Delayed and advanced backward stochastic differential equations

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Contents

Preface

Cette polycopie a comme objectif de présenter un cours spécialisé de Master 2 niveau recherche que j'aimerais donner à des doctorants et de jeunes chercheurs cherchant une spécialisation dans le sujet des équations différentielles stochastiques rétrogrades (EDSRs). Depuis le travail pionnier d'Etienne Pardoux et Shige Peng (1990) sur ce nouveau type d'équations différentielles stochastiques (EDS), ce sujet a trouvé un grand succès et a attiré beaucoup de chercheurs. Le développement rapide et dynamique du sujet des EDSRs a été notamment stimulé par des nombreuses applications en contrôle stochastique, en théorie des jeux, en théorie des équations aux dérivées partielles (EDPs) mais également en finance. Le souci d'avoir des modèles plus réalistes en finance et de tenir compte de l'influence des événements passés dans les décisions actuelles des investisseurs et des agents dans le marché a conduit à l'étude des EDSRs avec délai, tandis que l'équation adjointe dans le principe de Pontryagine dans les problèmes de contrôle stochastique, comme par exemple celui d'optimisation de portefeuille, est une EDSR dont le coefficient anticipe l'information sur la solution tout en restant adapté. Dans cette ploycopie nous nous intéressons à ces deux types d'EDSRs, celle avec délai et celle avec anticipation et nous y développons leur théorie. Nous les étudions d'abord dans un cadre brownien, pour les étendre ensuite à un cadre de grossissement de filtration. Ce cadre de grossissement de filtration, où la filtration brownienne est progressivement élargie par l'information venant d'un temps aléatoire, est souvent utilisé dans la modélisation en finance, et le temps aléatoire est interprété comme instant de faillite. Nous étudions les deux types d'EDSRs dans ce cadre, d'une part en utilisant la propriété de représentation de martingale, et d'autre part par une réduction de ces EDSRs à des EDSRs browniennes.

"S'il ne me restait qu'un jour dans ma vie, je l'utiliserais pour donner un cours de mathématiques."

Introduction

Backward stochastic differential equations (BSDEs) appear in their linear form as adjoint equation when dealing with stochastic control. Bismut [6] was the first who studied this type of linear BSDEs. After that, this theory has been developed also for the nonlinear case, we refer to namely to the seminal work by Pardoux and Peng [19] in 1990, but also to Pardoux [20], El Karoui, Peng and Quenez [10], Barles, Buckdahn and Pardoux [5] for more details about BSDEs. The first work applying in finance was made by El Karoui, Peng and Quenez [10].

Given a driven Brownian motion W, a generator $f : \Omega \times \mathbb{R}_+ \times \mathbb{R}^2 \to \mathbb{R}$ and a terminal condition ξ, solving a BSDE consists in finding a process $(Y_t, Z_t)_{t\geq 0}$ adapted to the considered filtration (the Brownian one), such that, at time t, $(Y_t, Z_t)_{t\geq 0}$ satisfies the equation

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s,\ 0 \leq t \leq T. \tag{1}$$

The crucial question now is which conditions should be satisfied by the generator f and the terminal value ξ in order to get the existence and the uniqueness of such a solution to (1).

In this note we are interested in a generalisation of this above BSDE, where at time s the coefficient f depends on either future or past information on the solution process. More precisely, we consider the BSDE of the form

$$Y_t = \xi + \int_t^T \mathbb{E}[f(s, Y_s, Y_{s+\delta'}, Z_s, Z_{s+\delta'})|\mathcal{F}_s]ds - \int_t^T Z_s dW_s, t \in [0, T],$$

where for $\delta' < 0$ information on the past of the solution process (Y, Z) is considered, and for $\delta' > 0$ future information enters into the coefficient. We also observe that for $\delta' > 0$ the coefficient at time s is considered under the conditional expectation knowing $\mathcal{F}_s$, in order to guarantee adaptedness (more precisely, what is considered here is the optional projection of the coefficient process $(f(s, Y_{s+\delta'}, Z_s, Z_{s+\delta'})0_{s\in[0,T]}$ with respect to the Brownian filtration $\mathbb{F} = (\mathcal{F}_s)_{s\in[0,T]})$. We also remark that in the case, where $\delta' < 0$, i.e., when the dependence of the coefficient f on past information is studied, the process

$f(s, Y_s, Y_{s+\delta'}, Z_s, Z_{s+\delta'})$ is already $\mathbb{F}$-adapted, and coincides dsdP-almost everywhere with its optional projection.

However, apart from the Brownian case we are interested in the above BSDE also in the context of enlargement of filtration. In order to be more precise, let us consider a random time τ which is not an $\mathbb{F}$-stopping time nor $\mathcal{F}_T$-measurable. Examples of such random times are, for instance, default times, where the default reason comes from outside the Brownian model. We denote $H_t = 1_{(\tau \le t)}$, $t \in [0,T]$, and consider the filtration $\mathbb{G}$ obtained by enlarging progressively the filtration $\mathbb{F}$ by the process H, i.e., $\mathbb{G}$ is the smallest filtration satisfying the usual assumptions of completeness and right-continuity, which contains the filtration $\mathbb{F}$ and has H as adapted process. The BSDE related with, we want to study in this note is the following:

$$\begin{cases} Y_t = \xi + \int_t^T \mathbb{E}\left[f(s, Y_s, Y_{s+\delta}, Z_s, Z_{s+\delta}, U_s) \mid \mathcal{G}_s\right] ds \\ \qquad\qquad - \int_t^T Z_s dW_s - \int_t^T U_s dH_s, & t \in [0,T], \\ Y_t = Y_0, Z_t = 0, U_t = 0, & t < 0, \\ Y_t = \xi, \ Z_t = \eta_t, & t > T. \end{cases} \tag{2}$$

Here $\eta = (\eta_t)_{t \in [T, T+\delta'^+]}$ is a square integrable process and the η_t's are all $\mathcal{F}_T$-measurable ($\delta'^+ = \max\{\delta', 0\}$).

For a positive constant δ, we have: if $\delta' > 0$ ($\delta' = \delta$), then the BSDE (2) is called advanced BSDE (ABSDE) and then equation (2) takes the form

$$\begin{cases} Y_t = \xi + \int_t^T \mathbb{E}\left[f(s, Y_s, Y_{s+\delta}, Z_s, Z_{s+\delta}, U_s) \mid \mathcal{G}_s\right] ds \\ \qquad\qquad - \int_t^T Z_s dW_s - \int_t^T U_s dH_s, & t \in [0,T], \\ Y_t = \xi, \ Z_t = \eta_t, & t > T. \end{cases}$$

This type of equations has been studied by Øksendal, Sulem and Zhang in [17] but with the one-point jump process H replaced by a compensated Poisson random measure. We refer also to Peng and Yang [21] for the study of such equations but both of them with respect to the filtration generated by a Brownian motion and a compensated Poisson random measure. If $\delta' < 0$, i.e., "$\delta' = -\delta$", BSDE (2) is called delayed and is equivalent to

$$\begin{cases} Y_t = \xi + \int_t^T f(s, Y_s, Y_{s-\delta}, Z_s, Z_{s-\delta}, U_s) ds \\ \qquad\qquad - \int_t^T Z_s dW_s - \int_t^T U_s dH_s, & t \in [0,T], \\ Y_t = Y_0, Z_t = 0, U_t = 0, & t < 0. \end{cases} \tag{3}$$

A more general case of DBSDEs has been studied by Delong and Imkeller [7], and is studied in a work in progress by Agram and Røse [2]. In both works the authors consider the coefficients depending on the full past of the path of the solution. The filtration is that generated by the Brownian motion and the independent Poisson random measure.

In this note we study BSDEs driven by a Brownian motion and a single random jump under a progressive enlargement of filtration. We show that these

BSDEs are linked with Brownian BSDEs. We prove that these equations (2) have a solution, if the the associated Brownian BSDEs have a solution.

Application for DBSDEs

If one wants to find an investment strategy and an investment portfolio which replicate a liability or meet a target which depends on the applied strategy or the portfolio depends on its past values, then the delayed backward stochastic differential equations (BSDEs) are the best tool to solve this financial problem.

BSDEs with delay can arise in portfolio management problems, variable annuities, unit-linked products and participating contract:

"...the benefits are based on the return on a specified pool of assets held by the insurer...The best estimate should be based on the current assets held by the undertaking. Future changes of the asset allocation should be taken into account ..." (Solvency II Directive).

For more details about applications of such equations, we refer to Delong [8].

Applications of ABSDEs

ABSDEs appear as the adjoint equation, if we use the maximum principle to study optimal control for delayed systems. This is a natural model in population growth, but also in finance, where people's memory plays a role in the price dynamics.

On the hand, one can study BSDEs with an adapted anticipation of market information which is a natural model of the risky asset price in an insider influenced market. The point is that if an insider is operating in the market, knowing, for example, from time $t = 0$ the terminal value L of the stock at time $t = T$, then it has been proved (Kyle, Back) that this will influence the price of the risky asset in such a way that it becomes a Brownian bridge terminating at L at time $t = T$. Such a Browninan bridge can be represented by this type of BSDEs.

We give an example of an optimal consumption from a cash flow with delay as was studied in [17], [1], [18] and even with respect to a Poisson measure.

Example 1 (Optimal consumption from a cash flow with delay)

Let us consider an example of an optimal consumption problem, where the wealth of an investor comes from his portfolio a part of which is invested in stock and another part in riskless bonds. The dynamics of his wealth is determined by the dynamics of the value of his portfolio and by his consumption. More precisely, in a financial market model with only two funds, one risky one (for instance, stocks) and one non risky one (for instance, bonds), we consider as model for the price of one unit of the risky fund

$$dS_t = \sigma S_t dW_t + \mu S_t dt,\, t \geq 0,$$

where $\sigma > 0$ *denotes the volatility and* $\mu \in \mathbb{R}$ *describes the tendency. As concerns the price of a unit of the riskless fund, we suppose for simplicity that* *the riskless interest rate is equal to zero. Then, with* $(\pi_t, c_t)_{t\geq 0}$ *as chosen self-financing portfolio strategy the wealth of the investor* $V = (V_t)_{t\geq 0}$ *(identified with the value of his portfolio) is given by the following dynamics:*

$$dV_t = \pi_t dS_t = \pi_t(\sigma S_t dW_t + \mu S_t dt),\ t \geq 0.$$

However, apart from optimising his investment for the time horizon $T > 0$*, the investor has also the objective of optimising his consumption. At time* t *he decides to consume in the (infinitesimally) small time interval* $[t, t+dt]$ *the part* c_t *of his wealth* $V_{t-\delta}$ *multiplied with the length of the time interval* dt*. The consumption rate* c_t *takes its values between* 0 *and* 1 *and is decided by the investor by considering his wealth* V *with a delay* $\delta > 0$*. Taking into account this consumption, the dynamics of the wealth obeys the following equation:*

$$\begin{cases} dV_t &= \pi_t \sigma S_t dW_t + (\pi_t \mu S_t - c_t V_{t-\delta})dt, \quad t \in [0,T], \\ V_t &= v_t^0, \qquad\qquad\qquad\qquad\qquad\quad\ t \in [-\delta, 0]. \end{cases}$$

We see in particular that the dynamics of the wealth $V = V^{\pi,c}$ *depends on the portfolio choice -namely the part of the portfolio invested in the risky asset- and on the consumption rate. The performance functional we want to maximize is*

$$J(\pi, c) = \mathbb{E}\left[\int_0^T U(t, c_t)\, dt + h(V_T^{\pi,c})\right], (\pi, c) \in \mathcal{U},$$

where $\mathcal{U}$ *is the set of admissible control values and* $U : \Omega \times [0,T] \times \mathbb{R}^+ \to \mathbb{R}$ *is a given stochastic utility function satisfying the following conditions*

$$\begin{aligned} &t \mapsto U(t,c) \ \text{ is } \ (\mathcal{F}_t) - \text{adapted, for every } c \geq 0, \\ &c \mapsto U(t,c) \ \text{ is } C^1 \ \text{ and } \ \frac{\partial U}{\partial c}(t,c) > 0, \\ &c \mapsto \frac{\partial U}{\partial c}(t,c) \text{ is strictly decreasing,} \\ &\lim_{c \to \infty} \frac{\partial U}{\partial c}(t,c) = 0, \ \text{ for all } \ (\omega, t) \in \Omega \times [0,T]. \end{aligned}$$

Put $v_t^0 = \frac{\partial U}{\partial c}(t, 0)$*, for all* $t \geq 0$*, and define*

$$I(t,v) = \begin{cases} 0 & \text{if } v \geq v_t^0, \\ \left(\dfrac{\partial U}{\partial c}(t,.)\right)_v^{-1} & \text{if } 0 \leq v < v_t^0. \end{cases}$$

Moreover, $h : \mathbb{R} \to \mathbb{R}$ *is a given real concave function which is continuously differentiable. Using the maximum principle, we get the first order maximum condition*

$$\begin{aligned} &\sigma S_t q_t + \mu S_t p_t = 0, t \in [0,T], \\ &\frac{\partial U}{\partial c}(t, \hat{c}_t) = V_{t-\delta} p_t, t \in [0,T], \end{aligned}$$

where (p, q) *is the solution of the following advanced BSDE*

$$\begin{cases} dp_t = \mathbb{E}\left[c_t p_{t+\delta} \mid \mathcal{F}_t\right] dt + q_t dW_t, & t \in [0, T], \\ p_t = h'(V_T), & t > T. \end{cases}$$

Let us now consider two filtrations $\mathbb{F}$ and $\mathbb{G}$ on our probability space $(\Omega, \mathcal{F}, P)$ and let us suppose that $\mathbb{F}$ is a subfiltration of $\mathbb{G}$, i.e., $\mathbb{F} \subset \mathbb{G}$ in the sense that $\mathcal{F}_t \subset \mathcal{G}_t$ that for all $t \geq 0$. We consider $\mathbb{G}$ as an enlargement of the original filtration $\mathbb{F}$, if the filtration $\mathbb{G}$ is obtained from $\mathbb{F}$ by injecting additional information either at once already at time 0 (the so-called initial enlargement) or by injecting the additional information continuously in time (progressive enlargement). This progressive enlargement is often considered as progressive adding of information given in form of a random time τ in a way which transforms τ to a stopping time with respect to the filtration $\mathbb{G}$.

The topic of enlargement of filtration was initiated by Jacod and Yor [13], Jeulin and Yor (see [12]). The following both questions may be asked here:

1. Does an $\mathbb{F}$-martingale X remain a $\mathbb{G}$-martingale?

2. If the answer to (1) is yes for all $\mathbb{F}$-martingales X and we have the martingale representation property with respect to the filtration $\mathbb{F}$, do we have also a martingale representation property with respect to the filtration $\mathbb{G}$?

If the answer to question (1) is positive for all $\mathbb{F}$-martingale X, then this is called the immersion property or Hypothesis (H). A weaker Hypothesis that (H) is that of the stability of the class of semimartingales when passing from the filtration $\mathbb{F}$ to $\mathbb{G}$. This Hypothesis is usually called (H') and says that any $\mathbb{F}$-semimartingale remains also a semimartingale with respect to $\mathbb{G}$.

Naturally, the enlargement of filtration appears in credit risk and it has also been related recently to stochastic optimal control by Pham [22] and to mean-variance hedging by Kharroubi, Lim and Ngoupeyou [16] where the optimal strategy is described by a non-standard BSDEs driven by a Brownian motion and a jump martingale in the enlarged filtration.

In the work in progress [3] the authors consider ABDEs in the enlarged filtration. For this the authors use the decomposition approach as in Kharroubi, Lim and Ngoupeyou [16] and because of the general nature of the generator coming from the decomposition, they use the Key Lemma and the density Hypothesis (DH). On the other hand they also derive the existence and the uniqueness from the martingale representation property (See Section 3.3).

Chapter 1

Mathematical background

The objective of this chapter is to give a short introduction to backward stochastic differential equations. Backward stochastic differential equations (BSDEs) and their extensions to advanced and delayed equations in a Brownian setting but also in the setting of enlargement of filtration will be the main subject of this note. In order to prepare this, we'll introduce in this short chapter the notion of BSDE and we'll state the fundamental result on the existence and the uniqueness given by Pardoux and Peng in their seminal paper [19] of 1990.

We consider a complete probability space $(\Omega, \mathcal{F}, P)$ equipped with a one-dimensional standard Brownian motion W. We denote by $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ the right-continuous complete filtration generated by W. Let us develop the basic idea which is behind BSDEs. For this let us consider a stochastic differential equation (SDE) driven by the Brownian motion W, which is of the following form

$$dY_t = f(Y_t)dt + \sigma(Y_t)dW_t,\ t \in [0,T].$$

Here we suppose for simplicity that the coefficients $f, \sigma : \mathbb{R} \to \mathbb{R}$ are Lipschitz. Then it is a classical result that, if $Y_0 = \xi \in \mathbb{R}$ is an imposed initial condition, the SDE, also called forward SDE, has a unique continuous $\mathbb{F}$-adapted solution Y. But what does happen, if we replace now the initial condition by a terminal one? As long as the condition ξ remains still a deterministic real value, we make a time inversion and write the equation with terminal condition for $V_t := Y_{T-t}$, $\tilde{W}_t := W_T - W_{T-t}$, $t \in [0,T]$, as an SDE with initial condition:

$$\begin{cases} dV_t = -f(V_t)dt - \sigma(V_t)d\tilde{W}_t,\ t \in [0,T], \\ V_0 = \xi. \end{cases}$$

We observe that $\tilde{W} = (\tilde{W}_t)_{t\in[0,T]}$ is a Brownian motion, and due to the Lipschitz condition on the coefficients, there is a unique continuous solution V adapted with respect to the filtration generated by $\tilde{W}$. But this means, that $Y_t = V_{T-t}$ is $\tilde{\mathcal{F}}_t = \sigma\{W_T - W_s,\ s \in [t,T]\}$-measurable, for all $t \in [0,T]$ (The σ-fields are considered as completed), and, for $t_i^n = T - t + t\frac{i}{n},\ 0 \leq i \leq n$, the stochastic integral of the SDE for the process Y can be described by

$$\begin{aligned}\int_0^t \sigma(Y_s)dW_s \quad &:= \int_{T-t}^T \sigma(V_s)d\tilde{W}_s \\ &= L^2 - \lim_{n\to+\infty} \sum_{i=0}^{n-1} \sigma\left(V_{t_i^n}\right)(\tilde{W}_{t_{i+1}^n} - \tilde{W}_{t_i^n}) \\ &= L^2 - \lim_{n\to+\infty} \sum_{i=0}^{n-1} \sigma\left(Y_{T-t_i^n}\right)(W_{T-t_i^n} - W_{T-t_{i+1}^n}) \\ &= L^2 - \lim_{n\to+\infty} \sum_{i=0}^{n-1} \sigma\left(Y_{t-t\frac{i}{n}}\right)(W_{t-t\frac{i}{n}} - W_{t-t\frac{i+1}{n}}) \\ &= L^2 - \lim_{n\to+\infty} \sum_{i=1}^{n} \sigma\left(Y_{t\frac{i+1}{n}}\right)(W_{t\frac{i+1}{n}} - W_{t\frac{i}{n}}),\end{aligned}$$

i.e., we have to do with the so-called Itô backward integral.

But how about a terminal condition $Y_T = \xi$ with $\xi \in L^2(\Omega, \mathcal{F}_T, P)$? We see that in this case a time inversion of our SDE for Y leads to a forward equation for $V_t = Y_{T-t}$ with an anticipating initial condition $V_0 = \xi$. But we are not interested in studying SDEs with anticipation.

In order to understand better what to do, let us first consider the special case where $\sigma \equiv 0$ and $f(y) = ay$, with $a \in \mathbb{R}$ is a real constant, i.e., we have the SDE

$$\begin{cases} dY_t = aY_t dt,\ t \in [0,T], \\ Y_T = \xi \in L^2(\Omega, \mathcal{F}_T, P). \end{cases}$$

The unique solution of this equation is the process $Y_t = \xi \exp\{-a(T-t)\}$, $t \in [0,T]$. This process is, obviously, not adapted to the Brownian filtration $\mathbb{F}$. Being interested in adapted solutions we replace this process Y_t, $t \in [0,T]$, by its best approximation in L^2 by a process which is adapted with respect to the filtration $\mathbb{F}$, i.e., we consider the process $U_t = \exp\{-a(T-t)\}\mathbb{E}[\xi|\mathcal{F}_t],\ t \in [0,T]$, which is juste the optional projection of the process Y.

In the special case, where $a = 0$, we see that $U_t = \mathbb{E}[\xi|\mathcal{F}_t],\ t \in [0,T]$, is just the martingale generated by $U_T = Y_T = \xi$, and from the martingale representation property in a Brownian setting we have the existence and the uniqueness of a square integrable $\mathbb{F}$-adapted process Z such that

$$\xi = \mathbb{E}[\xi] + \int_0^T Z_t dW_t,\ P-a.s.,$$

i.e.

$$U_t = \mathbb{E}[\xi|\mathcal{F}_t] = \mathbb{E}[\xi] + \int_0^t Z_s dW_s,\ t \in [0,T],$$

which shows that $dU_t = Z_t dW_t$, $t \in [0,T]$, $U_T = \xi$. This indicates that we have to reinterpret our equation for Y in the following sense:

$$\begin{cases} dY_t = (f(Y_t)dt + \sigma(Y_t)dW_t) + V_t dW_t\, t \in [0,T], \\ Y_T = \xi(\in L^2(\Omega, \mathcal{F}_T, P)), \end{cases}$$

where the solution we have to look for is a couple of square integrable $\mathbb{F}$-adapted processes (Y,V), where V has its origin from the martingale representation property. However, having (Y,V), we can define the process $Z = (Z_t)_{t\in[0,T]}$

by putting $Z_t := V_t + \sigma(Y_t)$, $t \in [0,T]$ (observe that, knowing (Y,Z) we can compute $V_t = Z_t - \sigma(Y_t)$), and this leads to the SDE,

$$\begin{cases} dY_t = f(Y_t)dt + Z_t dW_t, t \in [0,T], \\ Y_T = \xi. \end{cases}$$

Finally, in order to have the backward SDE in the general form studied by Pardoux and Peng in 1990, we replace the Lipschitz function $f : \mathbb{R} \to \mathbb{R}$ by a more general, adapted coefficient $f : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ which is now allowed to depend also on $(\omega, t) \in \Omega \times [0,T]$, but also on the solution component Z_t, and in order to be coherent with the notation which is usually used, we endow the function f with the negative sign. This leads to the so-called backward stochastic differential equation (BSDE) introduced and studied by Pardoux and Peng in their pioneering work of 1990 (see [19]):

$$\begin{cases} dY_t = -f(t, Y_t, Z_t)dt + Z_t dW_t,\ t \in [0,T], \\ Y_T = \xi \in L^2(\Omega, \mathcal{F}_T, P). \end{cases}$$

Let us finish this subsection with recalling the main result of Pardoux and Peng's work of 1990 in [19]. For this we shall introduce the corresponding spaces in which the solution processes live.

Let S^2_F denote the space of all real-valued continuous $\mathbb{F}$-adapted processes $Y = (Y_t)_{t\in[0,T]}$ with the property $\mathbb{E}\left[\sup_{t\in[0,T]} |Y_t|^2\right] < +\infty$. By $H^2_{\mathbb{F}}$ we denote the space of all $\mathbb{F}$-adapted processes $Z = (Z_t)_{t\in[0,T]}$ which are square integrable over the space $\Omega \times [0,T]$: $\mathbb{E}\left[\int_0^T |H_t|^2 dt\right] < +\infty$.

With these spaces Pardoux and Peng's result formulates as follows.

Proposition 2 . *Let $\xi \in L^2(\Omega, \mathcal{F}_T, P)$ and $f : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ a jointly measurable mapping satisfying the following assumptions:*

i) $f(.,.,0,0) \in H^2_{\mathbb{F}}$;

ii) $f(t,\omega,.,.)$ is uniformly Lipschitz, $dtP(d\omega)$-a.e., i.e., there is some constant $L \in \mathbb{R}$ such that, $dtP(d\omega)$-a.e., for all $y, y', z, z' \in \mathbb{R}$,

$$|f(t,\omega,y,z) - f(t,\omega,y',z')| \leq L(|y-y'| + |z-z'|).$$

Then the BSDE

$$\begin{cases} dY_t = -f(t, Y_t, Z_t)dt + Z_t dW_t,\ t \in [0,T], \\ Y_T = \xi, \end{cases}$$

possesses a unique solution $(Y,Z) \in S^2_{\mathbb{F}} \times H^2_{\mathbb{F}}$.

Chapter 2

Brownian backward stochastic differential equations

After an inition to BSDEs and Pardoux and Peng's fundamental existence and uniqueness result for BSDEs let us discuss in this chapter Delayed BSDEs (DBSDEs) and Advanced BSDEs (ABSDEs). These types of BSDEs generalise the classical BSDE studied by Pardoux and Peng [19] and have turned out to be useful in various applications, namely in finance and in stochastic control. While in finance the delay imposes in the modelling by the fact that agents have often only time-delayed information, advanced BSDEs appear as adjoint equation in the characterisation of optimal controls in stochastic control problems with delay.

This chapter is devoted to the study of DBSDEs and ABSDEs in a Brownian setting. If one considers financial models with DBSDEs and ABSDEs in a context of default times, then this leads to such BSDEs in a setting of enlargement of filtration, and this will be the object of the next chapter.

For the study of our Delayed and Advanced BSDEs in the Brownian setting let us introduce now the adequates spaces of processes and let us fix suitable hypotheses on the driver.

Framework

We consider a complete probability space $(\Omega, \mathcal{F}, P)$ which we suppose equipped with a one-dimensional standard Brownian motion W. We denote by $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ the right-continuous complete filtration generated by W. Let $\mathcal{P}(\mathbb{F})$ be the σ-algebra of $\mathbb{F}$-predictable subsets of $\Omega \times \mathbb{R}_+$, i.e., the σ-algebra generated by the left continuous $\mathbb{F}$-adapted processes. Moreover, for $k \geq 1$, we denote by $\mathcal{B}(\mathbb{R}^k)$ the Borel-σ-field over $\mathbb{R}^k$.

Throughout this chapter, we introduce some basic notations and spaces for any σ-algebra $\mathcal{F}$ and any filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$.

- $S^2_{\mathbb{F}}$ is the set of $\mathbb{R}$-valued $\mathbb{F}$-adapted processes Y essentially bounded, such that
$$\|Y\|^2_{S^2} := \mathbb{E}\left[\sup_{t\in[0,T]} |Y_t|^2\right] < \infty.$$
- $H^2_{\mathbb{F}}$ is the set of $\mathbb{R}$-valued $\mathcal{P}(\mathbb{F})$-measurable processes $(Z_t)_{t\in[0,T]}$, such that
$$\|Z\|^2_{H^2} := \mathbb{E}\left[\int_0^T |Z_t|^2\, dt\right] < \infty.$$
- $L^2(\Omega, \mathcal{F})$ is the space of all $\mathcal{F}$-measurable random variables ξ, such that
$$\mathbb{E}(|\xi|^2) < \infty.$$

2.1 Brownian delayed backward stochastic differential equations

The objective of this section is to give a theorem on the existence and the uniqueness of the following DBSDEs. For the proof we use the fixed point argument which is a classical tool to prove the existence and uniqueness of BSDEs (see for example [19], [20], [5] and [10]).

$$\begin{cases} Y_t = \xi + \int_t^T f(s, Y_s, Y_{s-\delta}, Z_s, Z_{s-\delta})ds - \int_t^T Z_s dW_s, & t \in [0,T]\ , \\ Y_t = Y_0\ ,\quad Z_t = 0, & t < 0\ , \end{cases} \tag{2.1}$$

where δ is a constant, $f : \Omega \times [0,T] \times \mathbb{R}^4 \to \mathbb{R}$ and the terminal condition ξ is an $\mathcal{F}_T$-measurable random variable.

Assumption H1

(i) $\xi \in (\Omega, \mathcal{F}_T)$ and $f\ (\cdot,\cdot,\cdot,\cdot,\cdot)$ is $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}^4)$-measurable.

(ii) Lipschitz condition: For all $t \in [0,T]$ and for all $u, u' \in \mathbb{R}^4$, we have for a constant $C > 0$
$$|f(t,u) - f(t,u')| \leq C\, |u - u'|\, .$$

(iii) Integrability condition:
$$\mathbb{E}\left[\int_0^T |f(t,0)|^2\, dt\right] < \infty,\ \text{where } 0 = (0,0,0,0) \in \mathbb{R}^4.$$

The following theorem gives the existence and the uniqueness of the solution of a DBSDE under Assumption H1.

Theorem 3 *Under Assumption H1, there is some* $\delta' > 0$ *such that, for all* $\delta \in (0, \delta')$, *DBSDE* (2.1) *admits a unique solution* $(Y, Z) \in S^2_{\mathbb{F}} \times H^2_{\mathbb{F}}$.

Proof. For any $\rho > 1$, we choose $\delta_\rho > 0$ such that the following relation is satisfied:

$$\frac{1}{2\rho}(1 + e^{(1+8\rho C^2)\delta_\rho}) = 1.$$

The constant C depends only on the Lipschitz constant of the the coefficient f and will be specified later. We observe that then, for all $\delta \in (0, \delta_\rho)$,

$$\frac{1}{2\rho}(1 + e^{(1+8\rho C^2)\delta}) < 1.$$

Let us define a mapping

$$\Phi : \left(L^2(\mathcal{F}_0) \times H^2_{\mathbb{F}}\right) \times H^2_{\mathbb{F}} \to \left(L^2(\mathcal{F}_0) \times H^2_{\mathbb{F}}\right) \times H^2_{\mathbb{F}}$$

by setting

$$\Phi\left(\left(U_0, U\right), V\right) := \left(\left(Y_0, Y\right), Z\right),$$

where for $U_0 \in L^2(\mathcal{F}_0)$, $U = (U_t) \in H^2_{\mathbb{F}}$, $U_t = U_0$, $t < 0$ and $V \in H^2_{\mathbb{F}}$, $V_t = 0$, $t < 0$, and the pair $(Y, Z) \in S^2_{\mathbb{F}} \times H^2_{\mathbb{F}}$ is the unique solution of the BSDE

$$\begin{cases} Y_t = \xi + \int_t^T f(s, U_s, U_{s-\delta}, V_s, V_{s-\delta})ds - \int_t^T Z_s dW_s, & t \in [0, T], \\ Y_t = Y_0, \ Z_t = 0, & t < 0. \end{cases}$$

We remark that this solution (Y, Z) exists indeed and is unique. For seeing this, let us put

$$h(s) := f(s, U_s, U_{s-\delta}, V_s, V_{s-\delta}), s \in [0, T],$$

and let us observe that under our assumptions $h \in H^2_{\mathbb{F}}$. Then, if (Y, Z) is a solution of the above equation, we get by taking the conditional expectation that

$$Y_t = \mathbb{E}\left[\xi + \int_t^T h(s)ds|\mathcal{F}_t\right] = \mathbb{E}\left[\xi + \int_0^T h(s)ds|\mathcal{F}_t\right] - \int_0^t h(s)ds, t \in [0, T],$$

and $Z \in H^2_{\mathbb{F}}$ is uniquely determined by the martingale representation of $\xi + \int_0^T h(s)ds \in L^2(\Omega, \mathcal{F}_T, P)$:

$$\xi + \int_0^T h(s)ds = \mathbb{E}\left[\xi + \int_0^T h(s)ds\right] + \int_0^T Z_s dW_s, P - a.s.$$

On the other hand, one can check easily that the pair $(Y, Z) \in S^2_{\mathbb{F}} \times H^2_{\mathbb{F}}$ defined by the above both relations solves our above BSDE. Now, for $\beta > 0$ and $((U_0, U), V) \in (L^2(\mathcal{F}_0) \times H^2_{\mathbb{F}}) \times H^2_{\mathbb{F}})$ we introduce the norm

$$\begin{aligned} \|((U_0, U), V)\| &:= \|((U_0, U), V)\|_\beta \\ &:= \left(\mathbb{E}\left[|U_0|^2\right] + \mathbb{E}\left[\int_0^T e^{\beta s}\left(|U_s|^2 + |V_s|^2\right)ds\right]\right)^{\frac{1}{2}}. \end{aligned}$$

Note that $(L^2(\mathcal{F}_0)\times H^2_{\mathbb{F}})\times H^2_{\mathbb{F}})$ endowed with this norm is a Banach space. We will show that for suitably chosen $\beta>0$ and for all $\delta\in(0,\delta_\delta)$, the mapping $\Phi:\left(\left(L^2(\mathcal{F}_0)\times H^2_{\mathbb{F}}\right)\times H^2_{\mathbb{F}},\|\cdot\|\right)\rightarrow\left(\left(L^2(\mathcal{F}_0)\times H^2_{\mathbb{F}}\right)\times H^2_{\mathbb{F}},\|\cdot\|\right)$ is contracting, i.e., there is a unique fixed point $((Y_0,Y),Z)\in\left(\left(L^2(\mathcal{F}_0)\times H^2_{\mathbb{F}}\right)\times H^2_{\mathbb{F}}\right)$ of Φ. Consequently,

$$\begin{cases} Y_t=\xi+\int_t^T f(s,Y_s,Y_{s-\delta},Z_s,Z_{s-\delta})ds-\int_t^T Z_s dW_s, & t\in[0,T],\\ Y_t=Y_0,\ Z_t=0, & t<0.\end{cases}$$

In particular Y has a continuous version, and

$$\begin{aligned}\mathbb{E}\left[\sup_{t\in[0,T]}|Y_t|^2\right]\leq 3\mathbb{E}\left[|\xi|^2\right]+3T\mathbb{E}\left[\int_0^T|f(t,U_t,U_{t-\delta},V_t,V_{t-\delta})|^2dt\right]\\ +12\mathbb{E}\left[\int_0^T|Z_t|^2dt\right]<\infty.\end{aligned}$$

Consequently, $Y\in S^2_{\mathbb{F}}$.
Let us consider $(U_0,U,V),(U_0',U',V')\in\left(L^2(\mathcal{F}_0)\times H^2_{\mathbb{F}}\right)\times H^2_{\mathbb{F}}$ and let us use the simplified notations:

$$\begin{aligned}\Phi(U_0,U,V)&:=(Y_0,Y,Z),\\ \Phi(U_0',U,V')&:=(Y_0',Y',Z'),\\ (\bar U_0,\bar U,\bar V)&=(U_0,U,V)-(U_0',U',V'),\\ \left(\bar Y_0,\bar Y,\bar Z\right)&=(Y_0,Y,Z)-(Y_0',Y',Z').\end{aligned}$$

Applying Itô's formula to $e^{\beta t}\left|\bar Y_t\right|^2$ and using the Lipschitz condition (ii) of Assumption H1, we get

$$\begin{aligned}&e^{\beta t}\left|\bar Y_t\right|^2+\int_t^T e^{\beta s}\left(\beta\left|\bar Y_s\right|^2+\left|\bar Z_s\right|^2\right)ds\\ &\leq 2\int_t^T e^{\beta s}\left|\bar Y_s\right|\cdot C\left(\left|\bar U_s\right|+\left|\bar U_{s-\delta}\right|+\left|\bar V_s\right|+\left|\bar V_{s-\delta}\right|\right)ds\\ &-2\int_t^T e^{\beta s}\bar Y_s\cdot\bar Z_s dW_s \qquad (2.2)\\ &=2C\int_t^T e^{\beta s}\left(\left|\bar Y_s\right|\cdot\left|\bar U_s\right|+\left|\bar Y_s\right|\cdot\left|\bar U_{s-\delta}\right|+\left|\bar Y_s\right|\cdot\left|\bar V_s\right|\right.\\ &\left.+\left|\bar Y_s\right|\cdot\left|\bar V_{s-\delta}\right|\right)ds-2\int_t^T e^{\beta s}\bar Y_s\cdot\bar Z_s dW_s.\end{aligned}$$

Using for every term in the integrand of the Lebesgue integral at the right-hand side of the above equality the estimate $2Cab \leq 2\rho C^2 a^2 + \frac{1}{2\rho} b^2$, we obtain

$$\begin{aligned}
& e^{\beta t} \left|\bar{Y}_t\right|^2 + \int_t^T e^{\beta s} \left(\beta \left|\bar{Y}_s\right|^2 + \left|\bar{Z}_s\right|^2\right) ds \\
& \leq \int_t^T e^{\beta s} \Big(2\rho C^2 \left|\bar{Y}_s\right|^2 + \frac{1}{2\rho} \left|\bar{U}_s\right|^2 + 2\rho C^2 \left|\bar{Y}_s\right|^2 + \frac{1}{2\rho} \left|\bar{U}_{s-\delta}\right|^2 \\
& + 2\rho C^2 \left|\bar{Y}_s\right|^2 + \frac{1}{2\rho} \left|\bar{V}_s\right|^2 + 2\rho C^2 \left|\bar{Y}_s\right|^2 + \frac{1}{2\rho} \left|\bar{V}_{s-\delta}\right|^2 \Big) ds \\
& -2 \int_t^T e^{\beta s} \bar{Y}_s \cdot \bar{Z}_s dW_s \\
& = \int_t^T e^{\beta s} \left(8\rho C^2 \left|\bar{Y}_s\right|^2 + \frac{1}{2\rho} \left(\left|\bar{U}_s\right|^2 + \left|\bar{U}_{s-\delta}\right|^2 + \left|\bar{V}_s\right|^2 + \left|\bar{V}_{s-\delta}\right|^2 \right) \right) ds \\
& -2 \int_t^T e^{\beta s} \bar{Y}_s \cdot \bar{Z}_s dW_s,
\end{aligned}$$

for $\rho > 1$. Choosing $\beta = 1 + 8\rho C^2$ and taking the conditional expectation, we get

$$\begin{aligned}
& e^{\beta t} \left|\bar{Y}_t\right|^2 + \mathbb{E}\left[\int_t^T e^{\beta s} \left(\left|\bar{Y}_s\right|^2 + \left|\bar{Z}_s\right|^2 \right) ds \mid \mathcal{F}_t \right] \\
& \leq \frac{1}{2\rho} \mathbb{E}\left[\int_t^T e^{\beta s} \left(\left|\bar{U}_s\right|^2 + \left|\bar{U}_{s-\delta}\right|^2 + \left|\bar{V}_s\right|^2 + \left|\bar{V}_{s-\delta}\right|^2 \right) ds \mid \mathcal{F}_t \right].
\end{aligned} \tag{2.3}$$

By changing of variables $r = s - \delta$, we have

$$\begin{aligned}
\int_0^T e^{\beta s} \left(\left|\bar{U}_{s-\delta}\right|^2 + \left|\bar{V}_{s-\delta}\right|^2 \right) ds & = e^{\beta\delta} \int_{-\delta}^{T-\delta} e^{\beta r} \left(\left|\bar{U}_r\right|^2 + \left|\bar{V}_r\right|^2 \right) dr \\
& \leq \delta e^{\beta\delta} \left|\bar{U}_0\right|^2 + e^{\beta\delta} \int_0^T e^{\beta r} \left(\left|\bar{U}_r\right|^2 + \left|\bar{V}_r\right|^2 \right) dr.
\end{aligned} \tag{2.4}$$

Combining (2.4) with (2.3) , taking $t = 0$ and taking the expectation, we obtain

$$\begin{aligned}
& \mathbb{E}\left[\left|\bar{Y}_0\right|^2\right] + \mathbb{E}\left[\int_0^T e^{\beta s} \left(\left|\bar{Y}_s\right|^2 + \left|\bar{Z}_s\right|^2 \right) ds \right] \\
& \leq \frac{1}{2\rho} (1 + e^{\beta\delta}) \left(\left|\bar{U}_0\right|^2 + \mathbb{E}\left[\int_0^T e^{\beta s} \left(\left|\bar{U}_s\right|^2 + \left|\bar{V}_s\right|^2 \right) ds \right] \right),
\end{aligned}$$

for $\delta < \delta_\rho \wedge 1$. Hence as

$$\frac{1}{2\rho} (1 + e^{(1+8\rho C^2)\delta}) < 1,$$

$$\Phi : \left(\left(L^2(\mathcal{F}_0) \times H^2_{\mathbb{F}} \right) \times H^2_{\mathbb{F}}, \|\cdot\|_\beta \right) \to \left(\left(L^2(\mathcal{F}_0) \times H^2_{\mathbb{F}} \right) \times H^2_{\mathbb{F}}, \|\cdot\|_\beta \right)$$

has a unique fixed point: $((Y_0, Y), Z) \in \left(\left(L^2(\mathcal{F}_0) \times H^2_{\mathbb{F}} \right) \times H^2_{\mathbb{F}} \right)$. ■

2.2 Brownian advanced backward stochastic differential equations

After having discussed in the preceding section Brownian BSDEs with delay, let us come now to BSDEs which driving coefficient anticipates the information on the solution process. More precisely, let us consider ABSDEs in the sense that, for a positive constant δ, it has the form

$$\begin{cases} Y_t = \xi + \int_t^T \mathbb{E}\left[f(s, Y_s, Y_{s+\delta}, Z_s, Z_{s+\delta}) \mid \mathcal{F}_s\right] ds - \int_t^T Z_s dW_s, & t \in [0,T], \\ Y_t = \xi,\ Z_t = \eta_t, & t > T. \end{cases} \tag{2.5}$$

Assumption H2

(i) *Assumption on the terminal condition:*

- $\xi \in L^2(\Omega, \mathcal{F}_T)$ is bounded,
  and $\eta = (\eta_t)_{t\in[T,T+\delta]} \in L^2(\Omega, \mathcal{F}_T \otimes \mathcal{B}([T, T+\delta]), dtdP)$.

(ii) *Assumptions on the generator function:* $f : \Omega \times \mathbb{R}_+ \times \mathbb{R}^4 \to \mathbb{R}$ is such that

- $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}^4)$-measurable and bounded by some constant K, such that

$$|f(t, y, \hat{y}, z, \hat{z})| \leq K,\ P - a.s,$$

for all $t \in [0,T]$ and all $y, z, , \hat{y}, \hat{z} \in \mathbb{R}$.

- Lipschitz in the sense that there exists $C > 0$ such that

$$\begin{aligned} &|f(t, y, \hat{y}, z, \hat{z}) - f(t, y', \hat{y}', z', \hat{z}')| \\ &\leq C\left(|y - y'| + |z - z'| + |\hat{y} - \hat{y}'| + |\hat{z} - \hat{z}'|\right), \end{aligned}$$

for all $t \in [0,T]$ and all $y, y', z, z', \hat{y}, \hat{y}', \hat{z}, \hat{z}' \in \mathbb{R}$.

The following theorem states the existence and the uniqueness of the solution of an ABSDEs under Assumption H2.

Theorem 4 *Under the above Assumption H2, the ABSDE* (2.5) *admits a unique solution* $(Y,Z) \in S^2_{\mathbb{F}} \times H^2_{\mathbb{F}}$.

Proof. The proof adapts the argument we have given for the DBSDEs. We define the mapping

$$\Phi : H^2_{\mathbb{F}} \times H^2_{\mathbb{F}} \to H^2_{\mathbb{F}} \times H^2_{\mathbb{F}}$$

by setting

$$\Phi(U, V) := (Y, Z),$$

where for $U \in H^2_{\mathbb{F}}$, $U_t = \xi$, $t > T$, $V \in H^2_{\mathbb{F}}$, $V_t = \eta$, $t > T$, $(Y, Z) \in S^2_{\mathbb{F}} \times H^2_{\mathbb{F}}$ $\left(\subset \left(H^2_{\mathbb{F}} \times H^2_{\mathbb{F}}\right)\right)$ is the unique solution of

$$\begin{cases} Y_t = \xi + \int_t^T \mathbb{E}\left[f(s, U_s, U_{s+\delta}, V_s, V_{s+\delta})ds \mid \mathcal{F}_s\right] - \int_t^T Z_s dW_s, & t \in [0, T], \\ Y_t = \xi,\ Z_t = \eta, & t > T. \end{cases}$$

For $\beta > 0$, let

$$\|(Y, Z)\|^2_\beta \quad := \mathbb{E}\left[\int_0^T e^{\beta t}\left(|Y_t|^2 + |Z_t|^2\right) dt\right].$$

We will show that for suitably chosen $\beta > 1$, the mapping

$$\Phi : \left(H^2_{\mathbb{F}} \times H^2_{\mathbb{F}}, \|\cdot\|_\beta\right) \to \left(H^2_{\mathbb{F}} \times H^2_{\mathbb{F}}, \|\cdot\|_\beta\right)$$

is contracting, i.e., there is a unique fixed point $(Y, Z) \in \left(H^2_{\mathbb{F}} \times H^2_{\mathbb{F}}\right)$ of Φ. Then,

$$\begin{cases} Y_t = \xi + \int_t^T \mathbb{E}\left[f(s, Y_s, Y_{s+\delta}, Z_s, Z_{s+\delta})ds \mid \mathcal{F}_s\right] - \int_t^T Z_s dW_s, & \in [0, T], \\ Y_t = \xi,\ Z_t = \eta, & t > T. \end{cases}$$

It follows that Y has a continuous version. By standard estimates, we have

$$\begin{aligned} \mathbb{E}\left[\sup_{t \in [0,T]} |Y_t|^2\right] \quad &\leq 3\mathbb{E}\left[|\xi|^2\right] + 3T\mathbb{E}\left[\int_t^T \left|\mathbb{E}\left[f(s, U_s, U_{s+\delta}, V_s, V_{s+\delta})ds \mid \mathcal{F}_s\right]\right|^2 ds\right] \\ &+12\mathbb{E}\left[\int_t^T |Z(s)|^2\, ds\right] < \infty. \end{aligned}$$

Consequently, $Y \in S^2_{\mathbb{F}}$. Let us consider that $(U, V), (U', V') \in H^2_{\mathbb{F}} \times H^2_{\mathbb{F}}$ and use the simplified notations:

$$\begin{aligned} (Y, Z) \quad &:= \Phi(U, V), \\ (Y', Z') \quad &:= \Phi(U', V'), \\ (\bar{U}, \bar{V}) \quad &= (U, V) - (U', V'), \\ \left(\bar{Y}, \bar{Z}\right) \quad &= (Y, Z) - (Y', Z'). \end{aligned}$$

Applying Itô's formula to $e^{\beta t}\left|\bar{Y}_t\right|^2$ and taking conditional expectation, we get

$$\begin{aligned} &e^{\beta t}\left|\bar{Y}_t\right|^2 + \mathbb{E}\left[\int_t^T e^{\beta s}\left(\beta\left|\bar{Y}_s\right|^2 + \left|\bar{Z}_s\right|^2\right) ds \mid \mathcal{F}_t\right] \\ &\leq 2\mathbb{E}\left[\int_t^T e^{\beta s}\left|\bar{Y}_s\right| \times\right. \\ &\left. \times\left|\mathbb{E}\left[f(s, U_s, U_{s+\delta}, V_s, V_{s+\delta}) - f(s, U'_s, U'_{s+\delta}, V'_s, V'_{s+\delta}) \mid \mathcal{F}_s\right]\right| ds \mid \mathcal{F}_t\right]. \end{aligned}$$

By Lipschitz condition of Assumption H1, integrating from 0 to T and taking the expectation, we obtain

$$\begin{aligned}
&\mathbb{E}\left[\left|\bar{Y}_0\right|^2\right]+\mathbb{E}\left[\int_0^T e^{\beta s}\left(\beta\left|\bar{Y}_s\right|^2+\left|\bar{Z}_s\right|^2\right)ds\right] \\
&\leq 2C\mathbb{E}\left[\int_0^T e^{\beta s}\left|\bar{Y}_s\right|\cdot\left(\left|\bar{U}_s\right|+\left|\bar{V}_s\right|\right.\right. \\
&\left.\left.+\left(\mathbb{E}\left[\left|\bar{U}_{s+\delta}\right|^2 \mid \mathcal{F}_s\right]\right)^{\frac{1}{2}}+\left(\mathbb{E}\left[\left|\bar{V}_{s+\delta}\right|^2 \mid \mathcal{F}_s\right]\right)^{\frac{1}{2}}\right)ds\right] \\
&\leq 8\rho C^2\mathbb{E}\left[\int_0^T e^{\beta s}\left|\bar{Y}_s\right|^2 ds\right]+\frac{1}{2\rho}\mathbb{E}\left[\int_0^T e^{\beta s}\left(\left|\bar{U}_s\right|^2+\left|\bar{V}_s\right|^2\right.\right. \\
&\left.\left.+\mathbb{E}\left[\left|\bar{U}_{s+\delta}\right|^2 \mid \mathcal{F}_s\right]+\mathbb{E}\left[\left|\bar{V}_{s+\delta}\right|^2 \mid \mathcal{F}_s\right]\right)ds\right] \\
&\leq 8\rho C^2\mathbb{E}\left[\int_0^T e^{\beta s}\left|\bar{Y}_s\right|^2 ds\right]+\frac{1}{2\rho}\mathbb{E}\left[\int_0^T e^{\beta s}\left(\left|\bar{U}_s\right|^2+\left|\bar{V}_s\right|^2\right.\right. \\
&\left.\left.+\left|\bar{U}_s\right|^2+\left|\bar{V}_s\right|^2\right)ds\right].
\end{aligned}$$

Choosing $\beta = 8\rho C^2 + 1$, we have

$$\mathbb{E}\left[\int_0^T e^{\beta t}\left(\left|\bar{Y}_t\right|^2+\left|\bar{Z}_t\right|^2\right)dt\right] \leq \frac{1}{\rho}\mathbb{E}\left[\int_0^T e^{\beta t}\left(\left|\bar{U}_t\right|^2+\left|\bar{V}_t\right|^2\right)dt\right].$$

Choose $\rho > 1$, we have that $\Phi : \left(H^2_{\mathbb{F}} \times H^2_{\mathbb{F}}, \|\cdot\|_\beta\right) \to \left(H^2_{\mathbb{F}} \times H^2_{\mathbb{F}}, \|\cdot\|_\beta\right)$ has a unique fixed point: $(Y, Z) \in \left(H^2_{\mathbb{F}} \times H^2_{\mathbb{F}}\right)$. ■

Chapter 3

BSDEs under progressive filtration enlargement

In this chapter we study delayed and advanced backward stochastic differential equations driven by a Brownian motion and a process with a single random jump. Such a jumps occurs in a natural way, for instance, in models with default risk, and the jump happens in such models at the default time. As the default can come from exogenous sources which are not described by the Brownian model, it is not a stopping time with respect to the Brownian filtration, and even, in general, not measurable with respect to the σ-field generated by the underlying Brownian motion. This makes necessary to enlarge the Brownian filtration progressively by the information coming from the default time, so that the default time becomes a stopping time under the enlarged filtration.

Our DBSDEs and ABSDEs will be studied in the context of this enlargment of the Brownian filtration. But we'll see that these BSDEs are linked with Brownian BSDEs through the decomposition of the terminal condition and the driving coefficient with respect to this progressive enlargement of filtration, as it was done in Kharroubi, Lim and Ngoupeyou [16] for standard BSDEs. We prove that the equation including the jump has a solution if the associated Brownian BSDEs have solutions.

Another approach, which we'll also exploit is the use of the property of martingale representation combined with standard arguments wellknown for BSDEs in a Brownian setting.

3.1 Framework

Let $(\Omega, \mathcal{G}, P)$ be a complete probability space. We assume that this space is equipped with a one-dimensional standard Brownian motion W and we denote by $\mathbb{F} := (\mathcal{F}_t)_{t\geq 0}$ the right continuous complete filtration generated by W. We also consider on this space a random time τ, which represents for example a default time in credit risk or in counterparty risk, or a death time in actuarial issues. The random time τ is not assumed to be an $\mathbb{F}$-stopping time. We therefore use in the sequel the standard approach of filtration enlargement by considering $\mathbb{G}$ the smallest right continuous extension of $\mathbb{F}$ that turns τ into a $\mathbb{G}$-stopping time (see e.g. [15, 14]). More precisely $\mathbb{G} := (\mathcal{G}_t)_{t\geq 0}$ is defined by

$$\mathcal{G}_t := \bigcap_{\varepsilon>0} \tilde{\mathcal{G}}_{t+\varepsilon} ,$$

for all $t \geq 0$, where $\tilde{\mathcal{G}}_s := \mathcal{F}_s \vee \sigma(1_{\tau\leq u} , u \in [0, s])$, for all $s \geq 0$.

We denote by $\mathcal{P}(\mathbb{F})$ (resp. $\mathcal{P}(\mathbb{G})$) the σ-algebra of $\mathbb{F}$ (resp. $\mathbb{G}$)-predictable subsets of $\Omega \times \mathbb{R}_+$, i.e. the σ-algebra generated by the left-continuous $\mathbb{F}$ (resp. $\mathbb{G}$)-adapted processes.

We now introduce a decomposition result for $\mathcal{P}(\mathbb{G})$-measurable processes.

Proposition 5 *Any $\mathcal{P}(\mathbb{G})$-measurable process $X = (X_t)_{t\geq 0}$ is represented as*

$$X_t = X^b_t 1_{t\leq\tau} + X^a_t(\tau)1_{t>\tau} ,$$

for all $t \geq 0$, where X^b is $\mathcal{P}(\mathbb{F})$-measurable and X^a is $\mathcal{P}(\mathbb{F})\otimes\mathcal{B}(\mathbb{R}_+)$-measurable.

This result is proved in Lemma 4.4 of [15] for bounded processes and is easily extended to the case of unbounded processes.

We then impose the following assumptions, which are classical in the filtration enlargement theory.

(H) The process W is a $\mathbb{G}$-Brownian motion. We observe that, since the filtration $\mathbb{F}$ is generated by the Brownian motion W, this is equivalent with the fact that all $\mathbb{F}$-martingales are also $\mathbb{G}$-martingales. Moreover, it also follows that the stochastic integral $\int_0^t X_s dW_s$ is well defined for all $\mathcal{P}(\mathbb{G})$-measurable processes X such that $\int_0^t |X_s|^2 ds < \infty$, for all $t \geq 0$.

(DH) Density hypothesis: the law of τ denoted by μ has no atoms and there exists a non-negative $\mathcal{O} \otimes \mathcal{B}(\mathbb{R})$-measurable function
$(\omega, t, u) \to \alpha_t(\omega, u) =: \alpha^u_t(\omega)$ càdlàg in t such that
(a) for every u, the process $(\alpha^u_t, t \geq 0)$ is an $\mathbb{F}$-martingale and
(b) for every $t \geq 0$, the measure $\alpha^u_t(\omega)\mu(du)$ is a version of $\mathbb{P}(\tau \in du|\mathcal{F}_t)(\omega)$.

Here $\mathcal{O}$ denotes the σ-field of all $\mathbb{F}$-optional subsets of $\Omega\times[0, T]$. Recall that this σ-field is generated by the set of all $\mathbb{F}$-adapted right-continuous processes.

We introduce the $\mathbb{F}$-supermartingale G (called Azéma's supermartingale) defined as

$$G_t = P(\tau > t \mid \mathcal{F}_t) .$$

In the sequel we denote by H the process $1_{\tau\leq\cdot}$.

- Under (H), the supermartingale G is non-increasing, and the compensated martingale of H is

$$M_t = H_t - \int_0^{t\wedge\tau} \frac{dA^p_s}{G_{s-}}, t \geq 0,$$

where A^p is the $\mathbb{F}$-predictable process in the Doob-Meyer decomposition of the $\mathbb{F}$-supermartingale G.

- Under (DH), the process

$$H_t - \int_0^{t\wedge\tau} \frac{\alpha_s(s)}{G_{s-}} \mu(ds), t \geq 0,$$

is a $\mathbb{G}$-martingale. Let us suppose that μ (the law of τ) possesses a density f_τ w.r.t. Lebesque measure. The $\mathbb{F}$-adapted process

$$\lambda_t = \tilde{\lambda}_t f_\tau(t) = \frac{\alpha_t\,(t)}{G_{t-}} f_\tau(t) 1_{(G_{t-}>0)}$$

is called the $\mathbb{F}$-intensity of τ. We assume that the process λ is bounded.

- Under (H) any square integrable $\mathbb{G}$ martingale Y admits a representation as

$$Y_t = y + \int_0^t \varphi_s dW_s + \int_0^t \gamma_s dM_s$$

where M is the compensated martingale of H, and φ, γ are square-integrable $\mathbb{G}$ predictable processes. (See Theorem 3.15 in [4]).

Remark 6
(i) The hypotheses (H) and (DH) imply that

$$\alpha^u_t = \alpha^u_u\ ,\ t \geq u\ .$$

(ii) Under (DH), $P(\tau = \theta < +\infty) = 0$, for all θ $\mathbb{F}$-stopping times, . Moreover, under (DH) the process G is continuous.
(iii) Note that $\tau \leq R$, where $R = \inf\{t : G_t = 0\} = \inf\{t : G_t = 0\}$ (Lemma 1.34 [4]).

Throughout this section, we introduce some basic notations and spaces for any filtration $\mathbb{A} = (\mathcal{A}_t)_{t\geq 0}$ where $\mathbb{A} \in \{\mathbb{F}, \mathbb{G}\}$.

- $S^2_{\mathbb{A}}$ is the subset of $\mathbb{R}$-valued $\mathbb{A}$-adapted càdlàg processes $(Y_t)_{t\in[0,T]}$, such that

$$\|Y\|^2_{S^2} := \mathbb{E}\left[\sup_{t\in[0,T]} |Y_t|^2\right] < \infty.$$

- $H^2_{\mathbb{A}}$ is the subset of $\mathbb{R}$-valued $\mathcal{P}(\mathbb{A})$-measurable processes $(Z_t)_{t\in[0,T]}$, such that
$$\|Z\|^2_{H^2} := \mathbb{E}\left[\int_0^T |Z_t|^2\,dt\right] < \infty.$$

- $L^2(\lambda)$ is the subset of $\mathbb{R}$-valued $\mathcal{P}(\mathbb{G})$-measurable processes $(U_t)_{t\in[0,T]}$, such that
$$\|U\|^2_{L^2(\lambda)} = \mathbb{E}\left[\int_0^{T\wedge\tau} \lambda_t\,|U_t|^2\,dt\right] < \infty.$$

3.2 DBSDEs under progressive enlargement of filtration

In this section we give the existence of a solution to a DBSDE in the enlarged filtration $\mathbb{G}$ as soon as an associated DBSDE in the filtration $\mathbb{F}$ admits a solution. Under Hypothesis (H), consider for a positive constant δ

$$\begin{cases} Y_t = \xi + \int_t^T f(s, Y_s, Y_{s-\delta}, Z_s, Z_{s-\delta}, U_s)1_{[0,\tau]}ds & \\ \qquad\qquad - \int_t^T Z_s dW_s - \int_t^T U_s dH_s, & t\in[0,T], \\ Y_t = Y_0,\ Z_t = U_t = 0, & t<0, \end{cases} \tag{3.1}$$

where $f:\Omega\times[0,T]\times\mathbb{R}^5\to\mathbb{R}$ satisfies the Lipschitz condition and the terminal condition ξ is a bounded $\mathcal{G}_{T\wedge\tau}$-measurable random variable of the form

$$\xi = \xi^b 1_{T<\tau} + \xi^a_\tau 1_{\tau\le T}, \tag{3.2}$$

where ξ^b is a bounded $\mathcal{F}_T$-measurable random variable and ξ^a is bounded and $\mathbb{F}$-predictable process.

Now by Proposition 2.1 and Remark 2.1 in [16], there exists a function f^b which is $\mathbb{F}$-predictable, such that

$$f(t,\cdot,\cdot,\cdot,\cdot,\cdot)1_{t\le\tau} = f^b(t,\cdot,\cdot,\cdot,\cdot,\cdot)1_{t\le\tau},\ t\ge 0, \tag{3.3}$$

given by

$$f^b(t,\cdot,\cdot,\cdot,\cdot,\cdot) = \frac{1}{G_t}\mathbb{E}(f(t,\cdot,\cdot,\cdot,\cdot,\cdot)1_{t\le\tau}|\mathcal{F}_t)1_{\{G_t>0\}}.$$

We want to prove the existence of the solution of the DBSDE (3.1) via a decomposition approach in [16], coming from the enlarged filtration $\mathbb{G}$ as soon as an associated DBSDE in the filtration $\mathbb{F}$ admits a solution.

We introduce then the DBSDE $\left(Y^b, Z^b\right)\in S^2_{\mathbb{F}}\times H^2_{\mathbb{F}}$ associated to the generator f^b and the terminal condition ξ^b as follows

$$\begin{cases} Y^b_t = \xi^b + \int_t^T \tilde{f}(s, Y^b_s, Y^b_{s-\delta}, Z^b_s, Z^b_{s-\delta})ds - \int_t^T Z^b_s dW_s, & t\in[0,T], \\ Y^b_t = Y^b_0,\ Z^b_t = 0, & t<0, \end{cases} \tag{3.4}$$

where $\tilde{f}(s,y,\tilde{y},z,\tilde{z}) = f^b(s,y,\tilde{y},z,\tilde{z},\xi^a_s - y)$.

In order to prove existence results for the DBSDE (3.4), we need that the generator $\tilde{f}$ and the terminal condition ξ^b satisfy the Assumption H1. Indeed

(i) $\xi^b \in L^2(\Omega, \mathcal{F}_T)$ is bounded and $\tilde{f}$ is $\mathcal{P}(\mathbb{F})$-measurable, by definition f and the fact that ξ^a is $\mathbb{F}$-predictable, respectively.

(ii) The *Lipschitz condition* also holds for $\tilde{f}$, since; for all $t \in [0,T]$, we have that

$$f^b(t,\cdot,\cdot,\cdot,\cdot,\cdot) = \frac{1}{G_t}\mathbb{E}(f(t,\cdot,\cdot,\cdot,\cdot,\cdot)1_{t\leq\tau}|\mathcal{F}_t)1_{\{G_t>0\}}. \tag{3.5}$$

Indeed, for all $(y,u), (\tilde{y},\tilde{u}) \in \mathbb{R}\times\mathbb{R}^3$, we have

$$\begin{aligned}
&\left|\tilde{f}(t,y,u) - \tilde{f}(t,\tilde{y},\tilde{u})\right| \\
&= \left|f^b(t,y,u,\xi_t^a - y) - f^b(t,\tilde{y},\tilde{u},\xi_t^a - \tilde{y})\right| \\
&= \left|\frac{\mathbb{E}\left[\left(f(t,y,u,\xi_t^a - y) - f(t,\tilde{y},\tilde{u},\xi_t^a - \tilde{y})\right)1_{t\leq\tau} \mid \mathcal{F}_t\right]}{G_t}1_{\{G_t>0\}}\right|,
\end{aligned} \tag{3.6}$$

and from the Lipschitz condition on f, we get that

$$\begin{aligned}
&\left|\tilde{f}(t,y,u) - \tilde{f}(t,\tilde{y},\tilde{u})\right| \\
&\leq \left|K(|y_-\tilde{y}| + |u-\tilde{u}|)\frac{\mathbb{E}\left[1_{t\leq\tau} \mid \mathcal{F}_t\right]}{G_t}1_{\{G_t>0\}}\right| \\
&= K(|y_-\tilde{y}| + |u-\tilde{u}|)1_{\{G_t>0\}} \\
&\leq K(|y_-\tilde{y}| + |u-\tilde{u}|).
\end{aligned} \tag{3.7}$$

(iii) *The integrability condition holds for $\tilde{f}$. Indeed,*

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T \left|\tilde{f}(t,0,0,0,0)\right|^2 dt\right] \\
&= \mathbb{E}\left[\int_0^T \left|f^b(t,0,0,0,0,\xi_t^a)\right|^2 dt\right] \\
&\leq 2\mathbb{E}\left[\int_0^T \left|f^b(t,0,0,0,0,\xi_t^a) - f^b(t,0,0,0,0,0)\right|^2 dt\right] \\
&+2\mathbb{E}\left[\int_0^T \left|f^b(t,0,0,0,0,0)\right|^2 dt\right] \\
&\leq \mathbb{E}\left[C\frac{\mathbb{E}\left[1_{t\leq\tau} \mid \mathcal{F}_t\right]}{G_t}1_{\{G_{t-}>0\}}\right] \\
&+2\mathbb{E}\left[\int_0^T \left|\frac{1}{G_t}\mathbb{E}(f(t,0,0,0,0,0)1_{t\leq\tau}|\mathcal{F}_t)1_{\{G_t>0\}}\right|^2 dt\right] \\
&\leq C + 2\mathbb{E}\left[\int_0^T |f(t,0,0,0,0,0)|^2 dt\right] < \infty.
\end{aligned}$$

Since f is Lipschitz, the last inequality holds by using the integrability condition on f, the $\mathbb{F}$-adaptedness of f, the definition of G and the boundedness of ξ^a.

Then, by Theorem 3, there exists some $\delta' > 0$, such that for all $\delta \in (0,\delta')$ the DBSDE (3.4) admits a unique solution.

Theorem 7 *Assume that the couple* $\left(Y^b, Z^b\right) \in S^2_{\mathbb{F}} \times H^2_{\mathbb{F}}$ *is a solution of* (3.4). *Then the DBSDE* (3.1) *admits a solution* $(Y, Z, U) \in S^2_{\mathbb{G}} \times H^2_{\mathbb{G}} \times L^2(\lambda)$, *given by*

$$\begin{aligned} Y_t &= Y^b_t 1_{t<\tau} + \xi^a_\tau 1_{t\geq\tau}, \\ Z_t &= Z^b_t 1_{t\leq\tau}, \\ U_t &= \left(\xi^a_t - Y^b_t\right) 1_{t\leq\tau}, \end{aligned} \tag{3.8}$$

for all $t \in [0, T]$.

Proof. We have

$$U_\cdot = \left(\xi^a_\cdot - Y^b_\cdot\right) 1_{[0,\tau]}.$$

We know that ξ^a and Y^b are $\mathbb{F}$-adapted and continuous, thus $\mathcal{P}(\mathbb{F})$-measurable and consequently, $\mathcal{P}(\mathbb{G})$-measurable. Since τ is a $\mathbb{G}$-stopping time, $1_{[0,\tau]}$ is left continuous and $\mathbb{G}$-adapted and hence $\mathcal{P}(\mathbb{G})$-measurable. We conclude that U is $\mathcal{P}(\mathbb{G})$-measurable. We will show that (Y, Z, U) defined by (3.8) satisfies the DBSDE (3.1). Note that for all $0 \leq t \leq T$, and since $\delta > 0$, we have that $t - \delta < t$, then

$$\begin{aligned} Y_{t-\delta} 1_{t<\tau} &= Y^b_{t-\delta} 1_{t<\tau} , \\ Z_{t-\delta} 1_{t<\tau} &= Z^b_{t-\delta} 1_{t<\tau} . \end{aligned} \tag{3.9}$$

The proof is decomposed into three steps.

Step 1 We can set three cases according to τ, for a fixed $t \in [0, T]$.

Case 1: On the set $\tau > T$: By (3.8), we have $Y_t = Y^b_t$, $Z_t = Z^b_t$ and $U_t = \xi^a_t - Y^b_t$, and by (3.9) we get $Y_{t-\delta} = Y^b_{t-\delta}$ and $Z_{t-\delta} = Z^b_{t-\delta}$. Using that $\left(Y^b, Z^b\right)$ is a solution of (3.4), we can rewrite (3.4) as

$$Y_t = \xi^b + \int_t^T f^b(s, Y_s, Y_{s-\delta}, Z_s, Z_{s-\delta}, U_s) ds - \int_t^T Z^b_s dW_s.$$

Since the predictable processes Z and Z^b are indistinguishable on $\{\tau > T\}$, we have from Theorem 12.23 of [11], $\int_t^T Z_s dW_s = \int_t^T Z^b_s dW_s$ on $\{\tau > T\}$. Moreover, since $\xi = \xi^b$ and by the definition of H, $\int_{t\wedge\tau}^{T\wedge\tau} U_s dH_s = 0$ on $\{\tau > T\}$, we obtain then by (3.3) and using that $T \wedge \tau = T$ and $t \wedge \tau = t$ on $\{\tau > T\}$

$$Y_t = \xi + \int_{t\wedge\tau}^{T\wedge\tau} f(s, Y_s, Y_{s-\delta}, Z_s, Z_{s-\delta}, U_s) ds - \int_{t\wedge\tau}^{T\wedge\tau} Z_s dW_s - \int_{t\wedge\tau}^{T\wedge\tau} U_s dH_s.$$

Case 2: For $\tau \in (t, T]$, by (3.8) we have $Y_t = Y^b_t$. Since $\left(Y^b, Z^b\right)$ is a solution to (3.4), we have

$$Y_t = Y^b_\tau + \int_t^\tau f^b(s, Y^b_s, Y^b_{s-\delta}, Z^b_s, Z^b_{s-\delta}, \xi^a_s - Y^b_s) ds - \int_t^\tau Z^b_s dW_s. \tag{3.10}$$

Using (3.3), (3.9), (3.8), adding and subtracting ξ^a_τ in (3.10), we get

$$Y_t = \xi^a_\tau + \int_t^\tau f^b(s, Y^b_s, Y^b_{s-\delta}, Z^b_s, Z^b_{s-\delta}, \xi^a_s - Y^b_s) ds - \int_t^\tau Z^b_s dW_s - \left(\xi^a_\tau - Y^b_\tau\right).$$

Since the predictable processes $Z\ 1_{.<\tau}$ and $Z^b\ 1_{.<\tau}$ are indistinguishable on $\{\tau > t\} \cap \{\tau \leq T\}$, we have from Theorem 12.23 of [11],
$\int_{t\wedge\tau}^{T\wedge\tau} Z_s dW_s = \int_{t\wedge\tau}^{T\wedge\tau} Z_s^b dW_s$ on $\{\tau > t\} \cap \{\tau \leq T\}$. Therefore, we get

$$Y_t = \xi_\tau^a + \int_t^\tau f(s, Y_s, Y_{s-\delta}, Z_s, Z_{s-\delta}, U_s) ds - \int_t^\tau Z_s^b dW_s - \left(\xi_\tau^a - Y_\tau^b\right).$$

Finally, we easily check from the definition of U that $\int_{t\wedge\tau}^{T\wedge\tau} U_s dH_s = \xi_\tau^a - Y_\tau^b$. Therefore, by using (3.2)

$$Y_t = \xi + \int_{t\wedge\tau}^{T\wedge\tau} f(s, Y_s, Y_{s-\delta}, Z_s, Z_{s-\delta}, U_s) ds - \int_{t\wedge\tau}^{T\wedge\tau} Z_s dW_s - \int_{t\wedge\tau}^{T\wedge\tau} U_s dH_s.$$

Case 3: When $\tau \leq t$, we have from (3.8) that $Y_t = \xi_\tau^a$, therefore by using (3.2) and because $t \wedge \tau = T \wedge \tau$, it holds that

$$Y_t = \xi + \int_{t\wedge\tau}^{T\wedge\tau} f(s, Y_s, Y_{s-\delta}, Z_s, Z_{s-\delta}, U_s) ds - \int_{t\wedge\tau}^{T\wedge\tau} Z_s dW_s - \int_{t\wedge\tau}^{T\wedge\tau} U_s dH_s.$$

Step 2 We notice that Y is a càdlàg, $\mathbb{G}$-adapted process and U is $\mathcal{P}(\mathbb{G})$-measurable since Y^b and ξ^a are continuous and $\mathbb{F}$-adapted. We also notice from its definition that the process Z is $\mathcal{P}(\mathbb{G})$-measurable, since Z^b is $\mathcal{P}(\mathbb{F})$-measurable.

Step 3 We prove now that the solution satisfies the integrability conditions.

- From the definition of Y, we have$|Y_t| \leq \left|Y_t^b\right| + |\xi_t^a|; t \in [0, T]$. Since $Y^b \in S_{\mathbb{F}}^2$ and $\xi^a \in S_{\mathbb{F}}^2$, we get that $\|Y\|_{S^2} < +\infty$.
- From the definition of the process Z, we have $Z \in H_{\mathbb{G}}^2$.
- From the definition of U, we have

$$|U_t| \leq \left|Y_t^b\right| + |\xi_t^a|; t \in [0, T].$$

- Since $Y^b \in S_{\mathbb{F}}^2$ and $\xi^a \in S_{\mathbb{F}}^2$, and λ is bounded, we get that $U \in L^2(\lambda)$.

■

3.3 ABSDEs under progressive enlargement of filtration

While we have discussed in the previous section the solution of the BSDEs under the enlargement of filtration $\mathbb{G}$ via a decomposition approach, we go now to the other direction, i.e., we prove the existence and the uniqueness of the ABSDEs under the filtration $\mathbb{G}$ by using the property of martingale representation (for short: PMR). Under (DH) we study the following equation for a positive constant δ

$$\begin{cases} Y_t = \xi + \int_t^T \mathbb{E}\left[f(s, Y_s, Y_{s+\delta}, Z_s, Z_{s+\delta}, U_s) \mid \mathcal{G}_s\right] ds \\ \qquad\qquad - \int_t^T Z_s dW_s - \int_t^T U_s dH_s, & t \in [0,T], \\ Y_t = \xi, \;\; Z_t = \eta_t, & t > T. \end{cases} \tag{3.11}$$

where f is $\mathcal{G}_t \otimes \mathcal{B}([0,T]) \otimes \mathcal{B}(\mathbb{R}^5)$-measurable, and the terminal condition ξ is $\mathcal{G}_T$-measurable.

Assumption H4

(i) *Assumption on the terminal condition:*

- $\xi \in L^2(\Omega, \mathcal{G}_T)$ is bounded, and
  $\eta = (\eta_t)_{t\in[T,T+\delta]} \in L^2(\Omega, \mathcal{G}_T \otimes \mathcal{B}([T, T+\delta]), dtdP)$.

(ii) *Assumptions on the generator function* $f : \Omega \times \mathbb{R}_+ \times \mathbb{R}^5 \to \mathbb{R}$ is

- $\mathbb{G}$-predictable and satisfies the integrability condition, such that

$$\mathbb{E}\left[\int_0^T |f(t,0,0,0,0,0)|^2 dt\right] < \infty,$$

for all $t \in [0,T]$.

- Lipschitz in the sense that, there exists $C > 0$ such that

$$\begin{aligned} &|f(t, y, \hat{y}, z, \hat{z}, u) - f(t, y', \hat{y}', z', \hat{z}', u')| \\ &\leq C\left(|y - y'| + |\hat{y} - \hat{y}'| + |z - z'| + |\hat{z} - \hat{z}'| + \lambda_t(1 - H_t)|u - u'|\right), \end{aligned}$$

for all $t \in [0,T]$ and all $y, y', \hat{y}, \hat{y}', z, z', \hat{z}, \hat{z}', u, u' \in \mathbb{R}$.

What we are going to do in this paper is to prove that there exists

$$(Y, Z, U) \in S^2_{\mathbb{G}} \times H^2_{\mathbb{G}} \times L^2(\lambda),$$

such that equation (3.11) is satisfied.

The existence follows from the property of martingale representation and a standard approach like any BSDE governed by a Brownian motion.

Under our assumptions we know that equation (3.11) is equivalent to

$$\begin{cases} Y_t = \xi + \int_t^T \mathbb{E}\left[F(s, Y_s, Y_{s+\delta}, Z_s, Z_{s+\delta}, U_s) \mid \mathcal{G}_s\right] ds \\ \qquad\qquad - \int_t^T Z_s dW_s - \int_t^T U_s dM_s, & t \in [0,T], \\ Y_t = \xi, \; Z_t = \eta_t, & t > T, \end{cases} \tag{3.12}$$

with $dH_s = dM_s + \lambda_s 1_{s<\tau} ds$, λ is $\mathbb{F}$-predictable, bounded, and

$$F(s, y, y', z, z', u) := f(s, y, y', z, z', u) - \lambda_s(1 - H_s)u. \tag{3.13}$$

In order to get existence and uniqueness for the ABSDE (3.12),let us check that the generator F satisfies the same assumption as f : The function F : $\Omega \times \mathbb{R}_+ \times \mathbb{R}^5 \to \mathbb{R}$ is such that

(i) $\mathbb{G}$-predictable and integrable in the sense that, for all $t \in [0,T]$, we have by equality (3.13)

$$\mathbb{E}\left[\int_0^T |F(t,0,0,0,0,0)|^2 dt\right] = \mathbb{E}\left[\int_0^T |f(t,0,0,0,0,0)|^2 dt\right] < \infty.$$

(ii) Lipschitz in the sense that there exists a constant $C' > 0$, such that

$$\begin{array}{l} |F(t,y,y',z,z',u) - F(t,\tilde{y},\tilde{y}',\tilde{z},\tilde{z}',u')| \\ = |f(t,y,y',z,z',u) - f(t,\tilde{y},\tilde{y}',\tilde{z},\tilde{z}',u') - \lambda_t(1-H_t)(u-u')| \\ \leq |f(t,y,y',z,z',u) - f(t,\tilde{y},\tilde{y}',\tilde{z},\tilde{z}',u')| + \lambda_t(1-H_t)|u-u'| \\ \leq C\left(|y-\tilde{y}| + |y'-\tilde{y}'| + |z-z'| + |\tilde{z}-\tilde{z}'| + \lambda_t(1-H_t)|u-u'|\right) + \lambda_t(1-H_t)|u-u'| \\ \leq C'\left(|y-\tilde{y}| + |y'-\tilde{y}'| + |z-z'| + |\tilde{z}-\tilde{z}'| + \lambda_t(1-H_t)|u-u'|\right), \end{array}$$

for all $t \in [0,T]$ and all $y, y', z, z', u, \tilde{y}, \tilde{y}', \tilde{z}', \tilde{z}', u' \in \mathbb{R}$ (Recall that the process λ_t is bounded).

The terminal value:

(iii) $\xi \in L^2(\Omega, \mathcal{G}_T)$ is bounded, and
$\eta = (\eta_t)_{t\in[T,T+\delta]} \in L^2(\Omega, \mathcal{G}_T \otimes \mathcal{B}([T,T+\delta]), dtdP)$.

Theorem 8 *Under the above assumptions (i-iii), ABSDE* (3.12) *admits a unique solution* $(Y,Z,U) \in S^2_{\mathbb{G}} \times H^2_{\mathbb{G}} \times H^2_{\mathbb{G}}$.

Proof. We define the mapping

$$\Phi : H^2_{\mathbb{G}} \times H^2_{\mathbb{G}} \times H^2_{\mathbb{G}} \to H^2_{\mathbb{G}} \times H^2_{\mathbb{G}} \times H^2_{\mathbb{G}},$$

for which we will show that it is contracting under a suitable norm. For this we note that for any $(Y,Z,U) \in H^2_{\mathbb{G}} \times H^2_{\mathbb{G}} \times H^2_{\mathbb{G}}$ there exists a unique pair $\left(\hat{Y},\hat{Z},\hat{U}\right) \in S^2_{\mathbb{G}} \times H^2_{\mathbb{G}} \times H^2_{\mathbb{G}}$; such that

$$\left\{\begin{array}{ll} \hat{Y}_t = \xi + \int_t^T h_s ds - \int_t^T \hat{Z}_s dW_s - \int_t^T \hat{U}_s dM_s, & t \in [0,T], \\ \hat{Y}_t = \xi, \;\; \hat{Z}_t = \eta_t, & t > T, \end{array}\right. \tag{3.14}$$

with $h_s := \mathbb{E}\left[F(s,Y_s,Y_{s+\delta},Z_s,Z_{s+\delta},U_s) \mid \mathcal{G}_s\right]$. Indeed: $h = (h_s)_{s\in[0,T]}$ is a square integrable, $\mathbb{G}$-adapted process. We put

$$\begin{array}{rl} \hat{Y}_t & := \mathbb{E}\left[\xi + \int_t^T h_s ds \mid \mathcal{G}_s\right] \\ & = \mathbb{E}\left[\xi + \int_0^T h_s ds \mid \mathcal{G}_s\right] - \int_0^t h_s ds, \quad t \in [0,T], \end{array} \tag{3.15}$$

with $\hat{Y}_t = \xi$, $\hat{Z}_t = \eta_t$, when $t > T$. Obviously, $\hat{Y} \in S^2_{\mathbb{G}}$. As we have the PMR with respect to (W,M), there are unique $\hat{Z} \in H^2_{\mathbb{G}}$, $\hat{U} \in H^2_{\mathbb{G}}$, such that

$$\left\{\begin{array}{ll} \xi + \int_0^T h_s ds = \mathbb{E}\left[\xi + \int_0^T h_s ds \mid \mathcal{G}_0\right] + \int_0^T \hat{Z}_s dW_s + \int_0^T \hat{U}_s dM_s, & t \in [0,T], \\ \hat{Y}_t = \xi, \;\; \hat{Z}_t = \eta_t, & t > T, \end{array}\right. \tag{3.16}$$

with $\xi + \int_0^T h_s ds \in L^2(\mathcal{G}_T)$ and by combining (3.16) with (3.15), we get

$$\hat{Y}_t - \int_0^t h_s ds = \hat{Y}_0 + \int_0^T \hat{Z}_s dW_s + \int_0^T \hat{U}_s dM_s, t \in [0,T].$$

Uniqueness of $\hat{Y}$ follows from (3.14)-(3.15) and of $\left(\hat{Z}, \hat{U}\right)$ follows from PMR. Let $\Phi(Y,Z,U) := (\hat{Y}, \hat{Z}, \hat{U})$. For given $\left(Y^i, Z^i, U^i\right) \in H^2_{\mathbb{F}} \times H^2_{\mathbb{F}} \times H^2_{\mathbb{F}}$, for $i = 1, 2$, we use the simplified notations:

$$\begin{array}{ll} \left(\hat{Y}^i, \hat{Z}^i, \hat{U}^i\right) & := \Phi\left(Y^i, Z^i, U^i\right), \\ \left(\tilde{Y}, \tilde{Z}, \tilde{U}\right) & := \left(\hat{Y}^1, \hat{Z}^1, \hat{U}^1\right) - \left(\hat{Y}^2, \hat{Z}^2, \hat{U}^2\right), \\ \left(\bar{Y}, \bar{Z}, \bar{U}\right) & := \left(Y^1, Z^1, U^1\right) - \left(Y^2, Z^2, U^2\right). \end{array}$$

The pair of processes $\left(\tilde{Y}, \tilde{Z}, \tilde{U}\right)$ satisfies the equation

$$\begin{aligned} \tilde{Y}_t = \int_t^T \mathbb{E}\left[F(s, Y^1_s, Y^1_{s+\delta}, Z^1_s, Z^1_{s+\delta}, U^1_s) - F(s, Y^2_s, Y^2_{s+\delta}, Z^2_s, Z^2_{s+\delta}, U^2_s) \mid \mathcal{G}_s\right] ds, \\ - \int_t^T \tilde{Z}_s dW_s - \int_t^T \tilde{U}_s dM_s, \qquad t \in [0,T]. \end{aligned}$$

We have that $M_t = H_t - \displaystyle\int_0^{t\wedge\tau} \lambda_s ds$ which is a pure jump martingale. Then,

$$[M]_t = \sum_{0 \le s \le t} (\triangle M_s)^2 = \sum_{0 \le s \le t} (\triangle H_s)^2 = H_t,$$

and

$$< M >_t = \int_0^{t\wedge\tau} \lambda_s ds = \int_0^t \lambda_s (1 - H_s) ds, t \ge 0,$$

$$\int_t^T \left|\tilde{U}_s\right|^2 d\left\langle M\right\rangle_s = \int_t^T \lambda_s (1 - H_s) \left|\tilde{U}_s\right|^2 ds.$$

Applying Itô's formula to $e^{\beta t} \left|\tilde{Y}_t\right|^2$, taking conditional expectation and using the Lipschitz condition, we get

$$
\begin{aligned}
&e^{\beta t}\left|\tilde{Y}_t\right|^2+\mathbb{E}\left[\int_t^T e^{\beta s}\left(\beta\left|\tilde{Y}_s\right|^2+\left|\tilde{Z}_s\right|^2+\lambda_s(1-H_s)\left|\tilde{U}_s\right|^2\right)ds\mid\mathcal{G}_t\right]\\
&\leq 2\mathbb{E}\left[\int_t^T e^{\beta s}\left|\tilde{Y}_s\right|\times\right.\\
&\left.\times\left|\mathbb{E}\left[F(s,Y_s^1,Y_{s+\delta}^1,Z_s^1,Z_{s+\delta}^1,U_s^1)-F(s,Y_s^2,Y_{s+\delta}^2,Z_s^2,Z_{s+\delta}^2,U_s^2)\mid\mathcal{G}_s\right]\right|ds\mid\mathcal{G}_t\right]\\
&\leq 8C'^2\mathbb{E}\left[\int_t^T e^{\beta s}\left|\tilde{Y}_s\right|^2ds\mid\mathcal{G}_t\right]\\
&+\frac{1}{4}\mathbb{E}\left[\int_t^T e^{\beta s}\left|\bar{Y}_s\right|^2+\left|\bar{Y}_{s+\delta}\right|^2+\left|\bar{Z}_s\right|^2+\left|\bar{Z}_{s+\delta}\right|^2+\lambda_s(1-H_s)\left|\bar{U}_s\right|^2ds\mid\mathcal{G}_t\right]\\
&\leq 8C'^2\mathbb{E}\left[\int_t^T e^{\beta s}\left|\tilde{Y}_s\right|^2ds\mid\mathcal{G}_t\right]\\
&+\frac{1}{2}\mathbb{E}\left[\int_t^T e^{\beta s}\left|\bar{Y}_s\right|^2+\left|\bar{Z}_s\right|^2+\lambda_s(1-H_s)\left|\bar{U}_s\right|^2ds\mid\mathcal{G}_t\right].
\end{aligned}
$$

Choosing $\beta=8C^2+1$ and $t=0$, we have

$$
\begin{aligned}
&\mathbb{E}\left[\int_0^T e^{\beta t}\left(\left|\tilde{Y}_s\right|^2+\left|\tilde{Z}_s\right|^2+\lambda_s(1-H_s)\left|\tilde{U}_s\right|^2\right)ds\right]\\
\leq\ &\frac{1}{2}\mathbb{E}\left[\int_0^T e^{\beta s}\left(\left|\bar{Y}_s\right|^2+\left|\bar{Z}_s\right|^2+\lambda_s(1-H_s)\left|\bar{U}_s\right|^2\right)ds\right].
\end{aligned}
$$

Then Φ is contracting and there exists a unique fixed point (Y,Z,U). ■

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