CHAPTER 3: MATRICES

1. Determinant of a square matrix

1.1. Definitions and properties.

Definition 1.1. Let $A = (a_{ij}) \in \mathcal{M}_n(\mathbb{K})$. We call determinant, the unique application noted by " det " defined from $\mathcal{M}_n(\mathbb{K})$ ro $\mathbb{K}$ as follows:
• For $n = 1$ such that $A = (a)$ with $a \in \mathbb{K}$, then $\det(A) = a$.
• For $n > 1$; $\det(A) = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij})$, for j fixed in certain value between 1 and n, where A_{ij} is the matrix obtained from A by deleting the i^{th} line and j^{th} column. Its determinant i.e. $\det(A_{ij})$ is called the minor of (a_{ij}) in A and the number $(-1)^{i+j} a_{ij} \det(A_{ij})$ is called the cofactor of (a_{ij}) in A.

Remark 1.2. • To compute $\det(A)$, we can use the previous formulae, by mean of the i^{th} line, i.e.,
$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij})$, for i fixed in certain value between 1 and n.
• The determinant of the matrix A can be also noted by $\mid A \mid$.

Example 1.3. Let $A = (6)$. Then, $\det(A) = 6$.

Example 1.4. Let $A = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$. Then, $\det(A) = 1 \times 4 - 2 \times 3 = -2$.

Example 1.5. Let $A = \begin{pmatrix} 1 & 1 & 2 \\ 3 & 1 & 0 \\ 0 & 1 & 4 \end{pmatrix}$. Then, $\det(A) = \sum_{i=1}^{3} (-1)^{i+j} a_{ij} \det(A_{ij})$.
Let us fix $j = 1$, we get

$$\begin{aligned}
\det(A) &= \sum_{i=1}^{3} (-1)^{i+1} a_{i1} \det(A_{i1}) \\
&= (-1)^{1+1} a_{11} \det(A_{11}) + (-1)^{2+1} a_{21} \det(A_{21}) + (-1)^{3+1} a_{31} \det(A_{31}) \\
&= a_{11} \det \begin{pmatrix} 1 & 0 \\ 1 & 4 \end{pmatrix} - a_{21} \det \begin{pmatrix} 1 & 2 \\ 1 & 4 \end{pmatrix} + a_{31} \det \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} \\
&= 1 \times 4 - 3 \times 2 + 0 = -2.
\end{aligned}$$

Properties
Let $A = (a_{ij}), B = (b_{ij}) \in \mathcal{M}_n(\mathbb{K})$ and p a natural number. Then, we have:

• if all the elements of a column or a line are equal to 0, then $\det(A) = 0$.
• if two column (or two lines) are proportional (or equivalent) then, $\det(A) = 0$.

Example 1.6. Let $A = \begin{pmatrix} 2 & -1 & 3 \\ 0 & 1 & 5 \\ 4 & -2 & 6 \end{pmatrix}$. Then, $\det(A) = 0$. Similarly for the matrix $B = \begin{pmatrix} 2 & 1 & 2 \\ -4 & 0 & -4 \\ 3 & 6 & 3 \end{pmatrix}$.

• If we add p times the corresponding elements of another column (line) to the elements of a line (column), the value of the determinant will not change.

Remark 1.7. This property is used to get zeros in a column or a line to make the computation easier.

Example 1.8. Let $A = \begin{pmatrix} 1 & 9 & -3 \\ 4 & 6 & -2 \\ -3 & 1 & 5 \end{pmatrix}$. Then,

$$\begin{aligned} \det(A) &= \det \begin{pmatrix} 1 & 9+3(-3) & -3 \\ 4 & 6+3(-2) & -2 \\ -3 & 1+3(5) & 5 \end{pmatrix} \\ &= \det \begin{pmatrix} 1 & 0 & -3 \\ 4 & 0 & -2 \\ -3 & 16 & 5 \end{pmatrix} = \det(B). \end{aligned}$$

Development of the determinant by taking la j^{th} column for $j = 1$.

$$\begin{aligned} \det(B) &= \sum_{i=1}^{3} (-1)^{i+1} a_{i1} \det(A_{i1}) \\ &= (-1)^{1+1} a_{11} \det(A_{11}) + (-1)^{2+1} a_{21} \det(A_{21}) + (-1)^{3+1} a_{31} \det(A_{31}) \\ &= \det \begin{pmatrix} 0 & -2 \\ 16 & 5 \end{pmatrix} - 4 \det \begin{pmatrix} 0 & -3 \\ 16 & 5 \end{pmatrix} - 3 \det \begin{pmatrix} 0 & -3 \\ 0 & 5 \end{pmatrix} \\ &= 32 - 4(48) = 32 - 192 = -160. \text{ Thus,} \\ \det(A) &= \det(B) = -160. \end{aligned}$$

Example 1.9. Let $A = \begin{pmatrix} 3 & 2 \\ -1 & 4 \end{pmatrix}$. We have $\det(A) = 14$. So that, $\det \begin{pmatrix} 2 & 3 \\ 4 & -1 \end{pmatrix} = \det \begin{pmatrix} -1 & 4 \\ 3 & 2 \end{pmatrix} = -14$.

• If A is an upper triangular matrix, lower triangular or diagonal, then its determinant is equal to the product of the diagonal coefficients, i.e., $\det(A) = a_{11}a_{22}...a_{nn}$.

Example 1.10. If $A = I_n$ (The identity matrix), then $\det(A) = 1$.

• if $A, B \in \mathcal{M}_n(\mathbb{K})$, then

$$\det(A \times B) = \det(A) \times \det(B) = \det(B) \times \det(B).$$

• If we multiply a column (or a line) of matrix by a scalar $\alpha \in \mathbb{K}$, the determinant of the new matrix is multiplied by α.
• If $A \in \mathcal{M}_n(\mathbb{K})$ and $\alpha \in \mathbb{K}$, then $\det(\alpha \cdot A) = \alpha^n \cdot \det(A)$.

Example 1.11. Let $A = \begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix}$, $B = \begin{pmatrix} 4 & 2 \\ 4 & 3 \end{pmatrix}$ and $C = \begin{pmatrix} 4 & 2 \\ 8 & 6 \end{pmatrix}$.
Then $\det(A) = 2$, $\det(B) = 4 = 2\det(A)$ and $\det(C) = 8 = 2^2 \det(A)$.

• Determinant of a sum of matrices:
There is no explicit formula, however we can generally confirm that $\det(A + B) \neq \det(A) + \det(B)$.

Example 1.12. let $A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$ and $B = \begin{pmatrix} -\lambda & 0 \\ 0 & -\lambda \end{pmatrix}$, with $\lambda \in \mathbb{R} - \{0\}$.
Then, $\det(A + B) = \det(0_2) = 0$, and $\det(A) + \det(B) = \lambda^2 + \lambda^2 = 2\lambda^2 \neq 0$.

Theorem 1.13 (Fundamental theorem)**.** *Let $A \in \mathcal{M}_n(\mathbb{K})$, Then*

$$A \textit{ is inverible } \Leftrightarrow \det(A) \neq 0\,.$$

Proposition 1.14. *If A is invertible, then*

$$\det(A^{-1}) = \frac{1}{\det(A)}\,.$$

Proposition 1.15 (Computaion of the inverse by the determinant)**.** *Let $A = (a_{ij}) \in \mathcal{M}_n(\mathbb{K})$ and i, j be two natural numbers.*

$$\textit{If } A \textit{ is invertible, then } A^{-1} = \frac{1}{\det(A)} C^t\,.$$

$C = (c_{ij}) = (-1)^{i+j} a_{ij} \det(A_{ij})$, C is the cofactor matrix (comatrix).

Example 1.16. We consider the matrix $A = \begin{pmatrix} 1 & 0 & -3 \\ 4 & 0 & -2 \\ -3 & 16 & 5 \end{pmatrix}$.

1- Show that A is invertible.

2- Compute the determinant of A^{-1}.
• We have seen from the example 1.8 that $\det(A) = -160 \neq 0$. Then, we conclude that A is invertible.
• To compute A^{-1}, we have $A^{-1} = \frac{1}{\det(A)} C^t$.
We firstly figure out the cofactor matrix $C = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix}$. We have:
The 1^{st} line:
$c_{11} = (-1)^{1+1} a_{11} \det(A_{11}) = \det \begin{pmatrix} 0 & -2 \\ 16 & 5 \end{pmatrix} = 32.$
$c_{12} = (-1)^{1+2} a_{11} \det(A_{12}) = 0.$
$c_{13} = (-1)^{1+3} a_{13} \det(A_{13}) = (-3) \det \begin{pmatrix} 4 & 0 \\ -3 & 16 \end{pmatrix} = -192.$
The 2^{nd} line:
$c_{21} = (-1)^{2+1} a_{21} \det(A_{21}) = (-1)(4) \det \begin{pmatrix} 0 & -3 \\ 16 & 5 \end{pmatrix} = -192.$
$c_{22} = (-1)^{2+2} a_{22} \det(A_{22}) = 0.$
$c_{23} = (-1)^{2+3} a_{23} \det(A_{23}) = (-1)(-2) \det \begin{pmatrix} 1 & 0 \\ -3 & 16 \end{pmatrix} = (2)(16) = 32.$
The 3^{rd} line:
$c_{31} = (-1)^{3+1} a_{31} \det(A_{31}) = (1)(-3) \det \begin{pmatrix} 0 & -3 \\ 0 & -2 \end{pmatrix} = 0.$
$c_{32} = (-1)^{3+2} a_{32} \det(A_{32}) = (-1)(16) \det \begin{pmatrix} 1 & -3 \\ 4 & -2 \end{pmatrix} = (-16)(10) = -160.$
$c_{33} = (-1)^{3+3} a_{33} \det(A_{33}) = (1)(5) \det \begin{pmatrix} 1 & 0 \\ -4 & 0 \end{pmatrix} = 0.$
Thus, $C = \begin{pmatrix} 32 & 0 & -192 \\ -192 & 0 & 32 \\ 0 & -160 & 0 \end{pmatrix}$. We conclude that
$A^{-1} = -\frac{1}{160} \begin{pmatrix} 32 & -192 & 0 \\ 0 & 0 & -160 \\ -192 & 32 & 0 \end{pmatrix} = \begin{pmatrix} -\frac{1}{5} & \frac{6}{5} & 0 \\ 0 & 0 & 1 \\ \frac{6}{5} & -\frac{1}{5} & 0 \end{pmatrix}.$

1.2. **Computation of the determinant by the Gauss pivot method.** We have seen before that the determinant of triangular matrix is equal to the product of the diagonal elements. The Gauss pivot method (also called Gauss-Jordan elimination) consists first of all in bringing a given matrix born to an upper triangular matrix, this can be done only by elementary operations on the lines. These operations are:
1- Exchange of two lines.
2- Multiplication of a line by a nonzero scalar.
3- Adding the multiple of a line to another line.

The principle of this method is as follows:

• We choose in the matrix A a term (scalar of $\mathbb{K}$) nonzero a_{ij}, in general the first term at the top left, which is called the pivot;

• if the term a_{11} does not suit to choose, we can, by permuting lines 1 and i (the columns 1 and j), put it in the correct position. We then obtain a matrix B such that $\det(A) = (-1)^{i+j}\det(B)$;

• we eliminate all the terms located under the pivot, a_{11}. The value of the determinant remains unchanged by this operation;

• we repeat the same process in the private sub-matrix of its first line and of its first column;

• at the last step we get a triangular matrix whose determinant is equal to the determinant of the given matrix.

Example 1.17. Let $A = \begin{pmatrix} 2 & 1 & -4 \\ 3 & 3 & -5 \\ 4 & 5 & -2 \end{pmatrix}$.

Calculate the determinant of A by the Gauss pivot method.
Indeed, we denote the lines of the matrix by L_1, L_2, L_3.
Step (1):
We have: $a_{11} = 2 \neq 0$. Then, we can choose 2 as the first pivot and add L_2, the first line L_1 multiplied by $\frac{-3}{2}$, i.e., $(L_2 \leftarrow L_2 - \frac{3}{2} \cdot L_1)$ and add the line L_3, the first line L_1 multiplied by -2, i.e., $(L_3 \leftarrow L_3 - 2 \cdot L_1)$. Then, we get the matrix

$$\begin{pmatrix} 2 & 1 & -4 \\ 0 & \frac{3}{2} & 1 \\ 0 & 3 & 6 \end{pmatrix}.$$

Step (2):
The second pivot is $\frac{3}{2}$. Add to the line L_3,the second line L_2 multiplied by -2, i.e., $(L_3 \leftarrow L_3 - 2 \cdot L_2.)$ Then, we get the matrix

$$\begin{pmatrix} 2 & 1 & -4 \\ 0 & \frac{3}{2} & 1 \\ 0 & 0 & 4 \end{pmatrix}.$$

The third pivot is $4 \neq 0$, where the matrix is upper triangular.
The determinant of A is equal to the product of the pivots, i.e.,

$$\det(A) = 2 \times \frac{3}{2} \times 4 = 12.$$

2. Matrices and linear applications

2.1. The matrix associated with a linear application.

Let E and F be two $\mathbb{K}$-vectorial spaces with dimensions, respectively, m and n, $\mathcal{B}' = \{w_1, w_2, ..., w_n\}$ a base of

F and $f \in \mathcal{L}(E,F)$. We call a matrix of f in $\mathcal{B}$ and $\mathcal{B}'$ and we write $M_{\mathcal{B},\mathcal{B}'}(f)$ the matrix of the family $f(\mathcal{B}) = \{f(v_1), f(v_2), ..., f(v_m)\}$ in the base $\mathcal{B}'$. In other words, $M_{\mathcal{B},\mathcal{B}'}(f)$ is the matrix with n lines and m columns with coefficients in $\mathbb{K}$, where the elements of j^{th} column are the coordinates of the vector $f(v_j)$ in the base $\mathcal{B}'$ where

$$M_{\mathcal{B},\mathcal{B}'}(f) = (a_{ij}), \forall j \in \{1,2,...,m\}; f(v_j) = \sum_{i=1}^{n} a_{ij} \cdot w_i = \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{nj} \end{pmatrix}.$$

If $E = F$ and $\mathcal{B} = \mathcal{B}'$, The matrix $M_{\mathcal{B},\mathcal{B}}(f)$ is simply noted by $M_{\mathcal{B}}(f)$.

Example 2.1. We consider E a $\mathbb{K}$-vectorial space with finite dimension n and, $\mathcal{B}$ a base of E. Then, $M_{\mathcal{B}}(Id_E) = I_n$.

Example 2.2. Let $f \in \mathcal{L}(\mathbb{R}^3)$ defined by;
$f(x,y,z) = (-x+y-z, -x+z, -2x+2y)$.
We consider $\mathcal{B}$ the canonical base of $\mathbb{R}^3$. Give the matrix $M_{\mathcal{B}}(f)$.
Indeed, we know that $\mathcal{B} = \{v_1 = (1,0,0), v_2 = (0,1,0), v_3 = (0,0,1)\}$. Then,
$f(v_1) = (-1,-1,-2), f(v_2) = (1,0,2), f(v_3) = (-1,1,0)$ and then

$$M_{\mathcal{B}}(f) = \begin{pmatrix} -1 & 1 & -1 \\ -1 & 0 & 1 \\ -2 & 2 & 0 \end{pmatrix}.$$

Example 2.3. Let $f \in \mathcal{L}(\mathbb{R}^2)$ defined by;
$f(x,y,z) = (x+y, x-y)$.
We consider $\mathcal{B} = \{v_1 = (1,2), v_2 = (-1,1)\}$ a base of $\mathbb{R}^2$ and $\mathcal{B}' = \{w_1 = (0,2), w_2 = (-2,1)\}$ a base of $\mathbb{R}^2$.
What is the matrix associated to f in the bases $\mathcal{B}$ and $\mathcal{B}'$?
Firstly, we compute $f(v_1)$ and $f(v_2)$ as a linear combination of w_1 et w_2.
We put; $f(v_1) = \alpha_1 \cdot w_1 + \alpha_2 \cdot w_2, f(v_2) = \beta_1 \cdot w_1 + \beta_2 \cdot w_2$. Then, we have
$(3,-1) = (-2\alpha_2, -2\alpha_1 + \alpha_2), (0,-2) = (-2\beta_2, 2\beta_1 + \beta_2)$. We conclude that
$\alpha_1 = \frac{1}{4}, \alpha_2 = \frac{-3}{2}, \beta_1 = -1, \beta_2 = 0$. Thus

$$M_{\mathcal{B},\mathcal{B}'}(f) = \begin{pmatrix} \frac{1}{4} & -1 \\ \frac{-3}{2} & 0 \end{pmatrix}.$$

Remark 2.4. In general, the matrix associated with a linear application depends on the bases chosen $\mathcal{B}$ and $\mathcal{B}'$.

2.2. Linear map associated with a matrix.

Proposition 2.5. *Let E and F be two $\mathbb{K}$-vectorial spaces with finite dimension m and n, respectively, $\mathcal{B} = \{v_1, v_2, ..., v_m\}$ a base of E, $\mathcal{B}' = \{w_1, w_2, ..., w_n\}$ a base of F. Then, the data of a matrix $A \in \mathcal{M}_{n,m}(\mathbb{K})$, gives a unique linear map f from E in F, where $A = M_{\mathcal{B},\mathcal{B}'}(f)$.*

Analytic expression of f:
We have: $\forall x \in E$, there exists $(x_1, x_2, ..., x_m) \in \mathbb{K}^m; x = \sum_{i=1}^m x_i \cdot v_i$.
Given the matrix $A = M_{\mathcal{B},\mathcal{B}'}(f)$.

• For any $x \in E$, we denote $X = M_{\mathcal{B}}(x) = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix}$.

• For any $y \in F$, we denote $Y = M_{\mathcal{B}'}(y) = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$.

Then if $y = f(x)$, we have: $Y = A \times X$. This equation can be written in the matrix form as follows:

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & ... & a_{1m} \\ a_{21} & a_{22} & ... & a_{2m} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & ... & a_{nm} \end{pmatrix} \times \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix}.$$

Thus, we get the following system:

$$\begin{cases} y_1 = a_{11}x_1 + a_{12}x_2 + ... + a_{1m}x_m \\ y_2 = a_{21}x_1 + a_{22}x_2 + ... + a_{2m}x_m \\ \vdots \\ y_n = a_{n1}x_1 + a_{n2}x_2 + ... + a_{nm}x_m. \end{cases}$$

We deduce that the linear map f associated with the matrix A is defined by:

$$f(x_1, x_2, ..., x_m) = (a_{11}x_1 + a_{12}x_2 + ... + a_{1m}x_m, ..., a_{n1}x_1 + a_{n2}x_2 + ... + a_{nm}x_m).$$

Example 2.6. Let $f \in \mathcal{L}(\mathbb{R}^3)$, both equipped with the canonical base $\mathcal{B} = \{e_1 = (1, 0, 0), e_2 = (0, 1, 0), e_2 = (0, 0, 1)\}$ of $\mathbb{R}^3$. Given the matrix

$$A = M_{\mathcal{B}}(f) = \begin{pmatrix} 1 & 0 & -1 \\ 3 & -2 & 0 \\ -1 & \frac{1}{2} & 4 \end{pmatrix}.$$

Give the analytic expression of f.
Indeed, we have for all the reals x, y and z:

$$\begin{pmatrix} 1 & 0 & -1 \\ 3 & -2 & 0 \\ -1 & \frac{1}{2} & 4 \end{pmatrix} \times \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x - z \\ 3x - 2y \\ -x + \frac{y}{2} + 4z \end{pmatrix}.$$

Then, the map f associated to the matrix A is defined by:

$$f(x,y,z) = (x-z, 3x-2y, -x+\frac{y}{2}+4z).$$

Properties:
Let $f \in \mathcal{L}(E,F)$ and $g \in \mathcal{L}(F,G)$ where E, F and G are three $\mathbb{K}$-vectorial spaecs with finite dimension m,n and p, respectively, with m,n,p three natural numbers diffrent from 0. Let $\mathcal{B} = \{v_1, v_2, ..., v_m\}$ a base of E, $\mathcal{B}' = \{w_1, w_2, ..., w_n\}$ a base of F and $\mathcal{B}'' = \{u_1, u_2, ..., u_p\}$ a base of G. Then, we have:
• The map $\varphi : f \to M_{\mathcal{B},\mathcal{B}'}(f)$ is an isomorphism from $\mathcal{L}(E,F)$ on $\mathcal{M}_{n,m}(\mathbb{K})$.
• $\mathcal{L}(E,F)$ is a $\mathbb{K}$-vectorial space with finite dimension such that

$$dim(\mathcal{L}(E,F)) = dim(\mathcal{M}_{n,m}(\mathbb{K})) = m \times n = dim(E) \times dim(F).$$

• We consider $m = n$ and $A = M_{\mathcal{B},\mathcal{B}'}(f)$. Then, we have:
f is an isomorphism from E on Fif and only if A is invertible. Moreover,

$$M_{\mathcal{B}',\mathcal{B}}(f^{-1}) = A^{-1}.$$

• The map $g \circ f$ is defined by:

$$M_{\mathcal{B},\mathcal{B}''}(g \circ f) = M_{\mathcal{B}',\mathcal{B}''}(g) \times M_{\mathcal{B},\mathcal{B}'}(f).$$

Remark 2.7. The order in which the product is made is the order in which the composition is writte.

Remark 2.8. If the matrix of f is $A = M_{\mathcal{B},\mathcal{B}'}(f) = (a_{ij}) \in \mathcal{M}_{n,m}(\mathbb{K})$, and the matrix of g is $B = M_{\mathcal{B}',\mathcal{B}''}(g) = (b_{ij}) \in \mathcal{M}_{p,n}(\mathbb{K})$, then the matrix $g \circ f$ is $C = M_{\mathcal{B},\mathcal{B}''}(g \circ f) = (c_{ij}) \in \mathcal{M}_{p,m}(\mathbb{K})$.

Example 2.9. Let $f \in \mathcal{L}(\mathbb{R}^3, \mathbb{R}^2)$ be defined by: $f(x,y,z) = (x+y+2z, x-y)$ and $g \in \mathcal{L}(\mathbb{R}^2)$ defined by: $f(x,y) = (x-y, 2x+y)$.
Let $\mathcal{B} = \{e_1 = (1,0,0), e_2 = (0,1,0), e_3 = (0,0,1)\}$ be the canonical base of $\mathbb{R}^3$ and $\mathcal{B}' = \mathcal{B}'' = \{e_1 = (1,0), e_2 = (0,1)\}$ lthe canonical base of $\mathbb{R}^2$.
Determine the map $g \circ f$.

For determining the map $g \circ f$, it is enough to go through the intermediate matrices. We firstly determine the matrix $M_{\mathcal{B},\mathcal{B}'}(g \circ f)$.
We have $M_{\mathcal{B}'}(g) = \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}$ and $M_{\mathcal{B},\mathcal{B}'}(f) = \begin{pmatrix} 1 & 1 & 2 \\ 1 & -1 & 0 \end{pmatrix}$. Then,

$$\begin{aligned} M_{\mathcal{B},\mathcal{B}'}(g\circ f) &= M_{\mathcal{B}'}(g)\times M_{\mathcal{B},\mathcal{B}'}(f) \\ &= \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 1 & 2 \\ 1 & -1 & 0 \end{pmatrix}. \\ &= \begin{pmatrix} 0 & 2 & 2 \\ 3 & 1 & 4 \end{pmatrix}. \end{aligned}$$

We conclude that $(g\circ f)(x,y,z) = M_{\mathcal{B},\mathcal{B}'}(g\circ f)\times \begin{pmatrix} x \\ y \\ z \end{pmatrix} = (2x+2z, 3x+y+4z)\,.$

Then, $g\circ f \in \mathcal{L}(\mathbb{R}^3, \mathbb{R}^2)$ is defined by: $(g\circ f)(x,y,z) = (2x+2z, 3x+y+4z)\,.$

2.3. **Passing matrix.** Let E be a $\mathbb{K}$-vectorial space with finite dimension $n$equipped with the two bases $\mathcal{B} = \{v_1, v_2, ..., v_n\}$ and $\mathcal{B}' = \{w_1, w_2, ..., w_n\}$.

Definition 2.10. We call a passing matrix from $\mathcal{B}$ to $\mathcal{B}'$ the square matrix $P = (p_{ij})$where j^{th} column is written in the base $\mathcal{B}$ in the form:

$$\forall j \in \{1,2,...,n\}, w_j = \sum_{i=1}^{n} p_{ij}\cdot v_i\,.$$

In other words, lThe columns of P are the coordinates of the vectors de la base $\mathcal{B}'$expressed in the base $\mathcal{B}$.

Sometimes we denote the matrix P by $M_{\mathcal{B}}(\mathcal{B}')$ where $P_{\mathcal{B},\mathcal{B}'}$.

Proposition 2.11. *We consider the identical endomorphism of E, the map $Id(E): v \to Id(v) = v$. Let P be the passing matrix from $\mathcal{B}$ to $\mathcal{B}'$. Then,*

$$P = M_{\mathcal{B}',\mathcal{B}}(Id_E).$$

Indeed, we have:

$$\begin{aligned} M_{\mathcal{B}',\mathcal{B}}(Id_E) &= M_{\mathcal{B}}(Id_E(w_1), Id_E(w_2), ..., Id_E(w_n)) \\ &= M_{\mathcal{B}}(w_1, w_2, ..., w_n) \\ &= M_{\mathcal{B}}(\mathcal{B}') = P. \end{aligned}$$

Example 2.12. $P = M_{\mathcal{B},\mathcal{B}}(Id_E) = I_n$.

Remark 2.13. Since $f = Id_E$ is an isomorphism from E on E (automorphism), then P the passing matrix from $\mathcal{B}$ to $\mathcal{B}'$ is invertible and its inverse P^{-1} is the passing matrix from $\mathcal{B}'$ to $\mathcal{B}$. In other words:

$$[M_{\mathcal{B}',\mathcal{B}}(Id_E)]^{-1} = M_{\mathcal{B},\mathcal{B}'}(Id_E).$$

Example 2.14. Let $\mathcal{B} = \{v_1 = (1,0), v_2 = (0,1)\}$ and $\mathcal{B}' = \{w_1 = (-1,2), w_2 = (2,3)\}$ be two bases of $\mathbb{R}^2$.
1- Give the passing matrix from $\mathcal{B}$ to $\mathcal{B}'$.
2- Give the passing matrix from $\mathcal{B}'$ to $\mathcal{B}$.
• We express the vectors of $\mathcal{B}'$ in $\mathcal{B}$:
We have

$$\begin{cases} w_1 = -v_1 + 2v_2 \\ \\ w_2 = 2v_1 + 3v_2. \end{cases}$$

Then, the passing matrix from $\mathcal{B}$ to $\mathcal{B}'$ is $M_{\mathcal{B}',\mathcal{B}}(Id_{\mathbb{R}^2}) = \begin{pmatrix} -1 & 2 \\ 2 & 3 \end{pmatrix}$.
• We express the vectors of $\mathcal{B}$ in $\mathcal{B}'$:
We have

$$\begin{cases} v_1 = \frac{-3}{7}w_1 + \frac{2}{7}w_2 \\ \\ v_2 = \frac{2}{7}w_1 + \frac{1}{7}w_2. \end{cases}$$

Then, the passing matrix from $\mathcal{B}'$ to $\mathcal{B}$ is $M_{\mathcal{B},\mathcal{B}'}(Id_{\mathbb{R}^2}) = \begin{pmatrix} \frac{-3}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{1}{7} \end{pmatrix}$.
Check: $M_{\mathcal{B}',\mathcal{B}}(Id_{\mathbb{R}^2}) \times M_{\mathcal{B},\mathcal{B}'}(Id_{\mathbb{R}^2}) = I_2$.

2.4. **Change of base.** Let E be a $\mathbb{K}$-vectorial space of dimension n equipped with two bases $\mathcal{B} = \{v_1, v_2, ..., v_n\}$ and $\mathcal{B}' = \{w_1, w_2, ..., w_n\}$.

2.4.1. *Change of base for a vector.*

Theorem 2.15. *Let x be an element of E, X and X' he column matrices of the coordinates of x in the bases $\mathcal{B}$ and $\mathcal{B}'$ respectively. Then,*

$$X = P \times X'.$$

Indeed, we have: $P = M_{\mathcal{B},\mathcal{B}'}(Id_{\mathbb{R}^2})$, $X = M_{\mathcal{B}}(x)$ and $X' = M_{\mathcal{B}'}(x)$. Then,

$$X = M_{\mathcal{B}}(x) = M_{\mathcal{B}}(Id_E(x)) = M_{\mathcal{B},\mathcal{B}'}(Id_E) \times M_{\mathcal{B}'}(x) = P \times X'.$$

Remark 2.16. The following formula can also be extracted: $X' = P^{-1} \times X$.

2.4.2. *Change of base for a linear map.* Let E be a $\mathbb{K}$-vectorial space with finite dimension m, $\mathcal{B}$ and $\mathcal{B}'$ be two bases of E and P be the passing matrix from $\mathcal{B}$ to $\mathcal{B}'$.
Let F be a $\mathbb{K}$-vectorial space with finite dimension n, $\mathcal{C}$ and $\mathcal{C}'$ e two bases of F and Q be the passing matrix from $\mathcal{C}$ to $\mathcal{C}'$.

Theorem 2.17. *Let $f \in \mathcal{L}(E,F)$, $A = M_{\mathcal{B},\mathcal{C}}(f)$ and $A' = M_{\mathcal{B}',\mathcal{C}'}(f)$. Then,*

$$A' = Q^{-1} \times A \times P.$$

Indeed, let $x \in E$, $X = M_{\mathcal{B}}(x)$ and $X' = M_{\mathcal{B}'}(x)$ be matrices columns of coordinates of x in the bases $\mathcal{B}$ and $\mathcal{B}'$ (resp.). Then, from Theorem 2.15, we get: $X = P \times X'$.
similarly, let $y \in F$, $Y = M_{\mathcal{C}}(x)$ and $Y' = M_{\mathcal{C}'}(y)$ be matrices columns of coordinates of y in the bases $\mathcal{C}$ and $\mathcal{C}'$ (resp.). then, we have: $Y = Q \times Y'$.
Then, the map $y = f(x)$ can be written in matrix form $Y = A \times X$ where

$$\begin{aligned} Y = A \times X \quad &\Leftrightarrow \quad Q \times Y' = A \times P \times X' \\ &\Leftrightarrow \quad Y' = Q^{-1} \times A \times P \times X'. \end{aligned}$$

According to the data, the matrix A' is the only matrix such that $y = f(x) \Leftrightarrow Y' = A' \times X'$. Thus, $A' = Q^{-1} \times A \times P$.

Corollary 2.18 (Case of endomorphism)**.** *If $f \in \mathcal{L}(E)$, $A = M_{\mathcal{B}}(f)$ and $A' = M_{\mathcal{B}'}(f)$. Then,*

$$A' = P^{-1} \times A \times P\,.$$

Example 2.19. Lett $f \in \mathcal{L}(\mathbb{R}^3)$, $\mathcal{B}$ be the canonical bases of $\mathbb{R}^3$ and the matrix $A = M_{\mathcal{B}}(f)$ be defined by: $\begin{pmatrix} 2 & 1 & 1 \\ -3 & -2 & -1 \\ 3 & 5 & 4 \end{pmatrix}$.
We consider $\mathcal{B}' = \{u_1 = (0,-1,1), u_2 = (1,-1,1), u_3 = (-1,1,-1)\}$ another base of $\mathbb{R}^3$.
1- Find the passing matrix P from $\mathcal{B}$ to $\mathcal{B}'$.
2- Find A' the matrix associates to f in the base $\mathcal{B}'$.

• The canonical base of $\mathbb{R}^3$ is $\mathcal{B} = \{e_1 = (1,0,0), e_2 = (0,1,0), e_3 = (0,0,1)\}$. We express the vectors of $\mathcal{B}'$ in $\mathcal{B}$ as follows:

$$\begin{cases} u_1 = -e_2 + e_3 \\ u_2 = e_1 - e_2 + e_3 \\ u_3 = -e_1 + e_2 - e_3. \end{cases}$$

Then, the passing matrix from $\mathcal{B}$ to $\mathcal{B}'$ is

$$P = M_{\mathcal{B}',\mathcal{B}}(Id_{\mathbb{R}^3}) = \begin{pmatrix} 0 & 1 & -1 \\ -1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}.$$

Consequently

$$P^{-1} = \begin{pmatrix} -1 & -1 & 0 \\ 1 & -1 & -1 \\ 0 & -1 & -1 \end{pmatrix}.$$

- A' the matrix associated to f in the base $\mathcal{B}'$ is

$$\begin{aligned} A' &= M_{\mathcal{B}'}(f) \\ &= P^{-1} \times A \times P \\ &= \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}. \end{aligned}$$

Definition 2.20 (Similar matrices)**.** We say that two matrices A and A' of $\mathcal{M}_n(\mathbb{K})$ are similar if there exists an invertible matrix P, i.e., $P \in GL_n(\mathbb{K})$ such that:

$$A' = P^{-1} \times A \times P.$$

Example 2.21. We consider E a $\mathbb{K}$-vectorial space with a finite dimension, $\mathcal{B}$ and $\mathcal{B}'$ two bases of E and $f \in \mathcal{L}(E)$. Then, the matrices $M_{\mathcal{B}}(f)$ et $M_{\mathcal{B}'}(f)$ are similar.

Definition 2.22 (Equivalent matrices)**.** Let $A, A' \in \mathcal{M}_{n,m}(\mathbb{K})$.We say that A' is equivalent to A if there exist two invertible matrices $P \in GL_m(\mathbb{K})$ et $Q \in GL_n(\mathbb{K})$ such that:

$$A' = Q^{-1} \times A \times P.$$

Example 2.23. We consider E and F two $\mathbb{K}$vectorial spaces with finite dimension, $\mathcal{B}$ and $\mathcal{B}'$ two bases of E, $\mathcal{C}$ and $\mathcal{C}'$ two bases of F and $f \in \mathcal{L}(E, F)$. Then, $M_{\mathcal{B},\mathcal{C}'}(f)$ et $M_{\mathcal{B}',\mathcal{C}'}(f)$ are equivalent.

2.5. Rank of a matrix.

Definition 2.24. Let A be a matrix of $\mathcal{M}_{n,m}(\mathbb{K})$. We call a rank of A, and we write $rg(A)$, the rank of this column vectors.

Example 2.25. Let $A = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$. The rank of A is the family rank of its column vectors, i.e., the family rank

$$\mathcal{H} = \{u_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, u_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}, u_3 = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \end{pmatrix}\}.$$

It is clear that the family of vectors $\{v_1, v_2\}$ is free. In the other hand, after solving the linear system $\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0$, we get $v_1 - v_2 + v_3 = 0$. Then, the family $\mathcal{H}$ is free. We deduce that $Vect(v_1, v_2, v_3) = Vect(v_1, v_2)$. Then, $rg(A) = rg(\mathcal{H}) = 2$.

The following proposition explains the link between the rank of a linear application and the rank of an associated matrix.

Proposition 2.26. *Let E and F be two $\mathbb{K}$-vectorial spaces with finite dimension, $\mathcal{B}$ a base of E, $\mathcal{B}'$ a base of F, $f \in \mathcal{L}(E, F)$ and $A = M_{\mathcal{B},\mathcal{B}'}(f)$. Then,*

$$rg(f) = rg(A)\,.$$

Indeed,

$$\begin{aligned} rg(A) &= rg(M_{\mathcal{B},\mathcal{B}'}(f)) \\ &= rg(M_{\mathcal{B}'}(f(\mathcal{B}))) \\ &= rg(f(\mathcal{B})) = dim(Vect(f(\mathcal{B}))) \\ &= dim(Im(f)) = rg(f)\,. \end{aligned}$$

Now we provide an invertibility criterion for matrices.

Theorem 2.27. *Let $A \in \mathcal{M}_n(\mathbb{K})$. Then, A if and only if the column matrices of A form a basis of $A \in \mathcal{M}_{n,1}(\mathbb{K})$. It also means that*

$$A \text{ est inversible } \Leftrightarrow rg(A) = n\,.$$

Remark 2.28. In practice, it is useful to identify $\mathcal{M}_{n,1}(\mathbb{K})$ with $\mathbb{K}^n$. Then, A is invertible if and only if its column vectors form a base of $\mathbb{K}^n$.

Theorem 2.29 (Characterisation)**.** *Let $A, B \in \mathcal{M}_{n,m}(\mathbb{K})$. Then,*

$$A \text{ et } B \text{ are equivalent } \Leftrightarrow rg(A) = rg(B)\,.$$

The following proposition explains the invariance of the rank by transposition.

Proposition 2.30. *Let A be a matrix of $\mathcal{M}_{n,m}(\mathbb{K})$. Then,*

$$rg(A^t) = rg(A)\,.$$

Remark 2.31. Since the row vectors of a matrix these are the columns of its transpose, so to determine the rank of a matrix consists in also determining the rank of its line vectors.

Proposition 2.32. *Let $C_1, C_2, ..., C_n$ be the columns of a matrix A. Then, the rank of A is not modified by the following three elementary operations on the vectors;*
1- We can exchange two columns ($C_i \leftrightarrow C_j$).
2- We can multiply a column by a non-zero scalar ($C_i \leftarrow \alpha \cdot C_i$, pour $\alpha \neq 0$).
3-We can add to column C_i a multiple of another column C_j. ($C_i \leftarrow C_i + \alpha \cdot C_j$).

3. Exercises

Let
$$A=\begin{pmatrix}-1&1\\1&0\\0&2\end{pmatrix},B=\begin{pmatrix}-3&0\\2&1\\-5&4\end{pmatrix},C=\begin{pmatrix}1&1&0\\4&-1&-2\\0&3&2\end{pmatrix}.$$
1- Compute $A+B, A-B, 2\cdot A-7\cdot B$.
2- Compute $A^t, B^t, (A\times B)^t$.
3- Compute $A\times B, B\times A, A^3$.

what can we say about a matrix $A\in\mathcal{M}_n(\mathbb{R})$ which satisfies $tr(A\times A^t)=0$?

Let $A=\begin{pmatrix}0&1&-1\\0&1&1\\1&0&1\end{pmatrix}$.
1- Compute $P(A)=A^3-2\cdot A^2+2\cdot A$.
2- Deduce from what precedes that A is invertible, then give A^{-1}.
3- Find A^{-1} by using the comatrix.

Let $A=\begin{pmatrix}3&0&1\\-1&3&-2\\-1&1&0\end{pmatrix}$.
1- Compute $(A-2\cdot I_3)^3$, then conclude that A is invertible.
2- Find A^{-1} by mean of I_3, A and A^2.

Let $A=\begin{pmatrix}\alpha&0&1\\0&\alpha&0\\0&0&\alpha\end{pmatrix}$, with $\alpha\in\mathbb{R}$.
1- Find a matrix X such that $A=\alpha\cdot I_3+X$.
2- Compute X^2, then deduce X^n, for a natural number n.

Compute the determinant of the following matrices.
$$A=\begin{pmatrix}\sin(x)&-\cos(x)\\\cos(x)&\sin(x)\end{pmatrix},B=\begin{pmatrix}1&0&1\\4&-1&0\\0&2&3\end{pmatrix},C=\begin{pmatrix}0&1&3&2\\4&1&2&0\\5&2&1&7\\0&3&1&0\end{pmatrix}.$$

Let $\mathcal{B}=\{e_1,e_2,e_3\}$ the canonical base of $\mathbb{R}^3$, $f\in\mathcal{L}(\mathbb{R}^3)$ whose canonical base is
$$A=\begin{pmatrix}1&2&-2\\2&1&-2\\2&2&-3\end{pmatrix}.$$
1- show that $\mathcal{F}=\{x\in\mathbb{R}^3, f(x)=x\}$ is a sub-space of $\mathbb{R}^3$ for which we give the base $\{v_1\}$.

2- We consider $v_2 = (0,1,1)$ et $v_3 = (1,1,2)$ two vectors of $\mathbb{R}^3$. Compute $f(v_2)$ et $f(v_3)$.
3- Show that $\mathcal{B}' = \{v_1, v_2, v_3\}$ is another base of $\mathbb{R}^3$.
4-Find the passing matrix P dfrom $\mathcal{B}$ to $\mathcal{B}'$.
5- Compute P^{-1}.
6- Find the matrix D of f in the base $\mathcal{B}'$.
7- Give the relation between A, P and D.

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