

Série I

Exercice 1 Soit les nombres complexes

$$z_1 = 0, \quad z_2 = -1, \quad z_3 = +1, \quad z_4 = i, \quad z_5 = -i, \\ z_6 = 1 + i, \quad z_7 = -1 + i, \quad z_8 = -1 - i, \quad z_9 = 1 - i.$$

- 1- Trouver les parties réelle et imaginaire de ces nombres.
- 2- Représenter ces nombres sur le plan complexe.
- 3- Trouver les conjugués de ces nombres.
- 4- Calculer (Forme algébrique)

$$z_2 + z_9, \quad z_4 + z_6, \quad z_2 - z_8, \quad z_5 - z_7, \quad z_4 z_6, \quad z_7 z_8, \quad \frac{z_3}{z_7}, \quad \frac{z_5}{z_9}, \quad \frac{z_7}{z_8}.$$

- 5- Calculer les modules de ces nombres.
- 6- Trouver les arguments principaux de ces nombres ($\text{Arg}(z) \in [0, 2\pi[$).
- 7- En déduire les arguments de ces nombres ($\arg(z) = \text{Arg}(z) + 2k\pi, \quad k \in \mathbb{Z}$).
- 8- Donner les formes trigonométrique et exponentielle de ces nombres.
- 9- Calculer $(z_6)^{25}$ et $\left(\frac{z_7}{z_8}\right)^{20}$.

Exercice 2 Soient a, b et c dans \mathbb{C} .

- 1- Montrer que

$$|1 + a| + |a + b| + |b| \geq 1.$$

- 2- Exprimer $\mathcal{R}e(a^2)$ en fonction de $|a|$ et de $\mathcal{R}e(a)$.
- 3- Si $|a| = 1$, calculer

$$|1 + a|^2 + |1 - a|^2.$$

4- Montrer que si $|a| = 1$ et $a \neq 1$ alors

$$i \left(\frac{a+1}{a-1} \right) \in \mathbb{R}$$

et

$$\operatorname{Re} \left(\frac{1}{1-a} \right) = \frac{1}{2}.$$

Exercice 3 Soit $\lambda \in \mathbb{C} \setminus \{-i\}$.

1- Déterminer le lieu des points M d'affixe λ tels que

$$\left| \frac{\lambda - i}{\lambda + i} \right| = 1.$$

2- Montrer l'équivalence

$$\lambda \in \mathbb{R} \iff \left| \frac{1 + \lambda i}{1 - \lambda i} \right| = 1.$$

Exercice 4 Montrer que pour tout $z \in \mathbb{C}$ tel que $|z| \neq 1$, on a

$$\left| \frac{1 - z^{2019}}{1 - z} \right| \leq \frac{1 - |z|^{2019}}{1 - |z|}.$$

Exercice 5 Calculer les racines

$$(-1)^{1/5}, \quad (+1)^{1/3}, \quad (i)^{1/4}, \quad (1+i)^{1/2}, \quad (-1+i)^{1/8}.$$

Résoudre dans \mathbb{C} ($z = x + iy$) les équations

$$(4 + 2i)x + (5 - 3i)y = 13 + i$$

$$4z^2 + 8|z|^2 - 3 = 0$$

$$z^2 + (2i - 3)z + 5 - i = 0$$

$$z^4 - (3 + 8i)z^2 - 16 + 12i = 0.$$

Solutions de la série I

Solution 1 Les solutions aux questions (1 + 3 + 5 + 6 + 7 + 8)

sont données dans le tableau suivant,

z	$\operatorname{Re}(z)$	$\operatorname{Im}(z)$	\bar{z}	$ z $	$\operatorname{Arg}(z)$	$\arg(z)$	forme trigo.	f. exp.
0	0	0	0	0	ind	ind	ind	ind
-1	-1	0	-1	1	π	$\pi + 2k\pi, k \in \mathbb{Z}$	$(\cos \pi + i \sin \pi)$	$e^{\pi i}$
+1	+1	0	+1	1	0	$2k\pi, k \in \mathbb{Z}$	$(\cos 0 + i \sin 0)$	e^{0i}
i	0	1	$-i$	1	$\frac{\pi}{2}$	$\frac{\pi}{2} + 2k\pi, k \in \mathbb{Z}$	$(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2})$	$e^{\frac{\pi}{2}i}$
$-i$	0	-1	$+i$	1	$\frac{3\pi}{2}$	$\frac{3\pi}{2} + 2k\pi, k \in \mathbb{Z}$	$(\cos \frac{3\pi}{2} + i \sin \frac{3\pi}{2})$	$e^{\frac{3\pi}{2}i}$
$1+i$	1	1	$1-i$	$\sqrt{2}$	$\frac{\pi}{4}$	$\frac{\pi}{4} + 2k\pi, k \in \mathbb{Z}$	$\sqrt{2}(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4})$	$\sqrt{2}e^{\frac{\pi}{4}i}$
$-1+i$	-1	1	$-1-i$	$\sqrt{2}$	$\frac{3\pi}{4}$	$\frac{3\pi}{4} + 2k\pi, k \in \mathbb{Z}$	$\sqrt{2}(\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4})$	$\sqrt{2}e^{\frac{3\pi}{4}i}$
$-1-i$	-1	-1	$-1+i$	$\sqrt{2}$	$\frac{5\pi}{4}$	$\frac{5\pi}{4} + 2k\pi, k \in \mathbb{Z}$	$\sqrt{2}(\cos \frac{5\pi}{4} + i \sin \frac{5\pi}{4})$	$\sqrt{2}e^{\frac{5\pi}{4}i}$
$1-i$	1	-1	$1+i$	$\sqrt{2}$	$\frac{7\pi}{4}$	$\frac{7\pi}{4} + 2k\pi, k \in \mathbb{Z}$	$\sqrt{2}(\cos \frac{7\pi}{4} + i \sin \frac{7\pi}{4})$	$\sqrt{2}e^{\frac{7\pi}{4}i}$

4- Forme algébrique,

$$z_2 + z_9 = -1 + (1 - i) = -i$$

$$z_4 + z_6 = i + (1 + i) = 1 + 2i$$

$$z_2 - z_8 = -1 - (-1 - i) = i$$

$$z_5 - z_7 = -i - (-1 + i) = 1 - 2i$$

$$z_4 z_6 = i(1 + i) = -1 + i$$

$$z_7 z_8 = (-1 + i)(-1 - i) = 2$$

$$\frac{z_3}{z_7} = \frac{1}{-1 + i} = \frac{-1 - i}{2} = -\frac{1}{2} - \frac{1}{2}i$$

$$\frac{z_5}{z_9} = \frac{-i}{1 - i} = \frac{-i(1 + i)}{2} = \frac{1 - i}{2} = \frac{1}{2} - \frac{1}{2}i$$

$$\frac{z_7}{z_8} = \frac{-1 + i}{-1 - i} = \frac{(-1 + i)(-1 + i)}{2} = \frac{-2i}{2} = -i.$$

9- On a ,

$$(z_6)^{25} = (1+i)^{25} = (2^{1/2}e^{\frac{\pi}{2}i})^{25} = 2^{25/2}e^{\frac{25\pi}{2}i} = 2^{12}\sqrt{2}e^{\frac{\pi}{2}i} = 2^{12}\sqrt{2}i.$$

$$\left(\frac{z_7}{z_8}\right)^{20} = \left(\frac{-1+i}{-1-i}\right)^{20} = (-i)^{20} = ((-i)^2)^{10} = (-1)^{10} = 1.$$

Solution 2

$$1 = |1+a-a-b+b| \leq |1+a| + |-a-b+b| \leq |1+a| + |a+b| + |b|.$$

$$\operatorname{Re}(a^2) = \operatorname{Re}((\alpha+i\beta)^2) = \alpha^2 - \beta^2 = -\alpha^2 - \beta^2 + 2\alpha^2 = 2(\operatorname{Re}(a))^2 - |a|^2.$$

$$|1+a|^2 + |1-a|^2 = (1+a)\overline{(1+a)} + (1-a)\overline{(1-a)} = (1+a)(1+\bar{a}) + (1-a)(1-\bar{a}) = 2+2a\bar{a} = 2+2|a|^2 = 4.$$

$$\overline{\left(i\left(\frac{a+1}{a-1}\right)\right)} = -i\left(\frac{\bar{a}+1}{\bar{a}-1}\right) = -i\left(\frac{\frac{1}{\bar{a}}+1}{\frac{1}{\bar{a}}-1}\right) = -i\left(\frac{a+1}{1-a}\right) = i\left(\frac{a+1}{a-1}\right) \Leftrightarrow i\left(\frac{a+1}{a-1}\right) \in \mathbb{R}$$

et

$$\begin{aligned} \operatorname{Re}\left(\frac{1}{1-a}\right) &= \frac{1}{2} \left[\left(\frac{1}{1-a}\right) + \overline{\left(\frac{1}{1-a}\right)} \right] = \frac{1}{2} \left[\left(\frac{1}{1-a}\right) + \left(\frac{1}{1-\bar{a}}\right) \right] = \frac{1}{2} \left[\frac{2-a-\bar{a}}{(1-a)(1-\bar{a})} \right] \\ &= \frac{1}{2} \left[\frac{2-a-\bar{a}}{(1-a-\bar{a}+a\bar{a})} \right] = \frac{1}{2} \left[\frac{2-a-\bar{a}}{2-a-\bar{a}} \right] = \frac{1}{2}. \end{aligned}$$

Solution 3

$$\left| \frac{\lambda-i}{\lambda+i} \right| = 1 \Leftrightarrow |\lambda-i| = |\lambda+i| \Leftrightarrow |x+i(y-1)| = |x+i(y+1)| \Leftrightarrow x^2 + (y-1)^2 = x^2 + (y+1)^2$$

$$\Leftrightarrow y = 0 \Leftrightarrow \lambda = x \in \mathbb{R}$$

$$\left| \frac{1+\lambda i}{1-\lambda i} \right| = 1 \Leftrightarrow \left| \frac{i(\lambda-i)}{-i(\lambda+i)} \right| = 1 \Leftrightarrow \left| \frac{\lambda-i}{\lambda+i} \right| = 1 \Leftrightarrow \lambda \in \mathbb{R}$$

Solution 4

$$\frac{1-z^n}{1-z} = \sum_{k=0}^{n-1} z^k$$

$$\left| \frac{1 - z^{2019}}{1 - z} \right| = \left| \sum_{k=0}^{2018} z^k \right| \leq \sum_{k=0}^{2018} |z^k| = \sum_{k=0}^{2018} |z|^k = \frac{1 - |z|^{2019}}{1 - |z|}.$$

Solution 5

$$(-1)^{1/5} = (e^{i(\pi+2k\pi)})^{1/5} = e^{i(\frac{2k+1}{5})\pi}, \quad 0 \leq k < 5.$$

$$(+1)^{1/3} = (e^{2k\pi i})^{1/3} = e^{i(\frac{2k}{3})\pi}, \quad 0 \leq k < 3.$$

$$(i)^{1/4} = (e^{i(\frac{\pi}{2}+2k\pi)})^{1/4} = e^{i(\frac{4k+1}{8})\pi}, \quad 0 \leq k < 4.$$

$$(1+i)^{1/2} = (2^{1/2}e^{i(\frac{\pi}{4}+2k\pi)})^{1/2} = 2^{1/4}e^{i(\frac{8k+1}{8})\pi}, \quad 0 \leq k < 2.$$

$$(-1+i)^{1/8} = (2^{1/2}e^{i(\frac{3\pi}{4}+2k\pi)})^{1/8} = 2^{1/16}e^{i(\frac{8k+3}{32})\pi}, \quad 0 \leq k < 8.$$

Solution 6

$$(4 + 2i)x + (5 - 3i)y = 13 + i$$

$$(4x + 5y - 13) + i(2x - 3y - 1) = 0$$

$$\begin{cases} 4x + 5y - 13 = 0 \\ 2x - 3y - 1 = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} x = 2 \\ y = 1 \end{cases}$$

$$\Leftrightarrow z = 2 + i.$$

$$4z^2 + 8|z|^2 - 3 = 0$$

$$\Leftrightarrow 4(x + iy)^2 + 8(x^2 + y^2) - 3 = 0$$

$$\Leftrightarrow 4(x^2 - y^2 + 2ixy) + 8(x^2 + y^2) - 3 = 0$$

$$\Leftrightarrow 12x^2 + 4y^2 - 3 + i8xy = 0$$

$$\Leftrightarrow \begin{cases} 12x^2 + 4y^2 - 3 = 0 \\ 8xy = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} x = 0 \\ y = \frac{\sqrt{3}}{2} \end{cases} ; \begin{cases} x = 0 \\ y = -\frac{\sqrt{3}}{2} \end{cases} ; \begin{cases} x = \frac{1}{2} \\ y = 0 \end{cases} ; \begin{cases} x = -\frac{1}{2} \\ y = 0 \end{cases} ;$$

$$\Leftrightarrow z_1 = -i\frac{\sqrt{3}}{2}; z_2 = i\frac{\sqrt{3}}{2}; z_3 = -\frac{1}{2}; z_4 = \frac{1}{2}.$$

$$z^2 + (2i - 3)z + 5 - i = 0$$

$$\Delta = (2i - 3)^2 - 4(5 - i) = -15 - 8i$$

$$\sqrt{\Delta} = \pm(1 - 4i)$$

$$z_1 = \frac{-(2i - 3) - (1 - 4i)}{2} = 2 - 3i$$

$$z_2 = \frac{-(2i - 3) + (1 - 4i)}{2} = 1 + i.$$

$$z^4 - (3 + 8i)z^2 - 16 + 12i = 0$$

$$X^2 - (3 + 8i)X - 16 + 12i = 0$$

$$\Delta = (3 + 8i)^2 - 4(-16 + 12i) = 9$$

$$X_1 = \frac{(3 + 8i) - 3}{2} = 4i$$

$$X_2 = \frac{(3 + 8i) + 3}{2} = 3 - 4i$$

$$\sqrt{X_1} = \pm(\sqrt{2} + i\sqrt{2})$$

$$\sqrt{X_2} = \pm(2 + i)$$

$$z_1 = -\sqrt{2} - i\sqrt{2}; z_2 = \sqrt{2} + i\sqrt{2}; z_3 = -2 - i; z_4 = 2 + i.$$