University of Biskra
Mathematics Department
Module: Analysis 2

First year license
2023/2024

Catch-up Exam

Exercise 1 (./06pts)

Calculate the following integrals:

1. $\int \ln(x^2 + 2x - 3)dx$ **(./04pts)**.
2. $\int \cos^3(x)\sin^2(x)dx$ **(./02pts)**.

Exercise 2 (./04pts)

Let I_n, $n \in \mathbb{R}_+^*$, defined as follows

$$I_n = \int_0^{\infty} x^n e^{-x} dx$$

1. Show that $I_n = nI_{n-1}$, for all $n \in \mathbb{R}_+^*$ **(./01.50pts)**.
2. Check that if $n \in \mathbb{N}$ then $I_n = n!$ **(./01.50pts)**.
3. If we know that the value of $I_{1/2} = \frac{\sqrt{\pi}}{2}$, then what is the value of $I_{7/2}$ **(./01pt)**.

Exercise 3 (./04pts)

Solve the following second order differential equation.

$$y'' + 4y' - 5y = e^x.$$

Exercise 4 (./06pts)

Let us consider the following first order differential equation.

$$y' + a(x)y + b(x)y^2 = c(x), \tag{1}$$

$$x^2\left(y' + y^2\right) = xy - 1, \tag{2}$$

1. Show that if y_0 is a particular solution of (1), then with using the substitution $z = y - y_0$ we can transform it to a Bernoulli's differential equation.
2. After verifying that $y_0 = \frac{1}{x}$ is a particular solution of (2), find its general solution.

Solution :

$\boxed{\text{Exo1 :}}$

$$F(x) = \int \ln(x^2+2x-3)\,dx = \int 1 * \ln(x^2+2x-3)\,dx$$

$$\begin{cases} u' = 1 \\ v = \ln(x^2+2x-3) \end{cases} \Rightarrow \begin{cases} u = x \\ v' = \dfrac{2x+2}{x^2+2x-3} \end{cases}$$ (0,5)

$$\Rightarrow F(x) = x\ln(x^2+2x-3) - \overbrace{\int \frac{2x^2+2x}{x^2+2x-3}\,dx}^{I}$$ (0,5)

$$x^2+2x-3 = 0 \Rightarrow \Delta = 16 \Rightarrow \begin{cases} x_1 = -3 \\ x_2 = 1 \end{cases}$$ (0,5) (0,5)

$$\Rightarrow I = \int 2 + \frac{a}{x+3} + \frac{b}{x-1}\,dx$$ (0,5)

$$\frac{2x^2+2x}{x^2+2x-3} = 2 + \frac{a}{x+3} + \frac{b}{x-1} \Rightarrow \begin{cases} a = -3 \\ b = 1 \end{cases}$$ (0,5)

$$\Rightarrow I = \int 2 + \frac{-3}{x+3} + \frac{1}{x-1}\,dx$$

$$= 2x - 3\ln|x+3| + \ln|x-1| + C \;/\; C \in \mathbb{R}$$ (0,5)

$$\Rightarrow F(x) = x\ln(x^2+2x-3) + 2x - 3\ln|x+3| + \ln|x-1| + C$$ (0,5) $/ C \in \mathbb{R}$

$$G(x) = \int \cos^3(x)\sin^2(x)\,dx = \int \cos^2(x)\sin^2(x)\cos(x)\,dx$$

we put $t = \sin(x) \Rightarrow dt = \cos(x)\,dx$. (0,5)

$$\Rightarrow G(t) = \int (1-t^2)t^2\,dt = \int t^2 - t^4\,dt$$ (0,5)

(0,5) $= \frac{1}{3}t^3 - \frac{1}{5}t^5 + C \;/\; C \in \mathbb{R}$ (1 than

$$\boxed{G(x) = \frac{1}{3}\sin^3(x) - \frac{1}{5}\sin^5(x) + C}$$ ok

EXO2 $I_n = \int_0^{+\infty} x^n e^{-x}, \quad n \in \mathbb{N}^*_+$ ok

(1) $I_n = \int_0^{+\infty} x^n e^{-x}\,dx = ?$ $\begin{cases} u = x^n \\ v' = e^{-x} \end{cases} \Rightarrow \begin{cases} u' = n x^{n-1} \\ v = -e^{-x} \end{cases}$

$$\Rightarrow I_n = \cancel{-x^n e^{-x}\Big|_0^{+\infty}} + \int_0^{+\infty} n x^{n-1} e^{-x}\,dx$$ ok

$$= n \int_0^{+\infty} x^{n-1} e^{-x}\,dx.$$

$$\boxed{I_n = n I_{n-1}}$$ ok

(2) Let $n \in \mathbb{N}$.

- we have for $n=0 \Rightarrow I_0 = \int_0^{+\infty} e^{-x}\,dx = 1 = 0!$ —— $(*)$ ok

- From the first question we have

$$I_n = n I_{n-1}$$

$$= n*(n-1) I_{n-2}$$

$$= n*(n-1)*\cdots*1*I_0 \quad / \text{As } I_0 = 1 \text{ than}$$

$$\boxed{I_n = n!} \quad \text{——} (**)$$ ok

ok

from $(*)$ and $(**)$ we deduce that $I_n = n!\ \forall n \in \mathbb{N}$.

3) we have $I_{1/2} = \sqrt{\pi}/2$

we have from the first question $I_n = n\, I_{(n-1)}$ so

$$I_{7/2} = \frac{7}{2} I_{5/2}$$
$$= \frac{7}{2} \times \frac{5}{2} I_{3/2}$$
$$= \frac{7}{2} \times \frac{5}{2} \times \frac{3}{2} I_{1/2}$$

1

$$= \frac{7}{2} \times \frac{5}{2} \times \frac{3}{2} \frac{\sqrt{\pi}}{2}$$

$$\boxed{I_{7/2} = \frac{105}{16}\sqrt{\pi}}$$

Exo 4:

$$x^2(y' + y^2) = xy - 1 \quad \text{———— } (1)$$

① $y = 1/x$ is a solution of (1) ?

$y = 1/x \Rightarrow y' = -1/x^2 \xrightarrow{\text{in } (1)} x^2\left(-\frac{1}{x^2} + \left(\frac{1}{x}\right)^2\right) = x\frac{1}{x} - 1.$

$\Rightarrow 0 = 0$ so $y = 1/x$ is a solution of (1).

0.5

② To find the general solution of (1) we must transform it to a Bernoulli equation $(n=2)$ using the substitution $z = y - y_0 = y - 1/x$.

0.5

$\Rightarrow y = z + 1/x \Rightarrow y' = z' - \frac{1}{x^2}.$

0.5

$\cancel{x^2\left(z' - \frac{1}{x^2} + \frac{1}{x^2}\right) = x\left(z + \frac{1}{x}\right) - 1}$

$(1) \Rightarrow x^2\left(\left(z' - \frac{1}{x^2}\right) + \left(z + \frac{1}{x}\right)^2\right) = x\left(z + \frac{1}{x}\right) - 1.$

$\Rightarrow z' + z/x = -z^2$

$\Rightarrow -z^{-2}z' - z^{-1}/x = 1.$ —— Bernoulli $n=2$. ——$(**)$

$u = z^{-1} \Rightarrow u' = -z^{-2}z'$ —— $(\cancel{***})$ 0,5

$(**) \Rightarrow u' - u/x = 1.$ —— $(***)$ 0,5

$\Rightarrow u' - u/x = 0$ —— $(****)$

$\Rightarrow u_H = kx \quad / k \in \mathbb{R}_+.$ 0,5

$u_G = k(x)x \Rightarrow u_G' = k'(x)x + k(x)$, substitute u_G and u_G' in $(***)$ we get:

$xk'(x) + \cancel{k(x)} - \cancel{\frac{k(x)x}{x}} = 1.$ 0,5

$\Rightarrow k'(x) = \frac{1}{x} \Rightarrow \boxed{k(x) = \ln(|x|) + c} \quad / \in \in \mathbb{R}$

so the solution of $(***)$ is given by

$\boxed{u_G = (\ln(|x|) + c)x} \quad c \in \mathbb{R}$ 0,5

hence $z = \frac{1}{u_G} = \left[(\ln(|x|) + c) \cdot x\right]^{-1}.$ 0,5

Consequently:

$y = \frac{1}{(\ln(|x|) + c)x} + \frac{1}{x} \quad / c \in \mathbb{R}.$ 0,5

end

EXO3: /04

$$y''+4y'-5y=e^{x}$$

① Homogenous solution

$$y''+4y'-5y=0 \Rightarrow R^{2}+4y-5=0$$

$$\Delta = 36 \Rightarrow \sqrt{\Delta}=6$$ 0,1 ✓

$$\Rightarrow \begin{cases} R_1 = \dfrac{-4-6}{2} = -5 \\ R_2 = \dfrac{-4+6}{2} = 1 \end{cases}$$ 0,1 ✓ 0,1 ✓

$$\Rightarrow \boxed{y_H = C_1 e^{x} + C_2 e^{-5x}}$$ 0,1 ✓

Homogenous

② Particular solution: $y_p = axe^{x}$ / as e^{x} is a solution 0,1 ✓

$$y_p = axe^{x}$$

$$\Rightarrow y'_p = axe^{x} + \cancel{2a}\, a e^{x}$$

$$y''_p = axe^{x} + 2ae^{x}$$ 0,1 ✓

$$\Rightarrow (axe^{x}+2ae^{x}) + 4(axe^{x}+ae^{x}) - 5axe^{x} = e^{x}$$

$$\Rightarrow 6a = 1 \Rightarrow \boxed{a = 1/6} \Rightarrow \cancel{y}$$

$$\Rightarrow \boxed{y_p = \tfrac{1}{6} x e^{x}}$$ 0,1 ✓

③ general solution: $y_G = y_H + y_p$

$$\Rightarrow \boxed{y_G = \left(C_1 + \tfrac{1}{6}x\right)e^{x} + C_2 e^{-5x}}$$ 0,1 ✓