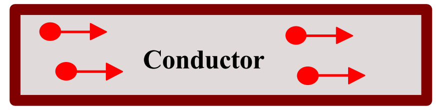
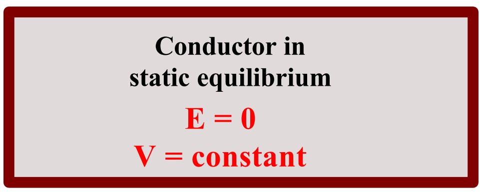
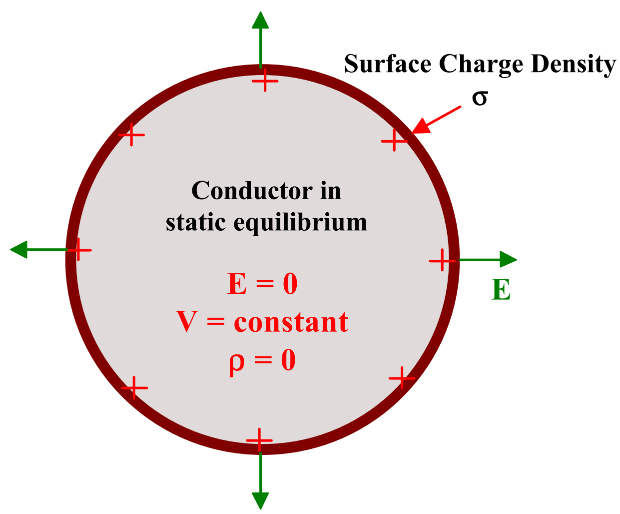
Conductors in Static Equilibrium



Conductor: In a conductor some electrons are free to move (**without restraint**) within the volumn of the material (**Examples:** copper, silver, aluminum, gold)



Conductor in Static Equilibrium: When the charge distribution on a conductor reaches **static equilibrium** (**i.e. nothing moving**), the **net electric field withing the conducting material is exactly zero** (**and the electric potential is constant**).



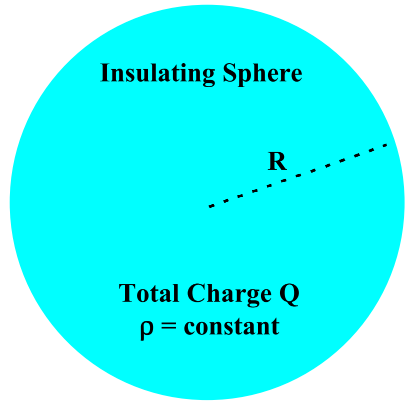
Excess Charge: For a conductor **in static equilibrium** all the (**extra**) electric charge **reside on the surface**. **There is no net electric charge within the volumn of the conductor (i.e. $\rho = 0$).**

Electric Field at the Surface: The electric field at the surface of a conductor **in static equilibrium** is **normal to the surface** and has a magnitude, $\mathbf{E = \sigma/\varepsilon_0}$, where σ is the surface charge density (**i.e. charge per unit area**) and the **net charge** on the conductor is

$$Q = \int_{Surface} \sigma \, dA \, .$$

Gauss' Law Examples

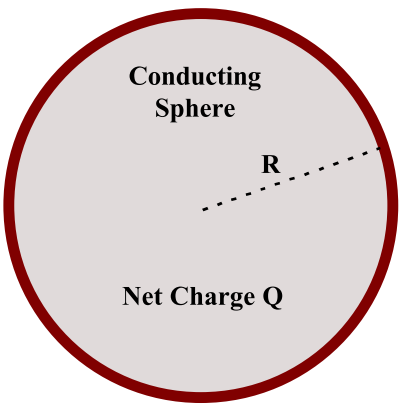
Problem: A solid insulating sphere of radius **R** has charge distributed uniformly throughout its volume. The total charge of the sphere is **Q**. What is the magnitude of the electric field inside and outside the sphere?



Answer:

$$\vec{E}_{out} = \frac{KQ}{r^2}\hat{r}$$

$$\vec{E}_{in} = \frac{KQr}{R^3}\hat{r}$$



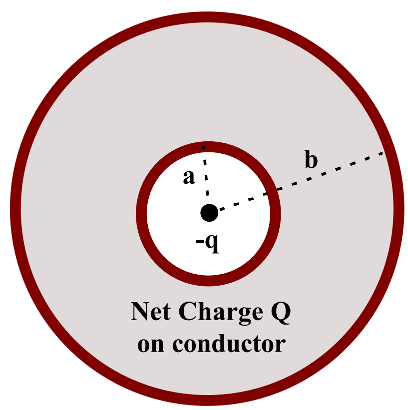
Problem: A solid conducting sphere of radius **R** has a net charge of **Q**. What is the magnitude of the electric field inside and outside the sphere? Where are the charges located?

Answer: **Charges are on the surface and**

$$\vec{E}_{out} = \frac{KQ}{r^2}\hat{r}$$

$$\vec{E}_{in} = 0$$

Problem: A solid conducting sphere of radius **b** has a spherical hole in it of radius **a** and has a net charge of **Q**. If there is a point charge **-q** located at the center of the hole, what is the magnitude of the electric field inside and outside the conductor? Where are the charges on the conductor located?



Answer: **Charges are on the inside and outside surface with Q_{in}=q and Q_{out}=Q-q and**

$$\vec{E}_{r>b} = \frac{K(Q-q)}{r^2}\hat{r}$$

$$\vec{E}_{a<r<b} = 0$$

$$E_{r<a} = \frac{-Kq}{r^2}\hat{r}$$

Gravitational Potential Energy

Gravitational Force: $F = G\, m_1 m_2 / r^2$

Gravitational Potential Energy GPE:

$U = GPE = mgh$ **(near surface of the Earth)**

Kinetic Energy: $KE = \frac{1}{2} m v^2$

Total Mechanical Energy: $E = KE + U$

Work Energy Theorem:

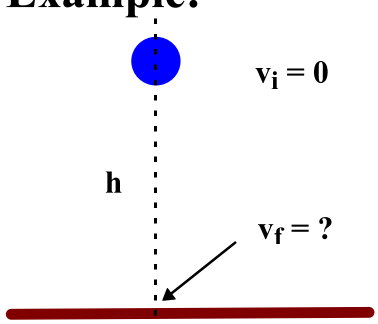
$$W = E_B - E_A = (KE_B - KE_A) + (U_B - U_A)$$

(work done on the system)

Energy Conservation: $E_A = E_B$

(if no external work done on system)

Example:



A ball is dropped from a height h. What is the speed of the ball when it hits the ground?

Solution: $E_i = KE_i + U_i = mgh$ $E_f = KE_f + U_f = mv_f^2/2$

$$E_i = E_f \Rightarrow v_f = \sqrt{2gh}$$

Electric Potential Energy

Gravitational Force: $F = K q_1 q_2/r^2$

Electric Potential Energy: $EPE = U$ (Units = Joules)

Kinetic Energy: $KE = \frac{1}{2} m v^2$ (Units = Joules)

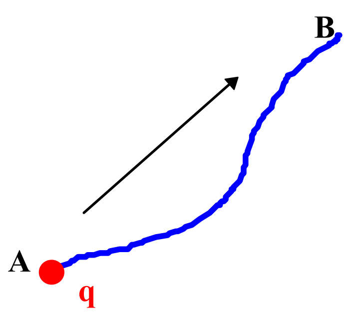
Total Energy: $E = KE + U$ (Units = Joules)

Work Energy Theorem: (work done on the system)

$$W = E_B - E_A = (KE_B - KE_A) + (U_B - U_A)$$

Energy Conservation: $E_A = E_B$ (if no external work done on system)

Electric Potential Difference $\Delta V = \Delta U/q$:



Work done (**against the electric force**) per unit charge in going from **A** to **B** (**without changing the kinetic energy**).

$$\Delta V = W_{AB}/q = \Delta U/q = U_B/q - U_A/q$$

(**Units = Volts** **1V = 1 J / 1 C**)

Electric Potential $V = U/q$: $U = qV$

Units for the Electric Field (Volts/meter):

$$N/C = Nm/(Cm) = J/(Cm) = V/m$$

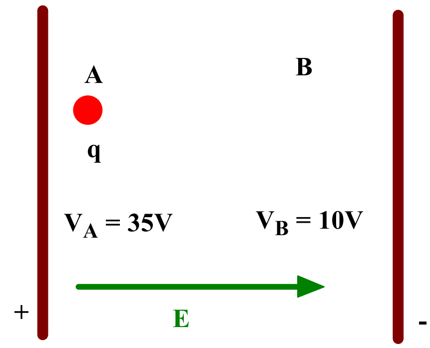
Energy Unit (electron-volt): One electron-volt is the amount of kinetic energy gained by an electron when it drops through one Volt potential difference

$$1\ eV = (1.6\times10^{-19}\ C)(1\ V) = 1.6\times10^{-19}\ Joules$$

1 MeV = 10^6 eV **1 GeV=1,000 MeV** **1 TeV=1,000 GeV**

Accelerating Charged Particles

Example Problem: A particle with mass **M** and charge **q** starts from rest a the point **A**. What is its speed at the point **B** if $\mathbf{V_A=35V}$ and $\mathbf{V_B=10V}$ ($M = 1.8\text{x}10^{-5}$kg, $q = 3\text{x}10^{-5}$C)?



Solution:

The total energy of the particle at **A** and **B** is

$$E_A = KE_A + U_A = 0 + qV_A$$

$$E_B = KE_B + U_B = \frac{1}{2}Mv_B^2 + qV_B\ .$$

Setting $\mathbf{E_A = E_B}$ (**energy conservation**) yields

$$\frac{1}{2}Mv_B^2 = q(V_A - V_B)$$

(Note: the particle gains an amount of kinetic energy equal to its charge, q, time the change in the electric potential.)

Solving for the particle speed gives

$$v_B = \sqrt{\frac{2q(V_A - V_B)}{M}}$$

(Note: positive particles fall from high potential to low potential $V_A > V_B$, while negative particles travel from low potential to high potential, $V_B > V_A$.)

Plugging in the numbers gives

$$v_B = \sqrt{\frac{2(3\quad 10^{-5}C)(25V)}{1.8\quad 10^{-5}kg}} = 9.1m/s\ .$$

Potential Energy & Electric Potential

Mechanics (last semester!):

Work done by force F in going from A to B:

$$W_{A\to B}^{byF} = \int_A^B \vec{F}\cdot d\vec{r}$$

Potential Energy Difference ΔU:

$$W_{A\to B}^{againstF} = \Delta U = U_B - U_A = -\int_A^B \vec{F}\cdot d\vec{r}$$

$$\vec{F} = -\vec{\nabla}U = -\frac{\partial U}{\partial x}\hat{x} - \frac{\partial U}{\partial y}\hat{y} - \frac{\partial U}{\partial z}\hat{z}$$

Electrostatics (this semester):

Electrostatic Force: $\vec{F} = q\vec{E}$

Electric Potential Energy Difference ΔU:

(work done against E in moving q from A to B)

$$\Delta U = U_B - U_A = -\int_A^B q\vec{E}\cdot d\vec{r}$$

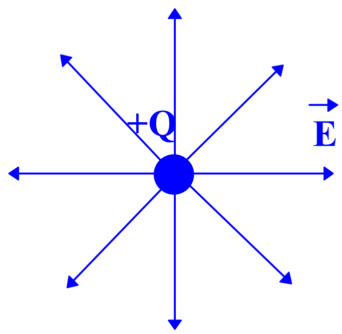
Electric Potential Difference ΔV=ΔU/q:

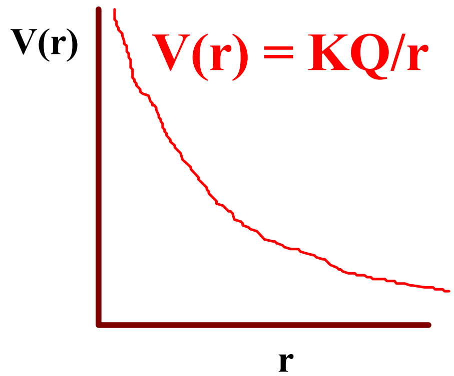
(work done against E per unit charge in going from A to B)

$$\Delta V = V_B - V_A = -\int_A^B \vec{E}\cdot d\vec{r}$$

$$\vec{E} = -\vec{\nabla}V = -\frac{\partial V}{\partial x}\hat{x} - \frac{\partial V}{\partial y}\hat{y} - \frac{\partial V}{\partial z}\hat{z}$$

The Electric Potential of a Point Charge

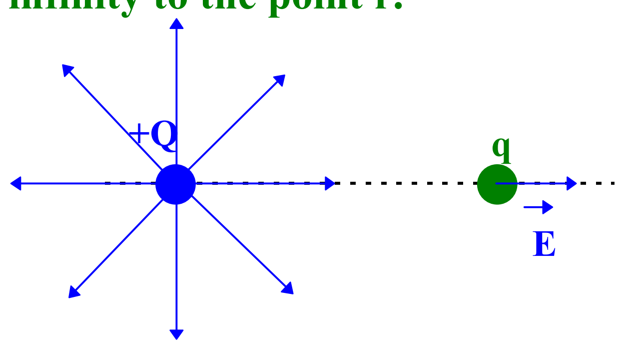




Potential from a point charge:

$$V(r) = \Delta V = V(r) - V(\text{infinity}) = KQ/r$$

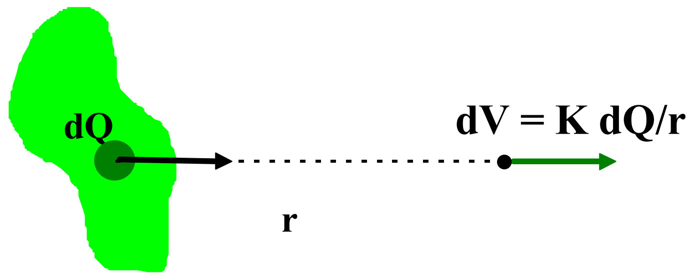
U = qV = work done against the electric force in bringing the charge q from infinity to the point r.



Potential from a system of N point charges:

$$V = \sum_{i=1}^{N} \frac{Kq_i}{r_i}$$

Electric Potential due to a Distribution of Charge

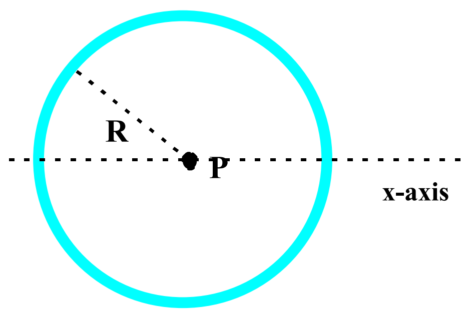


The electric potential from a continuous distribution of charge is the superposition (i.e. integral) of all the (infinite) contributions from each infinitesimal dQ as follows:

$$V = \int \frac{K}{r} dQ \quad \text{and} \quad Q = \int dQ$$

Example:

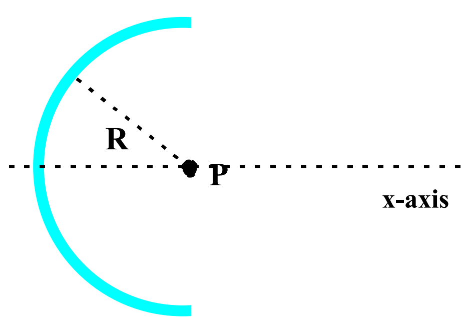
A total amount of charge Q is uniformly distributed along a thin **circle** of radius R. What is the electric potential at a point P at the center of the circle?



Answer: $V = \frac{KQ}{R}$

Example:

A total amount of charge Q is uniformly distributed along a thin **semicircle** of radius R. What is the electric potential at a point P at the center of the circle?

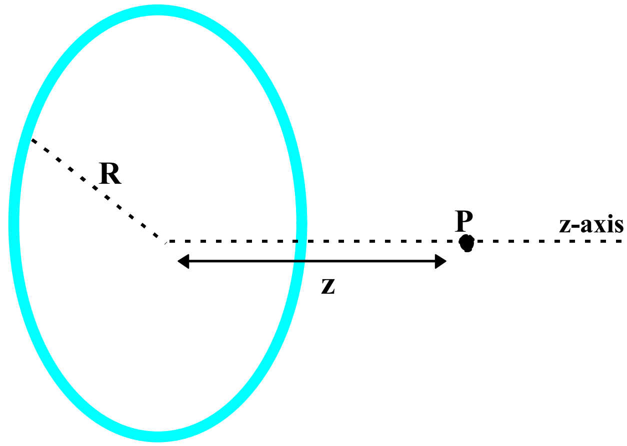


Answer: $V = \frac{KQ}{R}$

Calculating the Electric Potential

Example:

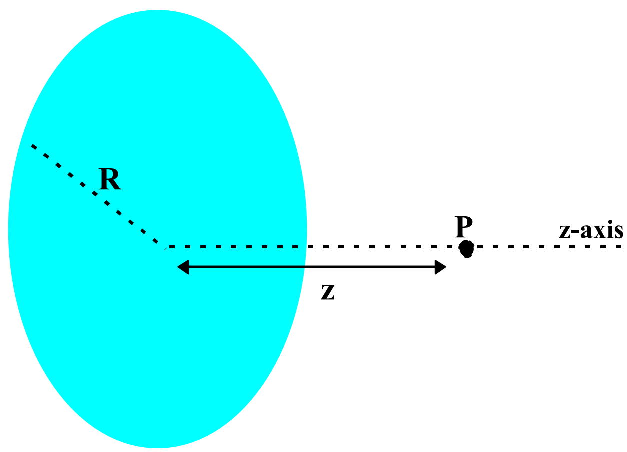
A total amount of charge Q is uniformily distributed along a thin **ring** of radius R. What is the electric potential at a point P on the z-axis a distance z from the center of the ring?



Answer: $V(z)=\dfrac{KQ}{\sqrt{z^2+R^2}}$

Example:

A total amount of charge Q is uniformily distributed on the surface of a **disk** of radius R. What is the electric potential at a point P on the z-axis a distance z from the center of the disk?



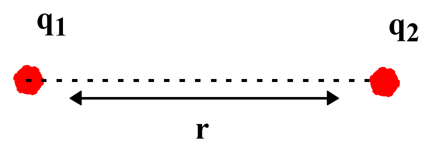
Answer: $V(z)=\dfrac{2KQ}{R^2}\left(\sqrt{z^2+R^2}-z\right)$

Electric Potential Energy

For a system of point charges:

The potential energy **U** is the **work** required to assemble the final charge configuration starting from an inital condition of infinite separation.

Two Particles:



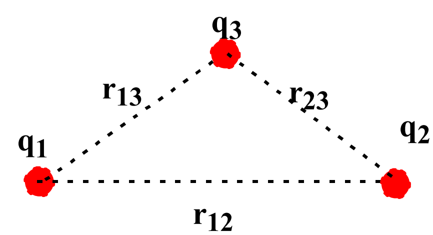
$$U = K\frac{q_1 q_2}{r} = \frac{1}{2}q_1\left(\frac{Kq_2}{r}\right) + \frac{1}{2}q_2\left(\frac{Kq_1}{r}\right)$$

so we see that

$$U = \frac{1}{2}\sum_{i=1}^{2} q_i V_i$$

where $\mathbf{V_i}$ is the electric potential at **i** due to the other charges.

Three Particles:



$$U = K\frac{q_1 q_2}{r_{12}} + K\frac{q_1 q_3}{r_{13}} + K\frac{q_2 q_3}{r_{23}}$$

which is equivalent to

$$U = \frac{1}{2}\sum_{i=1}^{3} q_i V_i$$

where $\mathbf{V_i}$ is the electric potential at **i** due to the other charges.

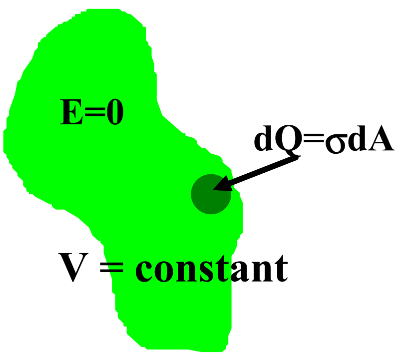
N Particles:

$$U = \frac{1}{2}\sum_{i=1}^{N} q_i V_i$$

Stored Electric Potential Energy

For a conductor with charge Q:

The potential energy **U** is the **work** required to assemble the final charge configuration starting from an inital condition of infinite separation.



For a conductor the total charge Q resides on the surface

$$Q = \int dq = \int \sigma \, dA$$

Also, V is constant on and inside the conductor and

$$dU = \frac{1}{2} dQ V = \frac{1}{2} V \sigma dA$$

and hence

$$U = \frac{1}{2} \int_{Surface} V \, dQ = \frac{1}{2} V \int_{Surface} \sigma \, dA = \frac{1}{2} V Q$$

Stored Energy: $$U_{conductor} = \frac{1}{2} Q V$$

where Q is the charge on the conductor and V is the electric potential of the conductor.

For a System of N Conductors:

$$U = \frac{1}{2} \sum_{i=1}^{N} Q_i V_i$$

where Q_i is the charge on the i-th conductor and V_i is the electric potential of the i-th conductor.

Capacitors & Capacitance

Capacitor:

Any arrangement of **conductors** that is used to store electric charge (**will also store electric potential energy**).

Capacitance: $C=Q/V$ or $C=Q/\Delta V$

Units: 1 farad = 1 F = 1 C/1 V   1 F=10^{-6} F   1 pF=10^{-9} F

Stored Energy:

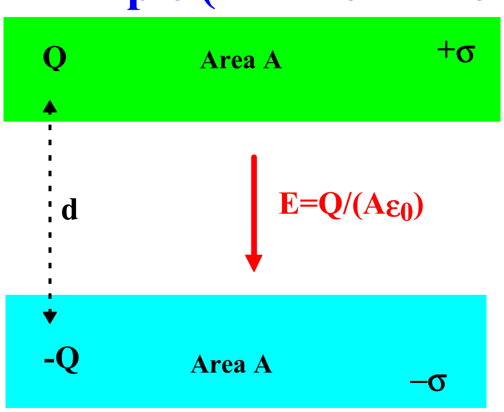
$$U_{conductor} = \frac{1}{2}QV = \frac{Q^2}{2C} = \frac{1}{2}CV^2$$

where **Q** is the charge on the conductor and **V** is the electric potential of the conductor and **C** is the capacitance of the conductor.

Example (Isolated Conducting Sphere):

For an isolated conducting sphere with radius R, $V=KQ/R$ and hence $C=R/K$ and $U=KQ^2/(2R)$.

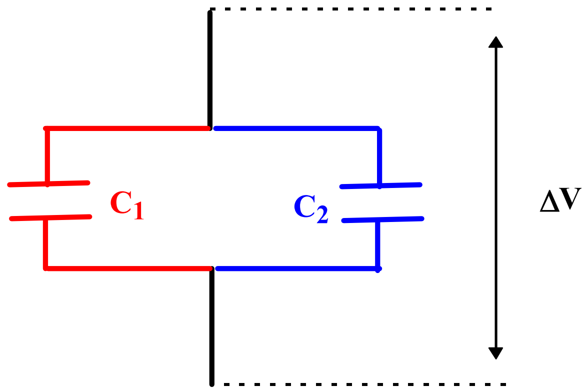
Example (Parallel Plate Capacitor):



For two parallel conducting plates of area A and separation d we know that $E = \sigma/\varepsilon_0 = Q/(A\varepsilon_0)$ and $\Delta V = Ed = Qd/(A\varepsilon_0)$ so that $C = A\varepsilon_0/d$. The stored energy is $U = Q^2/(2C) = Q^2d/(2A\varepsilon_0)$.

Capacitors in Series & Parallel

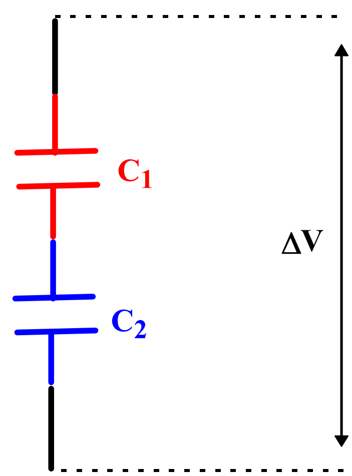
Parallel:



In this case $\Delta V_1=\Delta V_2=\Delta V$ and $Q=Q_1+Q_2$. Hence,
$Q = Q_1 + Q_2 = C_1\Delta V_1 + C_2\Delta V_2 = (C_1+C_2)\Delta V$
so $C = Q/\Delta V = C_1 + C_2$, where I used $Q_1 = C_1\Delta V_1$ and $Q_2 = C_2\Delta V_2$.

Capacitors in parallel add.

Series:

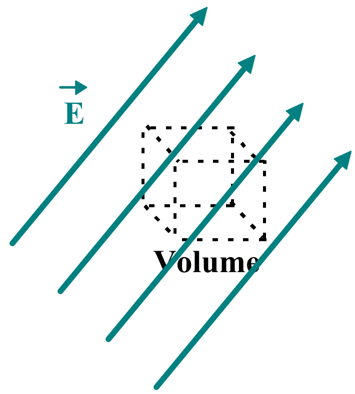


In this case $\Delta V=\Delta V_1+\Delta V_2$ and $Q=Q_1=Q_2$. Hence,
$\Delta V = \Delta V_1 + \Delta V_2 = Q_1/C_1+Q_2/C_2 = (1/C_1+1/C_2)Q$
so $1/C = \Delta V/Q = 1/C_1 + 1/C_2$, where I used $Q_1 = C_1\Delta V_1$ and $Q_2 = C_2\Delta V_2$.

Capacitors in series add inverses.

Energy Density of the Electric Field

Energy Density u:



Electric field lines contain **energy!** The amount of energy per unit volume is

$$u = e_0E^2/2,$$

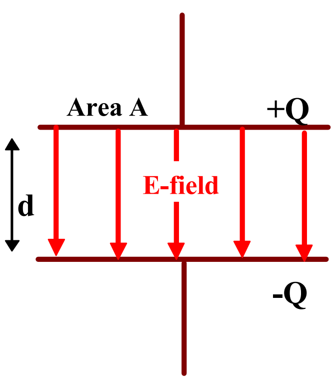
where E is the magnitude of the electric field. The energy density has **units of Joules/m^3**.

Total Stored Energy U:

The total energy strored in the electric field lines in an infinitessimal volume dV is **$dU = u\, dV$** and

$$U = \int_{Volume} u\, dV$$

If u is constant throughout the volume, V, then U = u V.



Example: Parallel Plate Capacitor

Think of the work done in bringing in the charges from infinity and placing them on the capacitor as the work necessary to produce the electric field lines and that the energy is **stored in the electric field!**

From before we know that **$C = A\varepsilon_0/d$** so that the stored energy in the capacitor is

$$U = Q^2/(2C) = Q^2d/(2A\varepsilon_0).$$

The energy stored in the electric field is **$U = uV = e_0E^2V/2$** with **$E = \sigma/e_0 = Q/(e_0A)$** and **$V = Ad$**, thus

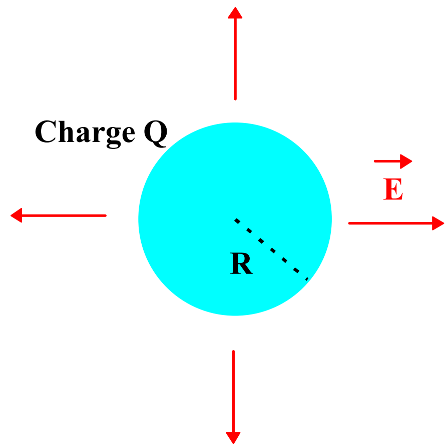
$$U=Q^2d/(2A\varepsilon_0),$$

which is the **same as the energy stored in the capacitor!**

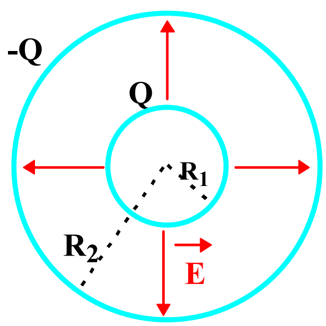
Electric Energy Examples

Example:

How much electric energy is stored by a **solid conducting sphere** of radius R and total charge Q?



Answer: $$U = \frac{KQ^2}{2R}$$



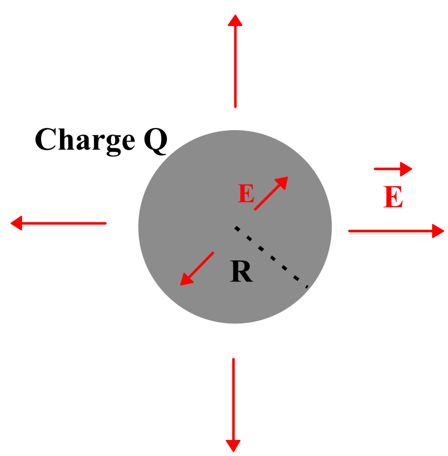
Example:

How much electric energy is stored by a two thin spherical conducting shells one of radius R_1 and charge Q and the other of radius R_2 and charge -Q (**spherical capacitor**)?

Answer: $$U = \frac{KQ^2}{2}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

Example:

How much electric energy is stored by a **solid insulating sphere** of radius R and total charge Q uniformly distributed throughout its volume?



Answer: $$U = \left(1 + \frac{1}{5}\right)\frac{KQ^2}{2R} = \frac{3}{5}\frac{KQ^2}{R}$$