CHAPTER 3: MATRICES

In this chapter, we present a very practical calculation tool that is matrices. In this course, we will present the definition of a matrices as well as their properties. We note that all the vectorial spaces considered here are with a finite dimension on a commutatif field $\mathbb{K}$ which can be $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$.

1. Definitions and examples

1.1. Definitions.

Definition 1.1. Let n and m be two natural numbers strictly big than 0. We call a matrix of type (n, m) or of n lines and m columns with coefficients in $\mathbb{K}$, all families $A = (a_{ij})_{1\leq i\leq n, 1\leq j\leq m}$ of elements in $\mathbb{K}$. Generally, the matrix A is represent in the form of table with n lines and m columns as follows:

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nm} \end{pmatrix}.$$

If there is no ambiguity, we write $A = (a_{ij})$, where $a_{ij} \in \mathbb{K}$ is the element correspondent to the intersection of the i^{th} line and the j^{th} column of A. The set of matrices of type (n, m) with coefficients in $\mathbb{K}$ is noted by $\mathcal{M}_{n,m}(\mathbb{K})$.

Example 1.2. Let

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 6 & 5 & 4 \\ -1 & -6 & 2 \\ 3 & -4 & 0 \end{pmatrix}.$$

A is a matrix type $(4, 3)$ of elements in $\mathbb{R}$. In other words, $A \in \mathcal{M}_{4,3}(\mathbb{R})$.

Definition 1.3. Let $A = (a_{ij})$ be a matrix in $\mathcal{M}_{n,m}(\mathbb{K})$.
• If $n = m = 1$, then the matrix A of $\mathcal{M}_{1,1}(\mathbb{K})$ is has only one element and it is of the form (a).
• If all the coefficients of A are equal to 0, then it is called a null matrix and we write $0_{n,m} = (0)$.
• If $n = 1$ and m is any natural number, the matrix A of $\mathcal{M}_{1,m}(\mathbb{K})$ is called line matrix and has the form

$$A = \begin{pmatrix} a_1 & a_2 & \dots & a_m \end{pmatrix}.$$

• If n is a natural number and $m = 1$, the matrix A of $\mathcal{M}_{n,1}(\mathbb{K})$ is called column matrix and has the form

$$A = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}.$$

Definition 1.4. Let $A = (a_{ij})$ and $B = (b_{ij})$ be two matrices of $\mathcal{M}_{n,m}(\mathbb{K})$. We say that $A = B$ if all the elements of A are equal to the elements of B, i.e.,

$$\forall i \in \{1, 2, ..., n\}, \forall j \in \{1, 2, ..., m\}; a_{ij} = b_{ij}.$$

Example 1.5. We define the matrices A and B by:

$$A = \begin{pmatrix} x+y & 0 \\ -1 & y \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 2 \\ -3 & 5 \end{pmatrix}.$$

Find the values of x and y so that $A = B$.
Indeed, $(A = B) \Leftrightarrow (x + y = 1 \text{ et } y = 5)$, thus $(x = -4 \text{ et } y = 5)$.

1.2. **Transpose of a matrix.**

Definition 1.6. Let $A = (a_{ij})$ be a matrix of $\mathcal{M}_{n,m}(\mathbb{K})$.
We call a transpose of A, the matrix (a_{ji}) of $\mathcal{M}_{m,n}(\mathbb{K})$ and we note it by A^t, i.e.,

$$A^t = (a_{ji}).$$

Example 1.7. Let $A = \begin{pmatrix} 0 & 1 & 2 & \frac{3}{4} \\ 5 & \sqrt{6} & 7 & 0 \end{pmatrix}$. Then

$$A^t = \begin{pmatrix} 0 & 5 \\ 1 & \sqrt{6} \\ 2 & 7 \\ \frac{3}{4} & 0 \end{pmatrix}.$$

Property:
Let $A = (a_{ij})$ be a matrix of $\mathcal{M}_{n,m}(\mathbb{K})$. Then

$$(A^t)^t = A.$$

1.3. **Square matrix.**

Definition 1.8. Let $A = (a_{ij})$ be a matrix of $\mathcal{M}_{n,m}(\mathbb{K})$.
If $m = n$, then A is called a square matrix. The set of square matrices of order n is noted by $\mathcal{M}_{n,n}(\mathbb{K})$ or simply by $\mathcal{M}_n(\mathbb{K})$.

Example 1.9. The matrix $A = \begin{pmatrix} 5 & 6 \\ 2 & 3 \end{pmatrix}$ is a square matrix of order 2.

Definition 1.10. Let $A = (a_{ij})$ be a matrix of $\mathcal{M}_n(\mathbb{K})$.
• The elements a_{ii} are called diagonal elements of A.
• The family $(a_{11}, a_{22}, ..., a_{nn})$ is called the main diagonal of A.
• A matrix $D \in \mathcal{M}_n(\mathbb{K})$ is called a diagonal matrix if all the coefficients are equal to 0 except those in the diagonal. This matrix has the form

$$D = \begin{pmatrix} a_{11} & 0 & 0 & .. & 0 \\ 0 & a_{22} & 0 & .. & 0 \\ 0 & 0 & a_{33} & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & .. & 0 & a_{nn} \end{pmatrix}.$$

Definition 1.11. Let $A = (a_{ij})$ be a matrix of $\mathcal{M}_n(\mathbb{K})$.
The trace of A, Tr, is given by $Tr = a_{11} + a_{22} + ... + a_{nn}$.

Definition 1.12. Let $A = (a_{ij})$ be a matrix of $\mathcal{M}_n(\mathbb{K})$.
We say that A is a symmetric matrix if and only if $A = A^t$.

Definition 1.13. Let $T = (a_{ij})$ be a matrix of $\mathcal{M}_n(\mathbb{K})$.
• We say that T is **upper triangular matrix** if all the coefficients coefficients bellow the diagonal are equal to 0. This matrix has the form

$$T = \begin{pmatrix} a_{11} & a_{12} & a_{13} & .. & a_{1n} \\ 0 & a_{22} & a_{23} & .. & a_{2n} \\ 0 & 0 & a_{33} & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & a_{nn-1} \\ 0 & 0 & .. & 0 & a_{nn} \end{pmatrix}.$$

• We say that T is **lower triangular matrix** if all the coefficients coefficients above the main diagonal are equal to 0. This matrix has the form:

$$T = \begin{pmatrix} a_{11} & 0 & 0 & .. & 0 \\ a_{21} & a_{22} & 0 & .. & 0 \\ a_{31} & a_{32} & a_{33} & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ a_{n1} & a_{n2} & .. & a_{nn-1} & a_{nn} \end{pmatrix}.$$

Remark 1.14. The intersection of the set of upper triangular matrices with the set lower triangular matrices of the same type (n, n), is equal the set of diagonal matrices of order n.

Definition 1.15. We call identity matrix of order n, the square matrix of order n whose elements of the diagonal are equal to 1 and all the others are equal to 0. It is noted by

I_n. We write

$$I_n = \begin{pmatrix} 1 & 0 & 0 & .. & 0 \\ 0 & 1 & 0 & .. & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & .. & 0 & 1 \end{pmatrix}.$$

2. Operations on matrices

2.1. **Sum of matrices.** Let $A = (a_{ij})$ and $B = (b_{ij})$ be two matrices of $\mathcal{M}_{n,m}(\mathbb{K})$. The sum of A and B is the matrix noted by $A + B$ of $\mathcal{M}_{n,m}(\mathbb{K})$ given by:

$$A + B = (a_{ij} + b_{ij}) = \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \dots & a_{1m} + b_{1m} \\ a_{21} + b_{21} & a_{22} + b_{22} & \dots & a_{2m} + b_{2m} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} + b_{n1} & a_{n2} + b_{n2} & \dots & a_{nm} + b_{nm} \end{pmatrix}.$$

Example 2.1. Let

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \text{ and } B = \begin{pmatrix} 7 & 8 & 9 \\ 0 & 1 & 2 \end{pmatrix}.$$

Thus, we have:

$$A + B = \begin{pmatrix} 8 & 10 & 12 \\ 4 & 6 & 8 \end{pmatrix}.$$

2.2. **Multiplication by a scalar.** Let $A = (a_{ij})$ be a matrix of $\mathcal{M}_{n,m}(\mathbb{K})$ and $\lambda \in \mathbb{K}$. The product of A by λ is the matrix noted by $\lambda \cdot A$ of $\mathcal{M}_{n,m}(\mathbb{K})$ given by:

$$\lambda \cdot A = (\lambda a_{ij}) = \begin{pmatrix} \lambda a_{11} & \lambda a_{12} & \dots & \lambda a_{1m} \\ \lambda a_{21} & \lambda a_{22} & \dots & \lambda a_{2m} \\ \vdots & \vdots & \vdots & \vdots \\ \lambda a_{n1} & \lambda a_{n2} & \dots & \lambda a_{nm} \end{pmatrix}.$$

Example 2.2. Let $A = \begin{pmatrix} 1 & 2 & 3 & 0 \\ 0 & 4 & 5 & 6 \end{pmatrix}$. Then,

$$3 \cdot A = \begin{pmatrix} 3 & 6 & 9 & 0 \\ 0 & 12 & 15 & 18 \end{pmatrix}.$$

Definition 2.3. Let $A = (a_{ij})$ be a matrix of $\mathcal{M}_{n,m}(\mathbb{K})$. The opposite of A is the matrix of type (n, m) noted by $-A$ such that

$$-A = (-a_{ij}).$$

Definition 2.4. Let $A = (a_{ij})$ be a matrix of $\mathcal{M}_{n,m}(\mathbb{K})$. We say that

$$A \text{ is antisymmetric if and only if } -A = A^t.$$

Property:
Let $A, B \in \mathcal{M}_{n,m}(\mathbb{K})$ and $\alpha \in \mathbb{K}$. Then,

$$(A+B)^t = A^t + B^t \text{ and } (\alpha \cdot B)^t = \alpha \cdot B^t \,.$$

Vectorial space of matrices:

Theorem 2.5. *By providing the set $\mathcal{M}_{n,m}(\mathbb{K})$ by the two operations:*
• *" + " addition of matrices (internal law),*
• *" · " multiplication of matrix by a scalar (external law).*
Then, $(\mathcal{M}_{n,m}(\mathbb{K}), +, \cdot)$ is a $\mathbb{K}$ vectorial space such that the zero vector is the matrix $0_{n,m}$.

2.3. **Product of matrices.**

Definition 2.6. Let $A = (a_{ij})$ be a matrix of $\mathcal{M}_{n,m}(\mathbb{K})$ and $B = (b_{ij})$ be a matrix of $\mathcal{M}_{m,p}(\mathbb{K})$. The product of A and B is the matrix $C = A \times B$ of $\mathcal{M}_{n,p}(\mathbb{K})$ defined by:

$$\forall i \in \{1, 2, ..., n\}, \forall j \in \{1, 2, ..., p\}; C = (c_{ij}) \text{ such that } c_{ij} = \sum_{k=1}^{m} a_{ik} b_{kj} \,.$$

Example 2.7. Let

$$A = \begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} 3 & -1 & 1 \\ 1 & 3 & -2 \end{pmatrix} .$$

Thus, we have

$$A \times B = \begin{pmatrix} 5 & 5 & -3 \\ -3 & 1 & -1 \end{pmatrix} .$$

Remark 2.8. For a multiplication of two matrices to be possible, it is necessary that the number of rows in the second is equal to the number of columns in the first.

General rule:
Let A, B and C be three matrices. Then,

$$A \text{ of type } (n, m) \times B \text{ of type } (m, p) = C \text{ of type } (n, p) \,.$$

Properties:
The multiplication of two matrices is not always commutative.

Example 2.9. Let

$$A = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 & 3 \\ 0 & 1 \end{pmatrix} .$$

Then

$$A \times B = \begin{pmatrix} 0 & 1 \\ 0 & 6 \end{pmatrix} \text{ and } B \times A = \begin{pmatrix} 6 & 0 \\ 2 & 0 \end{pmatrix} .$$

It is easy to see that $A \times B \neq B \times A$.

Proposition 2.10. *Let $A = (a_{ij}) \in \mathcal{M}_{n,m}(\mathbb{K})$, $B = (b_{ij}) \in \mathcal{M}_{m,p}(\mathbb{K})$ et $C = (c_{ij}) \in \mathcal{M}_{n,p}(\mathbb{K})$ such n, m and p are natural numbers strictly big than 0, and let $\alpha \in \mathbb{K}$. Then:*
- $A + B = B + A$.
- $(A + B) \times C = (A \times C) + (B \times C)$ *and* $A \times (B + C) = (A \times B) + (A \times C)$.
- $A \times (B \times C) = (A \times B) \times C$.
- $A \times I_m = A$ *and* $I_n \times A = A$.
- $\alpha \cdot (A \times B) = (\alpha \cdot A) \times B = A \times (\alpha \cdot B)$.

Definition 2.11. Let $A = (a_{ij}) \in \mathcal{M}_n(\mathbb{K})$ and k a natural number different of 0. We noe $A^0 = I_n$, $A^1 = A$, $A^2 = A \times A$ and $A^k = A \times A \times ... \times A(k \text{ termes })$. This matrice A^k, is the k^{th} power de A.

Let $A = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}$. Then, $A^0 = I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $A^1 = A$,

$$A^2 = A \times A = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix} \times \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix},$$

$$A^3 = A^2 \times A = \begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix} \times \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 6 & 5 \end{pmatrix}.$$

Ring of square matrices

Theorem 2.12. *By providing the set $\mathcal{M}_n(\mathbb{K})$ with the two law:*
- *" + " addition of matrices,*
- *" $\times$ " multiplications of matrices.*

Alors, $(\mathcal{M}_n(\mathbb{K}), +, \times)$ is an unitary ring, but it is not commutative.

2.4. **Inverse of a square matrix.**

Definition 2.13. Let $A = (a_{ij}) \in \mathcal{M}_n(\mathbb{K})$. We say that A is invertible (regular) if there exists a matrix B of $\mathcal{M}_n(\mathbb{K})$ such that $A \times B = B \times A = I_n$. If B exists, then B is unique and called the inverse matrix of A. Moreover, it is noted by A^{-1}.

Remark 2.14. If the square matrix A is not invertible, we say that it is singular.

We note by $GL_n(\mathbb{K})$ the set of invertible matrices of order n.

Consider the following matrix $A = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}$.

We show that A is invertible and this by looking for the matrix $B = \begin{pmatrix} x & z \\ y & t \end{pmatrix}$ such that $A \times B = B \times A = I_2$.

Indeed,

$$\begin{aligned} A \times B = I_2 &\Leftrightarrow \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix} \times \begin{pmatrix} x & z \\ y & t \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &\Leftrightarrow \begin{pmatrix} x & z \\ x+2y & z+2t \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &\Leftrightarrow \begin{cases} x = 1 \\ z = 0 \\ x + 2y = 0 \\ z + 2t = 1 \end{cases} \\ &\Leftrightarrow (x = 1, y = \frac{-1}{2}, z = 0, t = \frac{1}{2}). \end{aligned}$$

The same result for $B \times A = I_2$. Then, A is invertible and its inverse matrix is $A^{-1} = B = \begin{pmatrix} 1 & 0 \\ \frac{-1}{2} & \frac{1}{2} \end{pmatrix}$.

Properties:

Let $A, B \in GL_n(\mathbb{K})$. Then:

- $(A^t)^{-1} = (A^{-1})^t$.
- $(A^{-1})^{-1} = A$.
- $(A \times B)^{-1} = B^{-1} \times A^{-1}$.

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