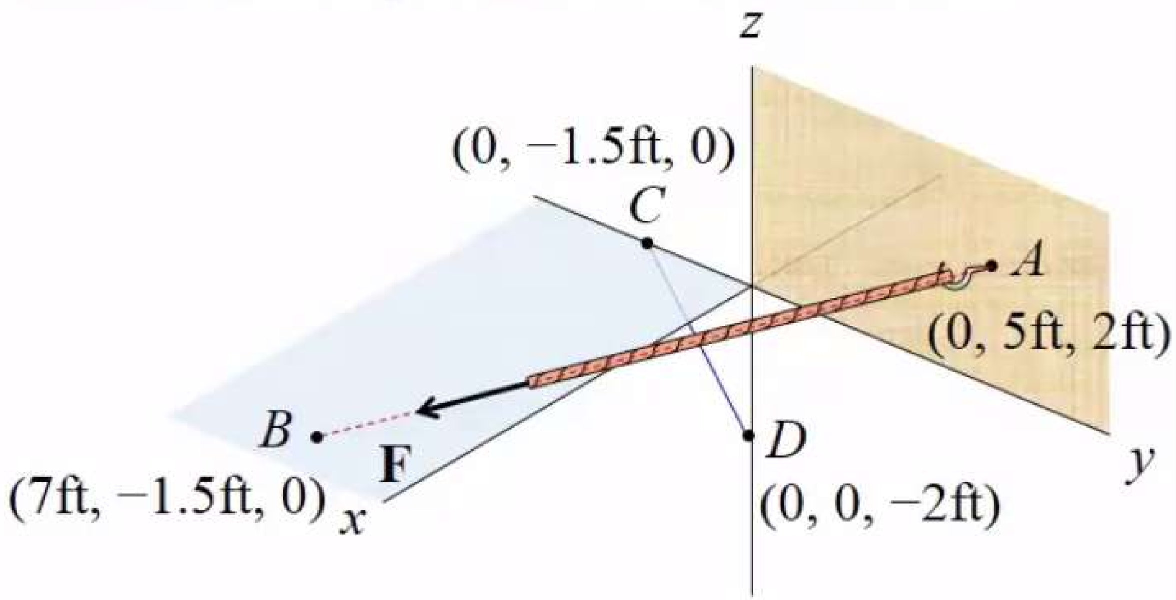
$$M_{aa} = \mathbf{u}_a \cdot (\mathbf{r} \times \mathbf{F}) = \begin{vmatrix} u_{a_x} & u_{a_y} & u_{a_z} \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{vmatrix}$$

$$= (r_y F_z - r_z F_y) u_{a_x}$$
$$- (r_x F_z - r_z F_x) u_{a_y}$$
$$+ (r_x F_y - r_y F_x) u_{a_z}$$



If you want to find the moment ***Maa*** as a vector: $\mathbf{M}_{aa} = M_{aa}\mathbf{u}_a$

Example: The line of action of force $\mathbf{F}$ directs from point A to point B. If the magnitude of the force is 120 lb, determine the magnitude of the moment caused by $\mathbf{F}$ about the CD axis.



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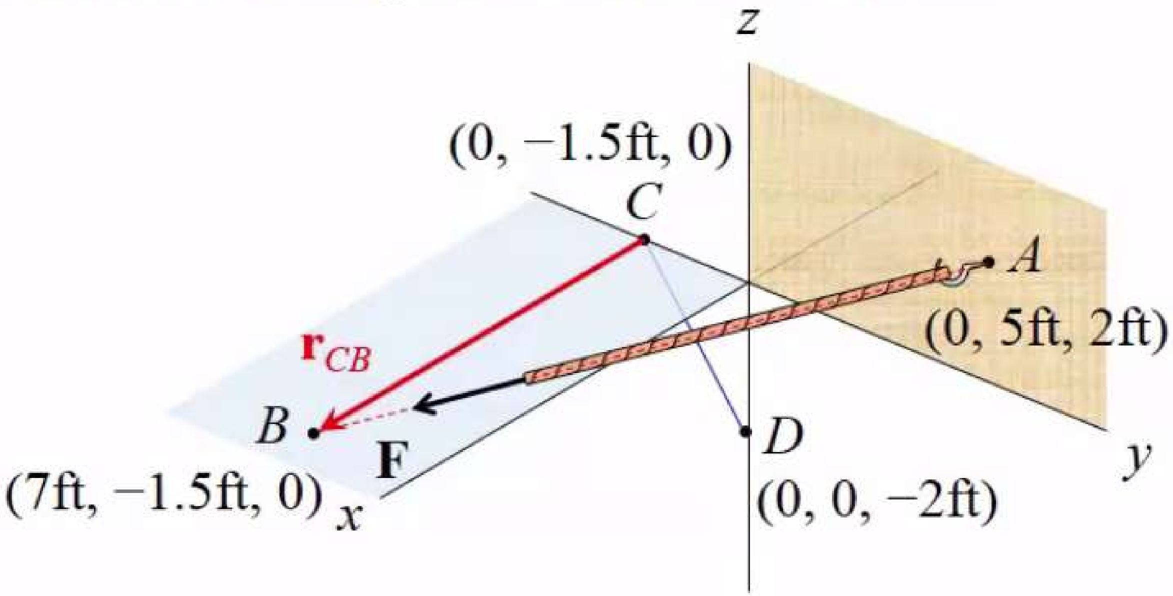


Force vector:

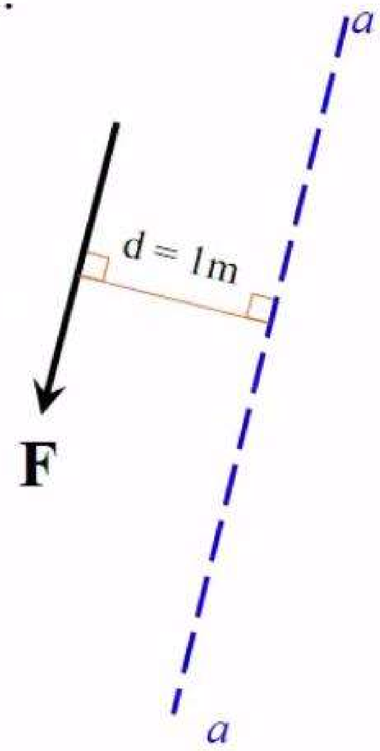
$$\mathbf{F} = \{86.1\mathbf{i} - 79.9\mathbf{j} - 24.6\mathbf{k}\}\ \text{lb}$$

Position vector: $\mathbf{r}_{CB} = 7\mathbf{i}\ \text{ft}$

Unit vector: $\mathbf{u}_{CD} = 0.6\mathbf{j} - 0.8\mathbf{k}$

$$M_{CD} = \mathbf{u}_{CD} \cdot (\mathbf{r}_{CB} \times \mathbf{F}) = \begin{vmatrix} 0 & 0.6 & -0.8 \\ 7 & 0 & 0 \\ 86.1 & -79.9 & -24.6 \end{vmatrix} = 551\,(\text{lb}\cdot\text{ft})$$ Ans.

Question 2: What is the moment of the 100-N force F about the *aa* axis when the force is parallel to the axis and the perpendicular distance between them is 1 meter.

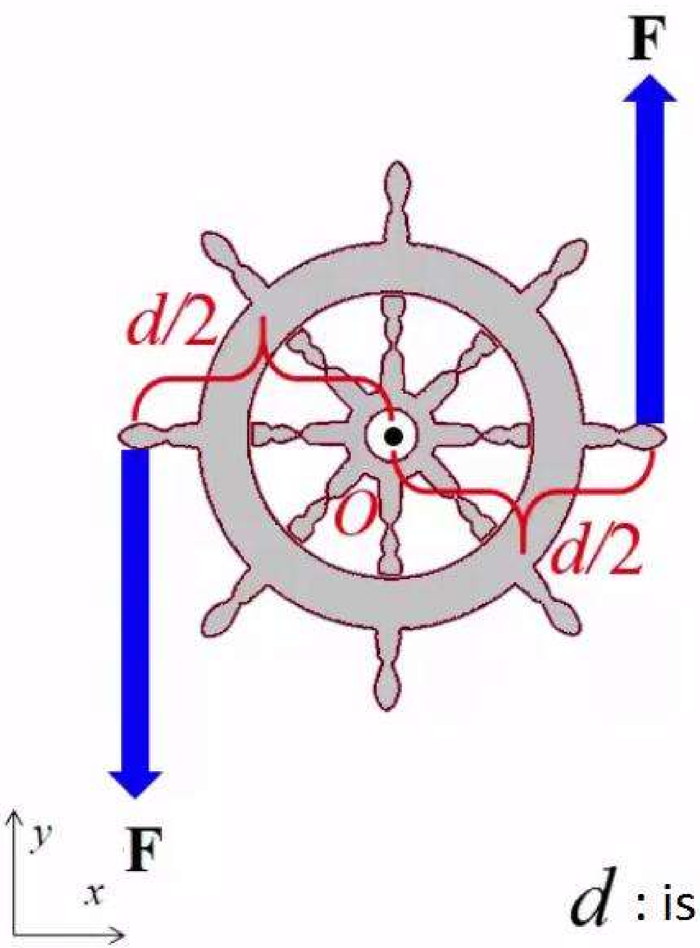


Moment of a Couple

Objectives :

- To define and explain the moment of a couple.
- To demonstrate the different calculation methods of couple moment through an example.

Question 1: When you are driving, how do you position your hands on the driving wheel? (Please be specific.) In your opinion why is that an optimal positioning?



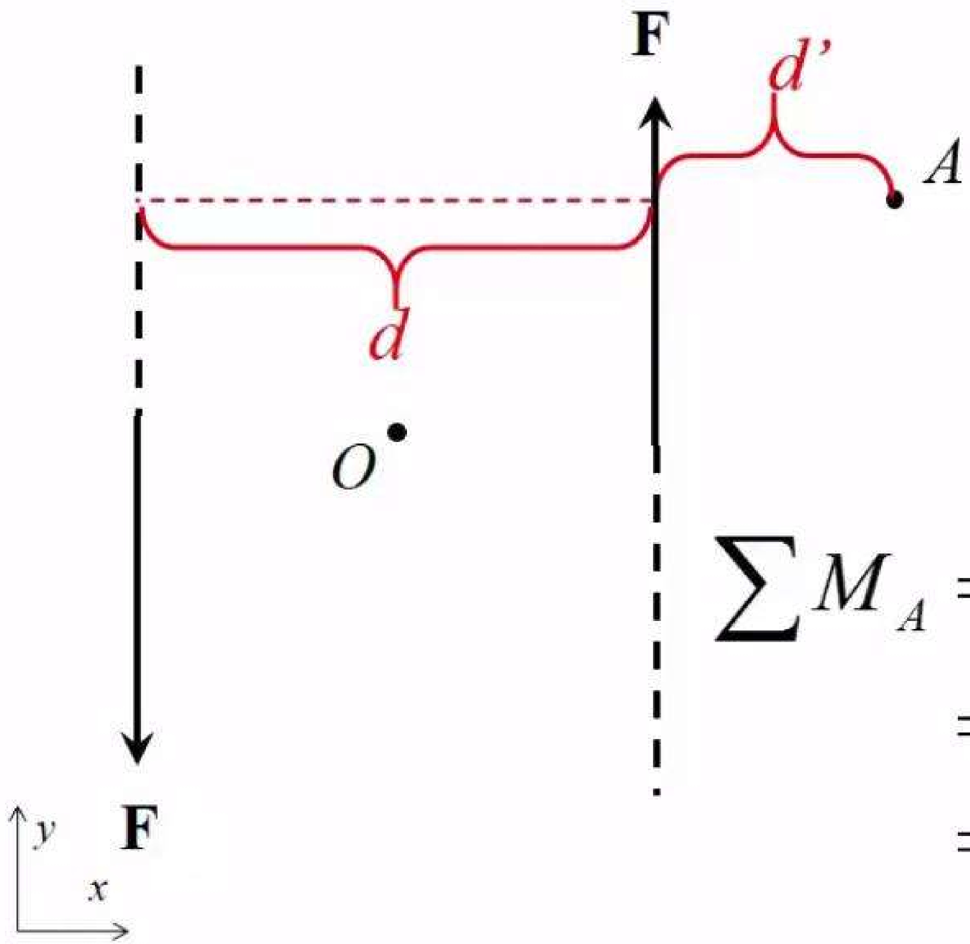
$$\sum F_y = F + (-F) = 0$$

The forces cancel each other out, therefore, have **no translational effect** on the wheel.

These two forces are known as a couple.

$$\sum M_O = F \cdot \frac{d}{2} + F \cdot \frac{d}{2}$$
$$= F \cdot d$$

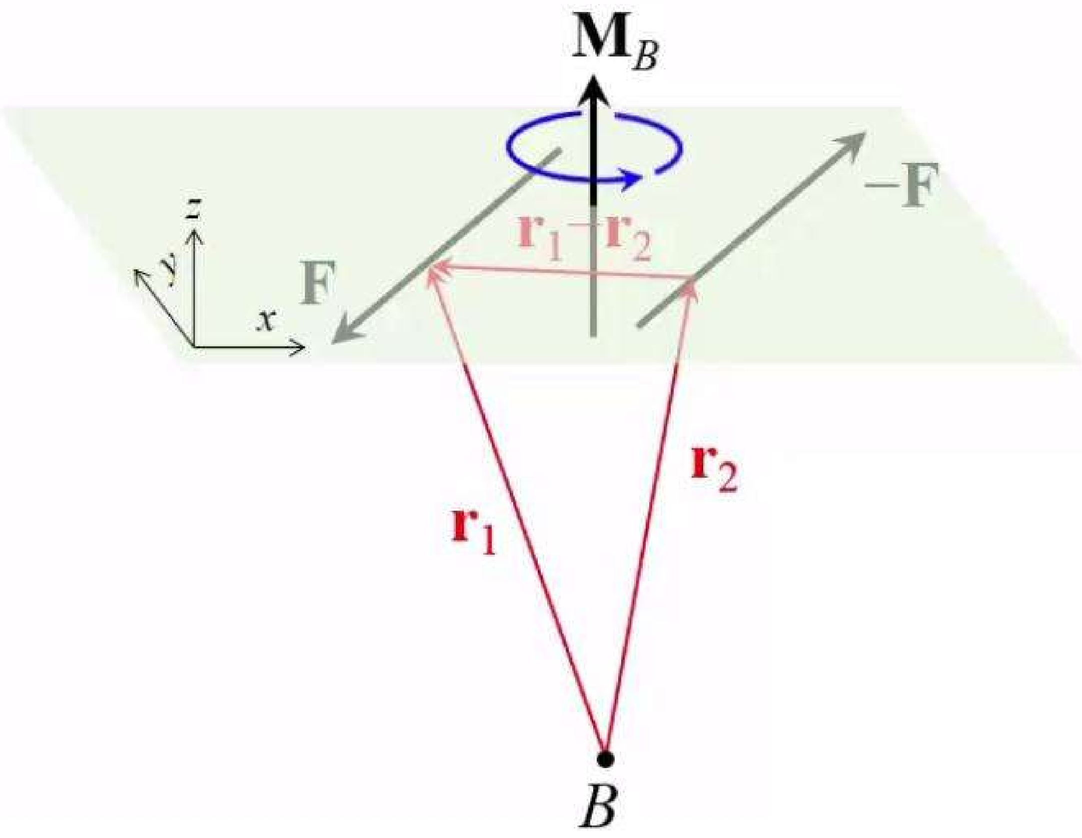
d : is the **perpendicular distance** between the lines of action of these two forces.



$$\sum M_A = F \cdot (d + d') - F \cdot d'$$
$$= F \cdot d$$
$$= \sum M_O$$

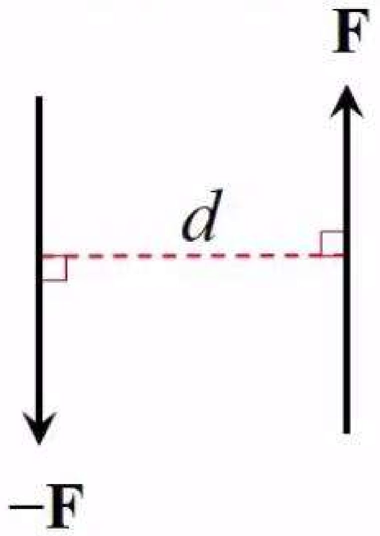
If we want to calculate the **total moment** caused by these **two forces** about point *B*, that is **not** even in the current xy plane,
We use **Vector Formulation**:

$$\sum \mathbf{M}_B = \mathbf{r}_1 \times \mathbf{F} + \mathbf{r}_2 \times (-\mathbf{F})$$
$$= (\mathbf{r}_1 - \mathbf{r}_2) \times \mathbf{F}$$
$$= (F \cdot d)\mathbf{k}$$



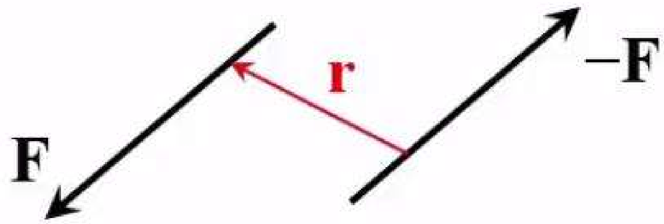
- The moment of a couple is a free vector because it does not depend on the reference point.
- The net external effect of a couple is that the net force equals zero and the magnitude of the net moment equals $F{\cdot}d$.
- Moments of couples are also vectors.

Scalar formulation



$$M = F \cdot d$$

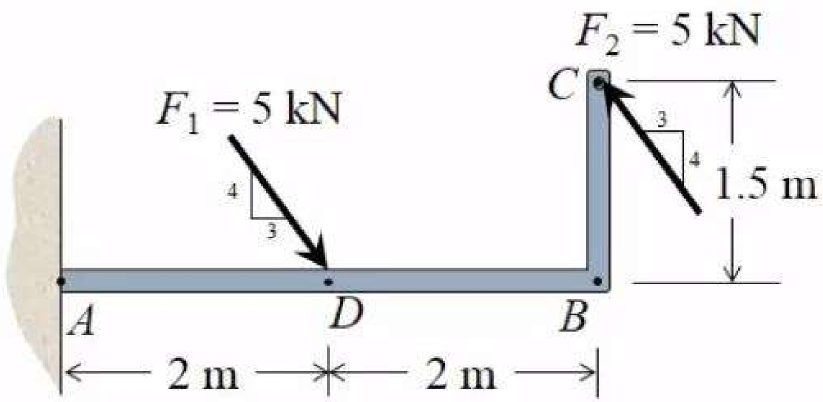
Vector formulation



$$\mathbf{M} = \mathbf{r} \times \mathbf{F}$$

We need to determine if the moment is positive or negative based on if the rotational effect is **counterclockwise** or **clockwise**.

Example: Determine the magnitude of the applied couple moment.
Neglect the thickness of the member.



Example: Determine the magnitude of the applied couple moment. Neglect the thickness of the member.

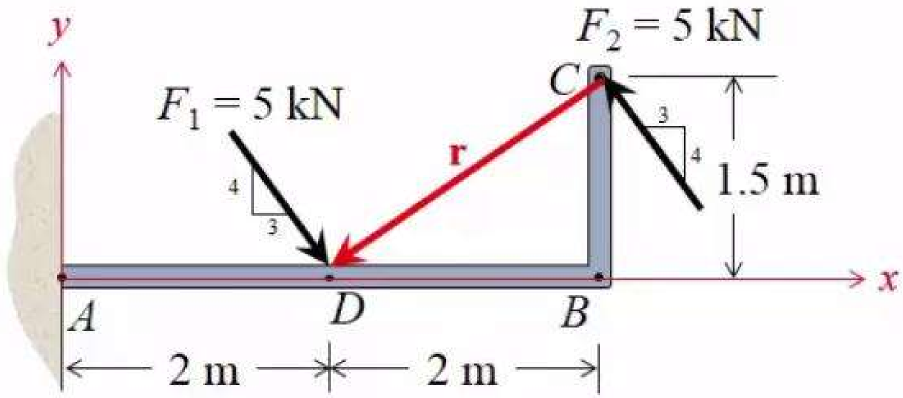
Vector formulation:

$$\mathbf{F}_1 = \{3\mathbf{i} - 4\mathbf{j}\}\ \text{kN}$$

$$\mathbf{r} = \{-2\mathbf{i} - 1.5\mathbf{j}\}\ \text{m}$$

$$\mathbf{M} = \mathbf{r} \times \mathbf{F}_1 = 12.5\mathbf{k}\ \text{kN} \cdot \text{m}$$

$$M = 12.5\ \text{kN} \cdot \text{m}$$ Ans.



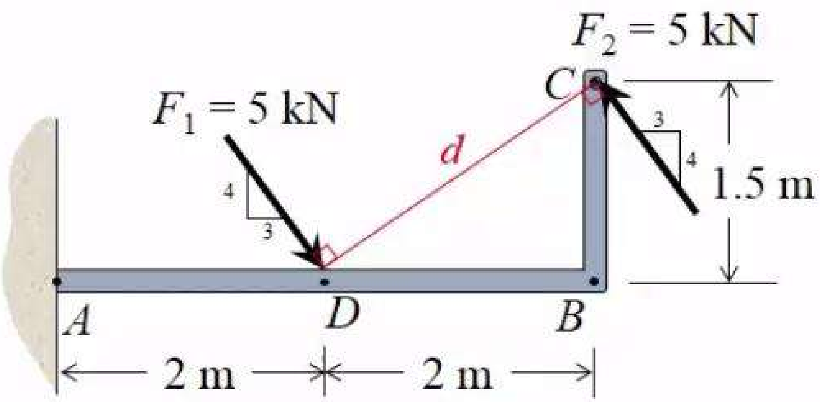
Example: Determine the magnitude of the applied couple moment.
Neglect the thickness of the member.

Scalar formulation:

$F = 5\ \text{kN}$

$d = 2.5\ \text{m}$

$M = F \cdot d = 12.5\ \text{kN} \cdot \text{m}$ Ans.

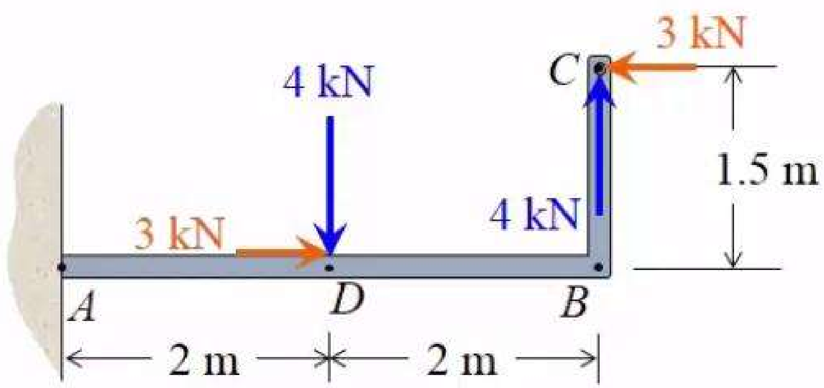


Example: Determine the magnitude of the applied couple moment. Neglect the thickness of the member.

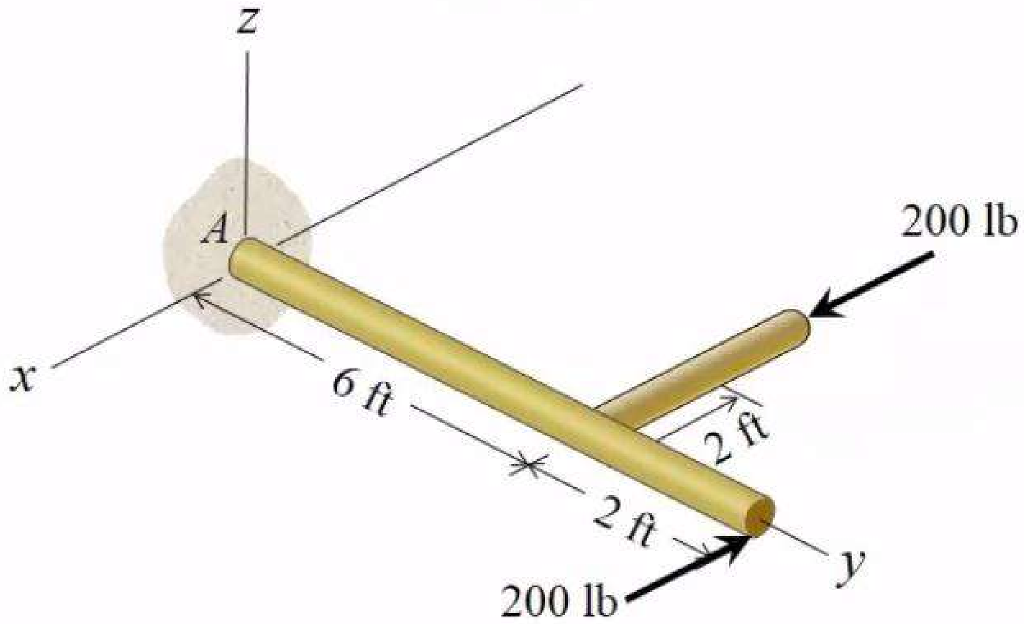
Principle of moments:

$$M = 4\ \text{kN} \cdot 2\ \text{m} + 3\ \text{kN} \cdot 1.5\ \text{m}$$

$$= 12.5\ \text{kN} \cdot \text{m}$$ Ans.



Question 2: What is the shown couple moment in Cartesian vector form?



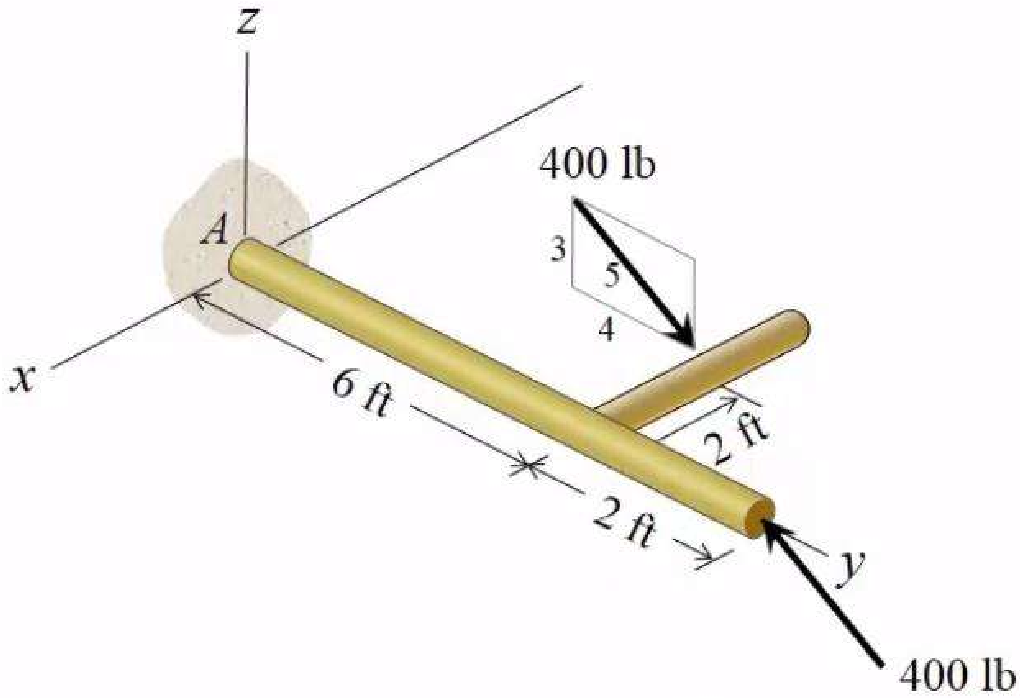
(a) $\{400\mathbf{k}\}\text{lb}\cdot\text{ft}$

(b) $\{-400\mathbf{k}\}\text{lb}\cdot\text{ft}$

(c) $\{800\mathbf{k}\}\text{lb}\cdot\text{ft}$

(d) $\{400\mathbf{j}+400\mathbf{k}\}\text{lb}\cdot\text{ft}$

Question 3: What is the shown couple moment in Cartesian vector form?



(a) $\{960\mathbf{i} - 960\mathbf{j} - 1280\mathbf{k}\}\,\text{lb}\cdot\text{ft}$

(b) $\{-480\mathbf{j} - 640\mathbf{k}\}\,\text{lb}\cdot\text{ft}$

(c) $\{-960\mathbf{j} - 1280\mathbf{k}\}\,\text{lb}\cdot\text{ft}$

(d) $\{480\mathbf{i} - 480\mathbf{j} - 640\mathbf{k}\}\,\text{lb}\cdot\text{ft}$

Simplification of Force and Moment System

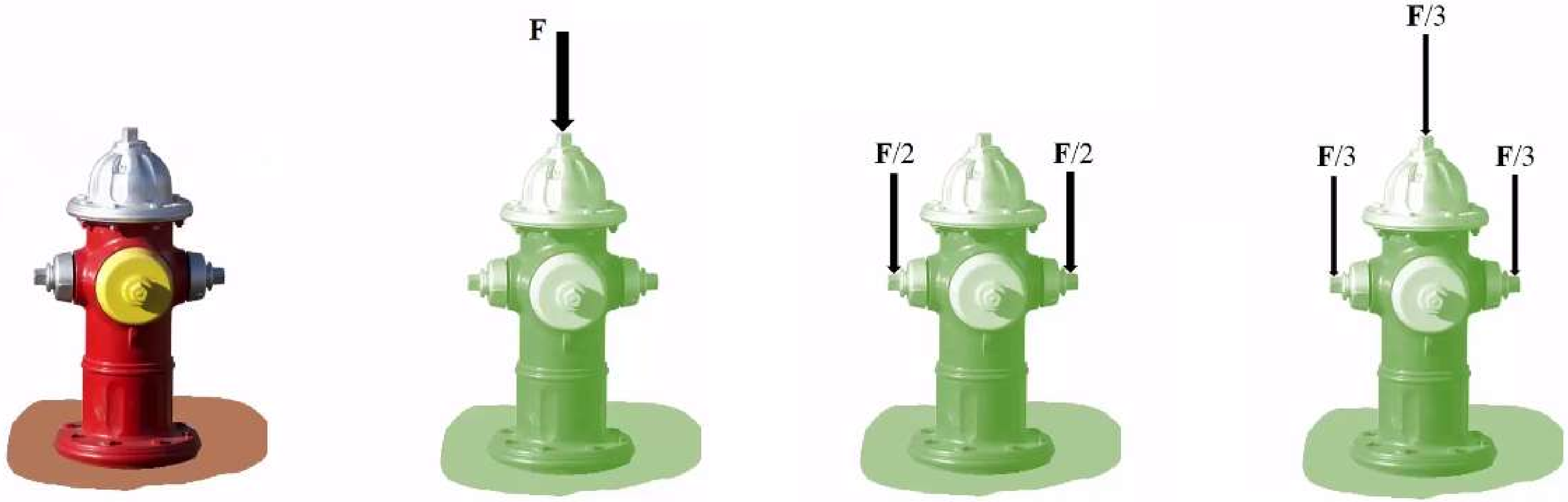
Dr. Abdelhak Khechai

Objectives :

- To calculate the resultant force and resultant moment of a given multiple-force-moment system.
- To replace the original multiple-force-moment system by its equivalent single-force-moment system, or a single-force-only system in some cases.

Question 1: In particle equilibrium you've learned how to find the resultant force of multiple forces (and subsequently apply Newton's 1st law to solve problems). Similarly, for a rigid body subjected to forces AND moments, how can you find the resultant moment? For any given system, is there only one correct resultant moment?

Imagine this fire hydrant, fixed to the ground, and there is a force **F** acting on it (Fig.1).



From experience, we can say that if we replace this force **F** by **two forces** (Fig.2), each **with half the magnitude**, placed **symmetrically** about the **central axis**, these two forces will create the same effect as the original **F** force. Even if we replace the forces by these three (Fig.3), again, they create the same effect.

By the **same effect**, I mean that the forces create the **same tendency to push** the fire hydrant down, and also, the **ground** will generate **the same** force **to support** the fire hydrant, preventing it from going down.

These several force systems are known to be equivalent systems.

Now, let's imagine the force **F** acts on the fire hydrant this way. Now the force creates a **translational tendency to push** it to the right, and also it creates **a clockwise rotational tendency** for the fire hydrant **to fall** to the right.

For this fire hydrant to **stay still**, as a response, the ground must **create a force** supporting the fire hydrant, pointing to the left, and also a moment to cancel out **the rotational effect**.

We can add **a pair of cancelling forces** to this fire hydrant without changing the load **status**.

These two forces now create a **Couple moment.**

Now this force still provides a **translational tendency** to push the fire hydrant to the right, while **the couple moment** creates the **clockwise rotational effect**.

So, in order to keep the fire hydrant **statics**, the ground must create **a force pointing to the left**, and **counterclockwise moment** to cancel out the rotational effect.

This **force-moment system** is the **equivalent system** as the previous **single force system**.

Equivalent system

- A system is equivalent if the external effects it produces on a body are the same as those caused by the original force and couple moment system.

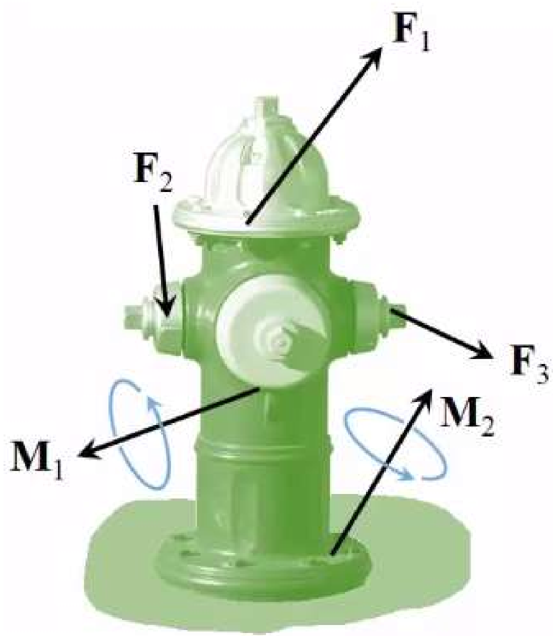
- A load with multiple forces and couple moments acting on multiple locations can be replaced by a single force and a single couple moment acting on a single point.

We want to do so to help calculate the support reactions.

Let's imagine the fire hydrant subjected to **multiple forces** and **multiple moments** acting on **multiple points**.

We want to replace all of these by **a single force** and **a single moment** placed at **a certain point** say point *O*.

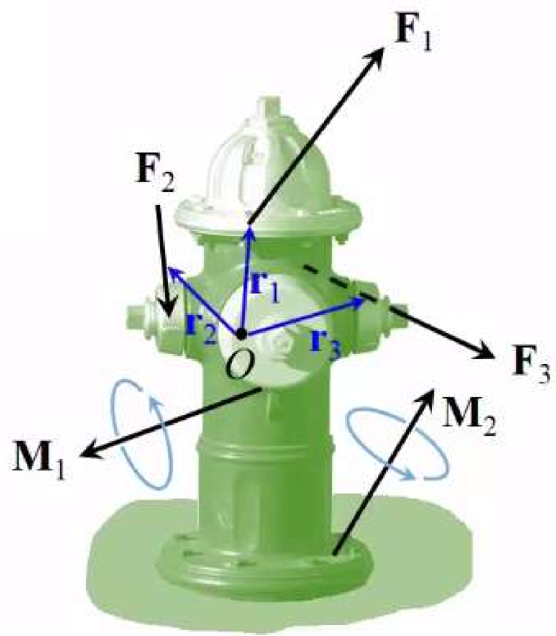
The single force is simply **the resultant force**:



$$\mathbf{F}_R = \sum \mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3$$

For the resultant moment, we need to first calculate **the individual moment** caused by **each force** about point *O*, add them together:

$$\sum \mathbf{M}_{F,O} = \mathbf{r}_1 \times \mathbf{F}_1 + \mathbf{r}_2 \times \mathbf{F}_2 + \mathbf{r}_3 \times \mathbf{F}_3$$

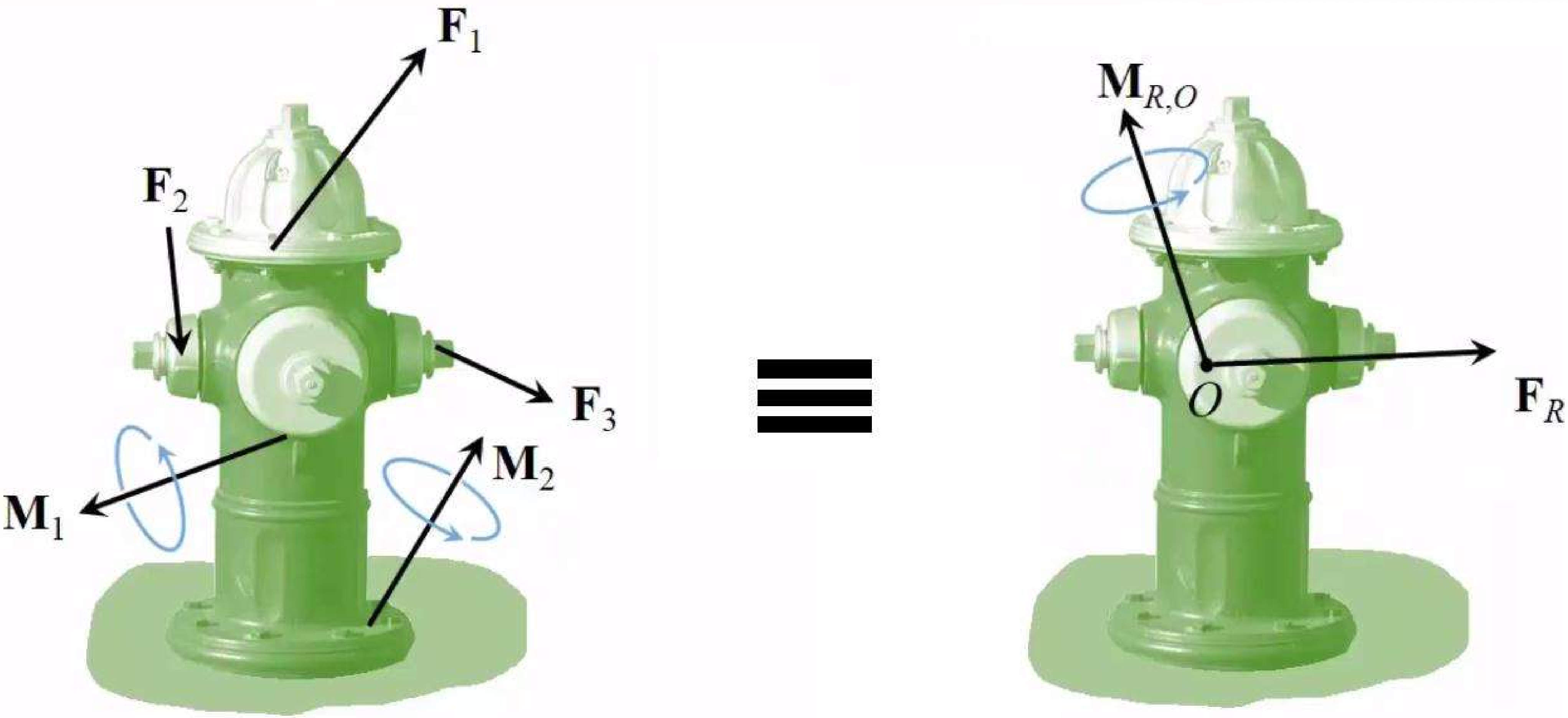


Then, add all of the free couple moments together **M1** and **M2**:

$$\sum \mathbf{M} = \mathbf{M}_1 + \mathbf{M}_2$$

Then, we add **the total moment caused by the forces** and the **couple moments** together:

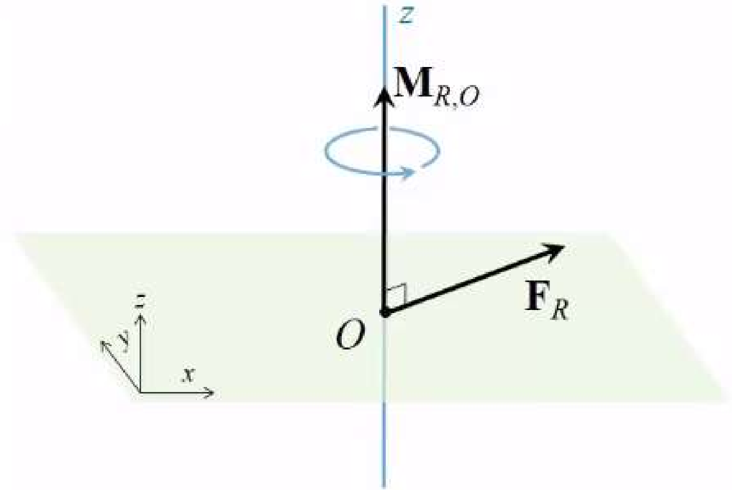
$$\mathbf{M}_{R,O} = \sum \mathbf{M}_{F,O} + \sum \mathbf{M}$$



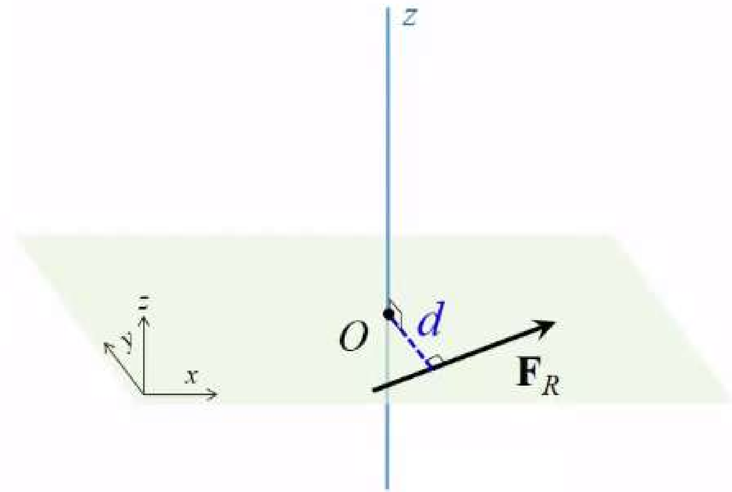
We replaced the **original multi-force multi-moment load system** by **a single force single moment system**.

In some special situation, the **resultant force vector** and **the resultant moment vector** are perpendicular to each other.

We can further reduce the moment by placing the force away from point *O*, say at distance *d*:



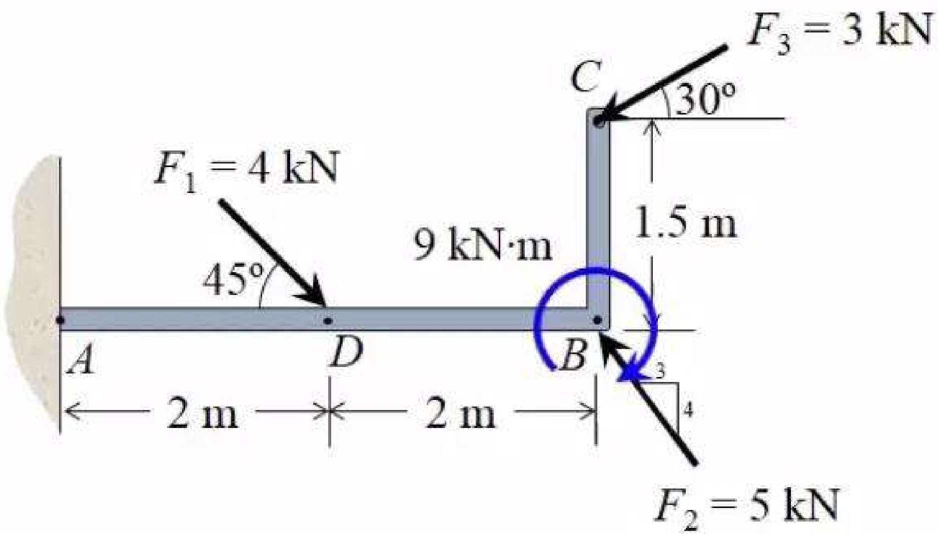
$$d = \frac{M_{R,O}}{F_R}$$



The reason is because, this way, the resultant force is creating a moment about point *O* that equals to :

$$M_{R,O} = F_R\, d$$

Example: Replace the shown force-moment system with an equivalent single force placed on the AB segment. Neglect the thickness of the member.

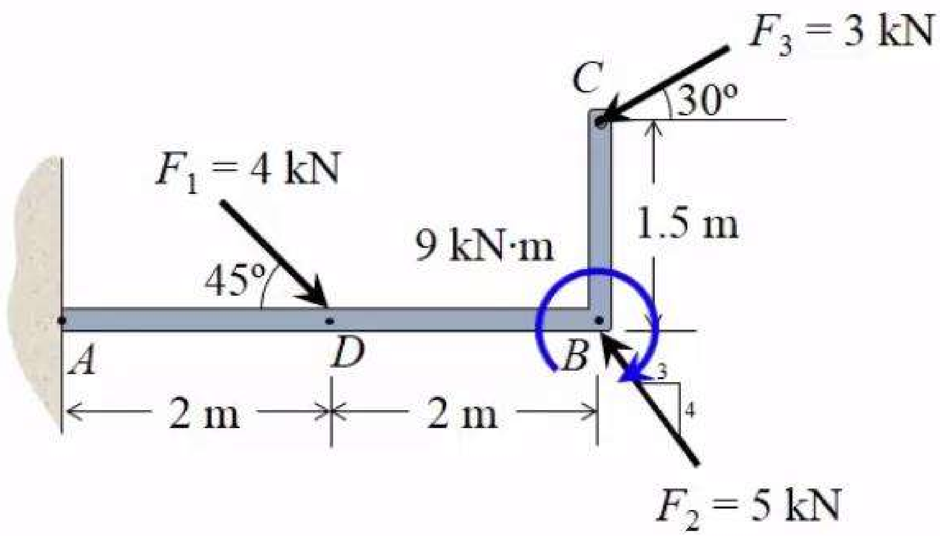


Example: Replace the shown force-moment system with an equivalent single force placed on the AB segment. Neglect the thickness of the member.

A force & a couple moment:

$$\mathbf{F}_R = \sum \mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3$$

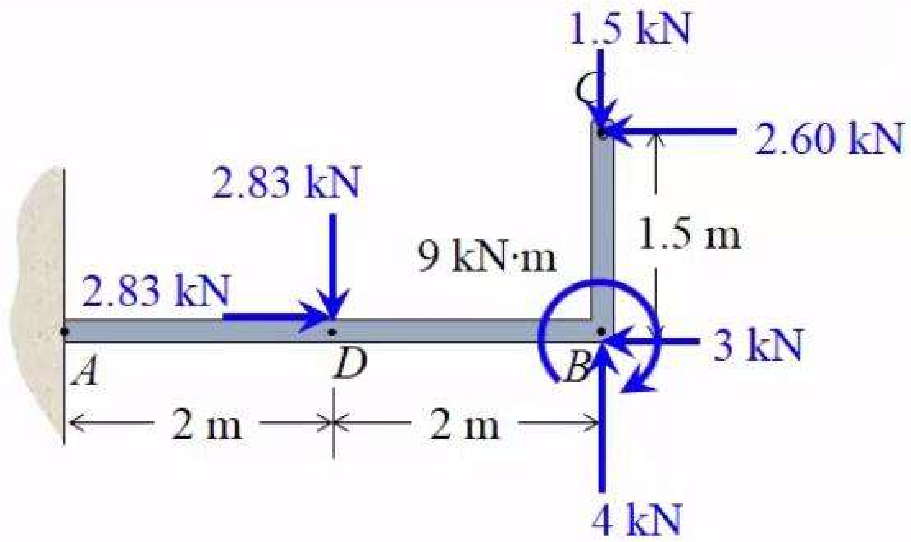
$$\mathbf{M}_{R,A} = \sum \mathbf{M}_{F,A} + \sum \mathbf{M}$$



$$\xrightarrow{+} F_x = 2.83 - 2.60 - 3 = -2.77\ (\text{kN})$$

$$+\uparrow F_y = -2.83 - 1.5 + 4 = -0.33\ (\text{kN})$$

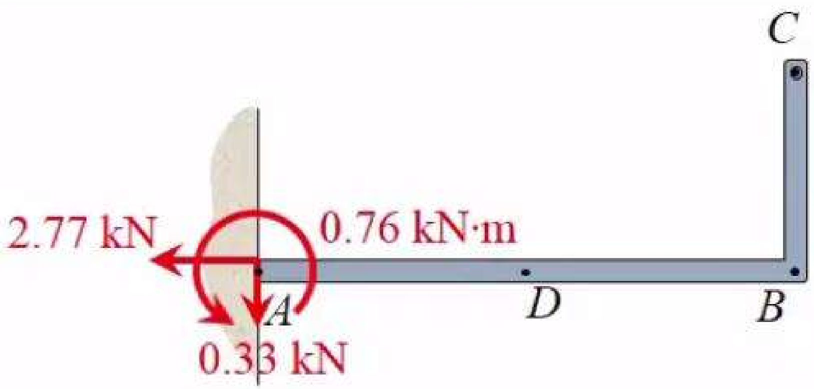
$$\circlearrowleft + M_A = 8.24 - 9 = -0.76\ (\text{kN} \cdot \text{m})$$



$$\xrightarrow{+} F_x = -2.77\ \text{kN}$$

$$+\uparrow F_y = -0.33\ \text{kN}$$

$$\circlearrowleft + M_A = -0.76\ \text{kN} \cdot \text{m}$$



$$\xrightarrow{+} F_x = -2.77 \text{ kN}$$

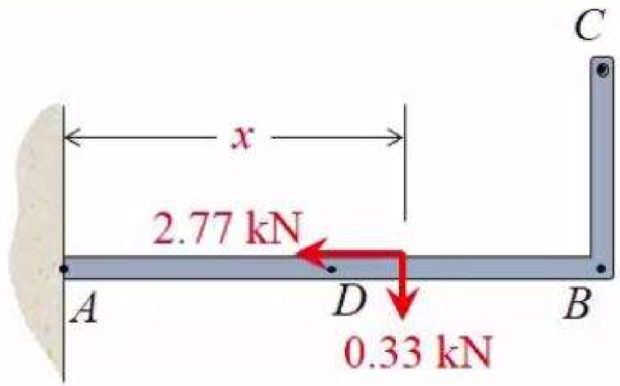
$$+\uparrow F_y = -0.33 \text{ kN}$$

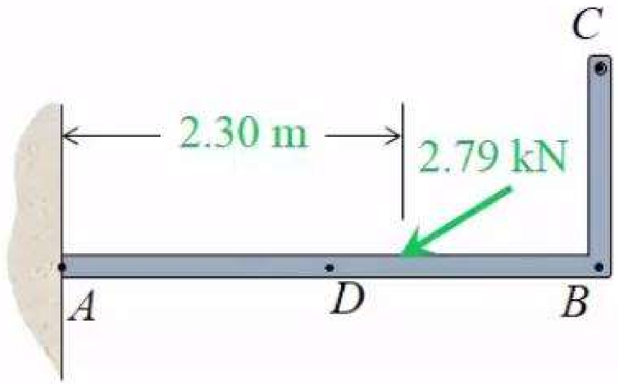
$$\circlearrowleft + M_A = -0.76 \text{ kN}\cdot\text{m}$$

$$x = \left|\frac{M_A}{F_y}\right| = \left|\frac{-0.76}{-0.33}\right| = 2.30\,(\text{m})$$

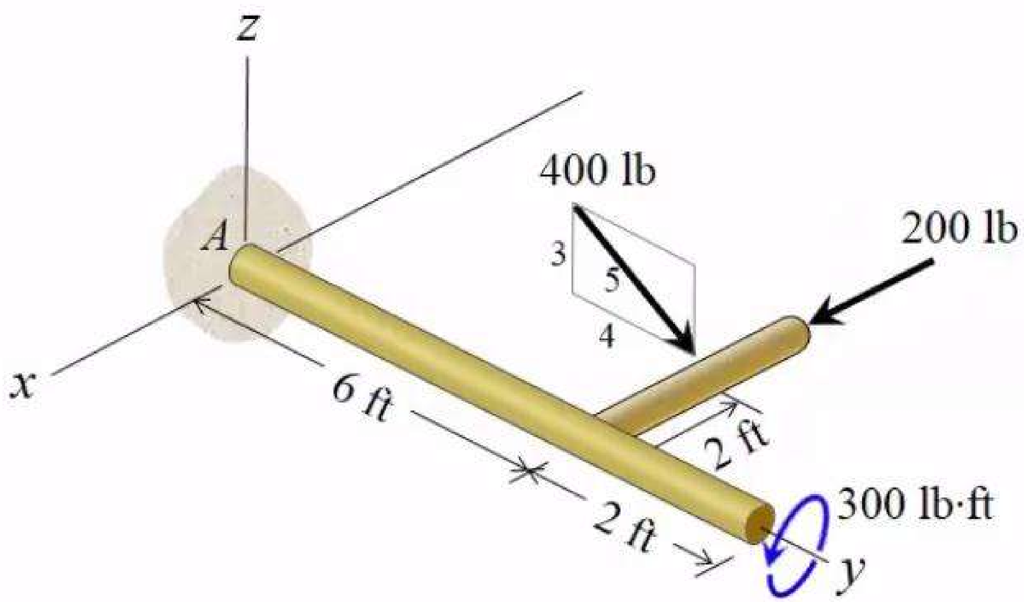
$$F = \sqrt{F_x^2 + F_y^2} = 2.79\,(\text{kN})$$

Ans.





Question 2: What is the resultant **applied** force (not including support) in Cartesian vector form?



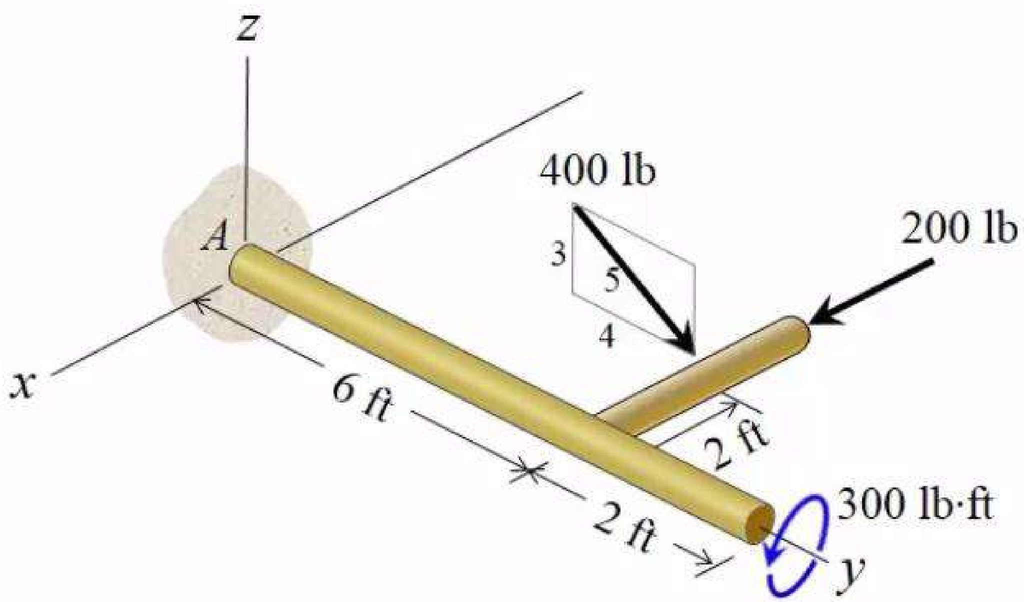
(a) $\{200\mathbf{i} - 20\mathbf{j} - 240\mathbf{k}\}$ lb

(b) $\{200\mathbf{i} - 320\mathbf{j} - 240\mathbf{k}\}$ lb

(c) $\{200\mathbf{i} + 620\mathbf{j} - 240\mathbf{k}\}$ lb

(d) $\{200\mathbf{i} + 320\mathbf{j} - 240\mathbf{k}\}$ lb

Question 3: What is the resultant **applied** moment (not including support) about point A in Cartesian vector form?



(a) $\{-1440\mathbf{i}-480\mathbf{j}-1840\mathbf{k}\}\text{lb}\cdot\text{ft}$ (b) $\{-1440\mathbf{i}-480\mathbf{j}-640\mathbf{k}\}\text{lb}\cdot\text{ft}$

(c) $\{-1440\mathbf{i}-180\mathbf{j}-1840\mathbf{k}\}\text{lb}\cdot\text{ft}$ (d) $\{-1440\mathbf{i}-180\mathbf{j}-640\mathbf{k}\}\text{lb}\cdot\text{ft}$

Rigid body equilibrium: Conditions

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Objective :

To introduce the general conditions for 2D and 3D rigid body equilibrium problems.

Particle equilibrium

First let's recall the conditions for **particle equilibrium**.

According to **Newton's first law**, an object will have **a linear acceleration of zero** when there is **no unbalanced force** acting on it.

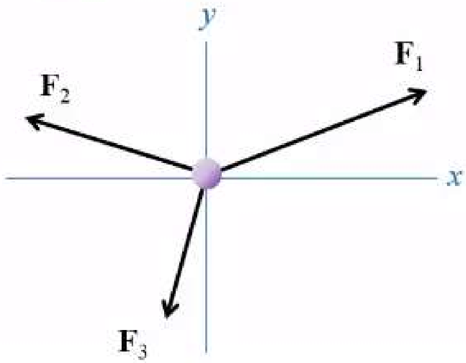
$$\mathbf{F}_R = \sum \mathbf{F} = \mathbf{0}$$

Particle equilibrium $\mathbf{F}_R = \sum \mathbf{F} = \mathbf{0}$

Since **Particle** is an idealized object with **no size** or **shape,** and is only represented by **a dot** in space, the forces acting on the particle will be **concurrent**.

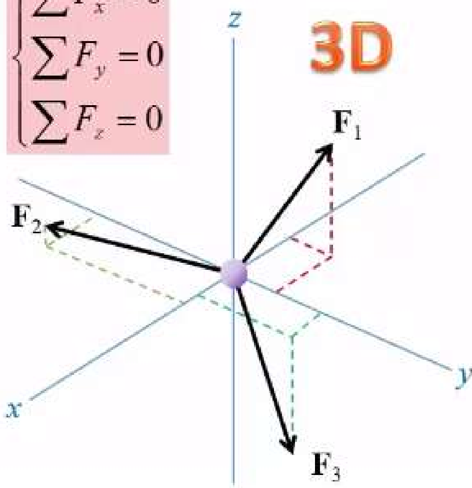
$$\begin{cases} \sum F_x = 0 \\ \sum F_y = 0 \end{cases}$$

2D



$$\begin{cases} \sum F_x = 0 \\ \sum F_y = 0 \\ \sum F_z = 0 \end{cases}$$

3D



For a 2D problem, the vector equation can be written as **2 scalar equations**.

For a 3D problem, the vector equation becomes **3 scalar equations**.

2D probem = Solve for 2 unknowns

3D probem = Solve for 3 unknowns

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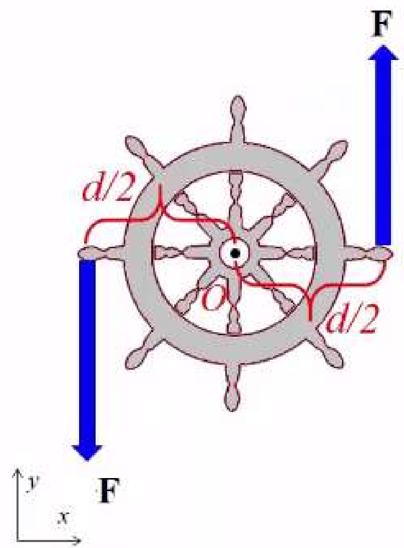
Recall: moment of a couple

However, **a rigid body** has **shape** and **size** and it is **not necessarily static** even if **the resultant force** acting on it is indeed **zero**.



The two forces acting on this wheel are indeed in **equilibrium**.

$$\sum F_y = F + (-F) = 0$$

But this only means that they don't cause **translational motion.** We already learned that these two forces make **a couple moment.**

$$\sum M_O = F \cdot \frac{d}{2} + F \cdot \frac{d}{2}$$
$$= F \cdot d$$

This moment causes **rotational effect** on this wheel.

Therefore, for **a rigid body** to be **static**, it is not enough to only have **unbalanced force**, but the **resultant moment** summarized about **any arbitrary point** must be **Zero** as well.

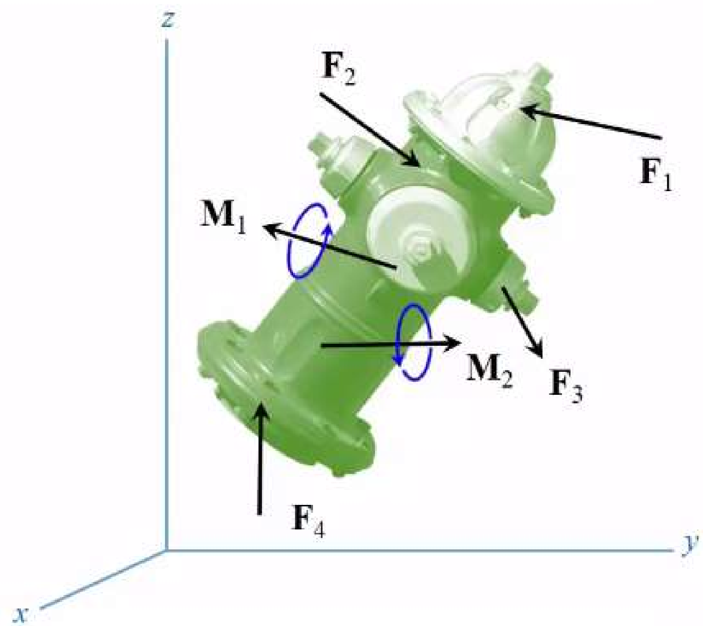
Otherwise, the object will rotate.

For **a rigid body** that is subjected to multiple forces and couple moments, **the first condition for equilibrium is**:

$$\mathbf{F}_R = \sum \mathbf{F} = \mathbf{0}$$

Then, **the resultant moment** summarized about any point, must also be **zero**, includes both the **total moment** caused **by the forces** and the **total couple moments**.

$$\mathbf{M}_{R,O} = \sum \mathbf{M}_{F,O} + \sum \mathbf{M} = \mathbf{0}$$



Conditions for rigid body equilibrium

As a summary, for rigid body equilibrium, we can have **two vector equations**, one for force and one for moment.

$$\begin{cases} \mathbf{F}_R = \sum \mathbf{F} = \mathbf{0} \\ \mathbf{M}_{R,O} = \sum \mathbf{M}_{F,O} + \sum \mathbf{M} = \mathbf{0} \end{cases}$$

For a 2D problem, based on one free body diagram, we can write a maximum of **3 independent scalar equations** and then solve for **3 unknowns**.

2-D problems:

$$\begin{cases} \sum F_x = 0 \\ \sum F_y = 0 \\ \sum M_O = 0 \end{cases}$$

2-D problems:

$$\begin{cases} \sum F_x = 0 \\ \sum M_A = 0 \\ \sum M_B = 0 \end{cases}$$

2-D problems:

$$\begin{cases} \sum M_A = 0 \\ \sum M_B = 0 \\ \sum M_C = 0 \end{cases}$$

Conditions for rigid body equilibrium

As a summary, for rigid body equilibrium, we can have **two vector equations**, one for force and one for moment.

$$\begin{cases} \mathbf{F}_R = \sum \mathbf{F} = \mathbf{0} \\ \mathbf{M}_{R,O} = \sum \mathbf{M}_{F,O} + \sum \mathbf{M} = \mathbf{0} \end{cases}$$

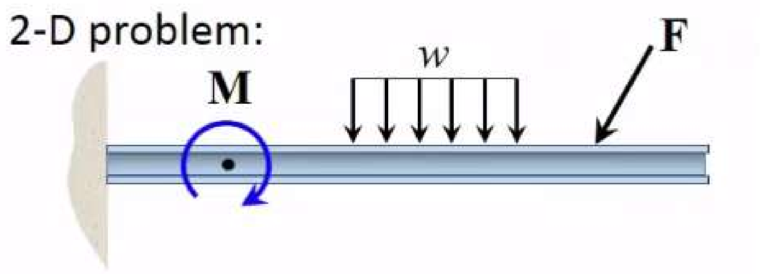
For a 3D problem, based on one free body diagram, we can write a maximum of **6 independent scalar equations** and solve for a maximum of **6 unknowns**.

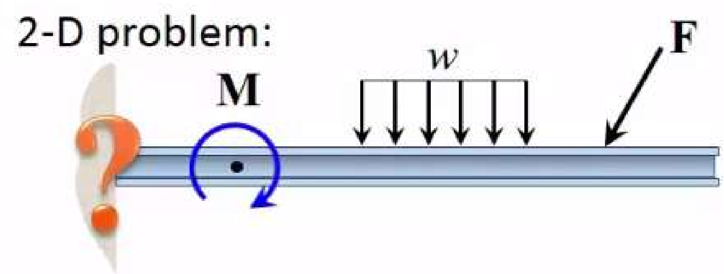
3-D problems:

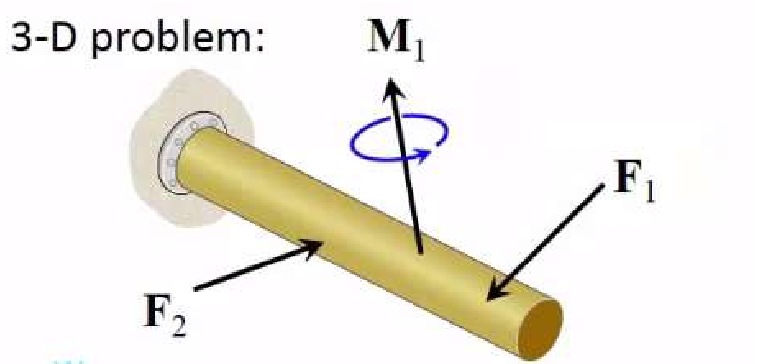
$$\begin{cases} \sum F_x = 0 \\ \sum F_y = 0 \\ \sum F_z = 0 \end{cases} \quad \begin{cases} \sum M_x = 0 \\ \sum M_y = 0 \\ \sum M_z = 0 \end{cases}$$

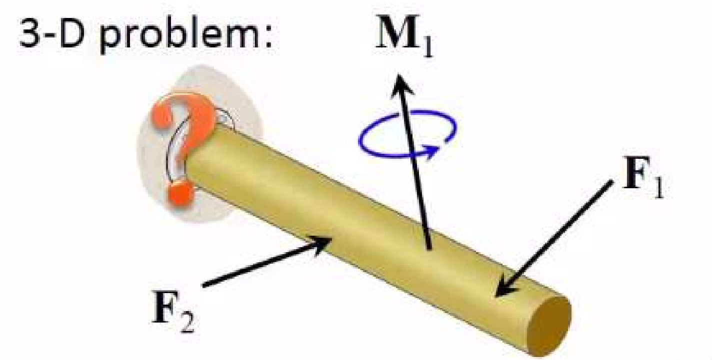
Here are **two examples** of rigid body **equilibrium problems**. Normally the **applied loadings** are **known**, and we will need to use **the equilibrium equations** to find the **unknown support reactions**.

The support reactions are also external force or moment acting on the body.









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