# Tutorial N°6: Exercises on Bernoulli's law (hydrodynamics)

## Exercise 6.1:

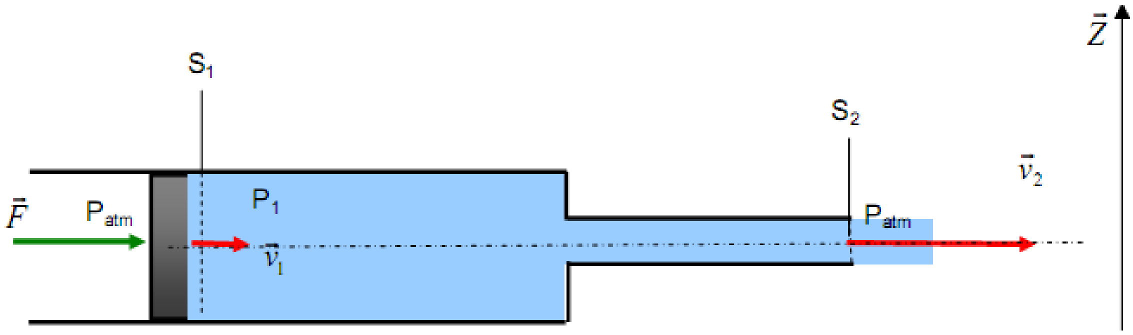
The figure below shows a piston that moves without friction in a cylinder of section $S_1$ and diameter $d_1 = 4\ cm$ filled with a perfect fluid of density$\rho = 1000\ kg/m^3$. A force F with an intensity of 62.84 N acts on the piston, at a constant speed $V_1$ .The fluid can escape to the outside through a cylinder of section $S_2$ and diameter $d_2 = 1\ cm$ at a speed $V_2$ and a pressure $P_2 = P_{atm} = 1\ bar$.

1- By applying the Fundamental Principle of Dynamics to the piston, determine the pressure $P_1$ of the fluid at section $S_1$ as a function of $F$, $P_{atm}$ and $d$?

2- Write the continuity equation and determine the expression of the speed $V_1$ as a function of $V_2$?

3- By applying the Bernoulli equation, determine the flow speed $V_2$ as a function of $P_1$, $P_{atm}$ and $\rho$?

We assume that the cylinders are in a horizontal position ($Z_1 = Z_2$)




## Solution

1- Applying the Fundamental Principle of Dynamics, we obtain:

$P_1 = \frac{4F}{\pi d_1^2} + P_{atm}$=1.5bar

2- The continuity equation:

$\pi {d_1}^2 V_1 = \pi {d_2}^2 V_2$ and $d_1 = 4d_2$

$$\pi 16 {d_2}^2 V_1 = \pi {d_2}^2 V_2 \Rightarrow V_1 = \frac{1}{16} V_2$$

3- By applying the Bernoulli equation

$$\frac{P_1}{\rho}+\frac{v_1^2}{2}=\frac{P_{atm}}{\rho}+\frac{v_2^2}{2}\Longrightarrow\frac{P_1}{\rho}+\frac{v_2^2}{2\times 256}=\frac{P_{atm}}{\rho}+\frac{v_2^2}{2}$$

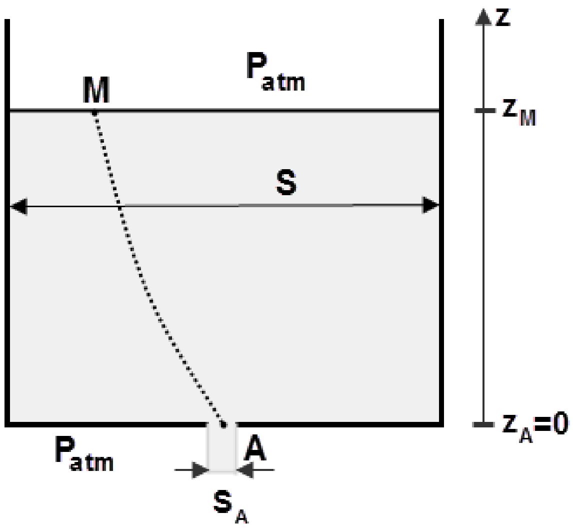
$$\frac{P_1}{\rho}-\frac{P_{atm}}{\rho}=\frac{v_2^2}{2}-\frac{v_2^2}{2\times 256}\Longrightarrow\frac{(P_1-P_{atm})}{\rho}=\frac{(256-1)\times v_2^2}{512}$$

$$\frac{(P_1-P_{atm})}{\rho}=\frac{(255)\times v_2^2}{512}$$

$$v_2^2=\frac{512\times(P_1-P_{atm})}{255\times\rho}\Rightarrow v_2=\sqrt{\frac{512\times(P_1-P_{atm})}{255\times\rho}}=10m/s.$$

## Exercise 6.2:

A reservoir, cubic in shape and section $S=4m^2$ $and$ $a=2m$ .The reservoir is filled with liquid that can be emptied through an opening $A$ pierced at its horizontal bottom and opening into the open air. A is section $S_A=8\ cm^2$. We will assume that when it is drained, the liquid is perfect, incompressible and its flow speed is constant.




1- When emptying this reservoir, consider the streamline joining points $M$ and $A$. By applying the Bernoulli relation between these two points, give the expression for the flow speed $v_A$ liquid to the point $A$ depending on the acceleration of gravity g and the altitude $Z_M$ of the point M.
2- Give the relation of the volume flow$Q_v$ at the orifice $A$ as a function of g, $Z_M$ and M.
3- Establish the relationship of the flow speed $v_M$ at point $M$ according to $g, Z_M, S_A$ $and$ $S$
4- Calculate the time necessary for the total emptying of this reservoir.

## Solution

1- The Bernoulli relation between the two points $M\ and\ A$ is written:

$$z_A + \frac{P_A}{\rho g} + \frac{v_A^2}{2g} = z_M + \frac{P_M}{\rho g} + \frac{v_M^2}{2g}$$

$$z_A = 0$$

The points $M\ and\ A$ and being in direct contact with the air, their pressures are equal to atmospheric pressure: $P_A = P_M = P_{atm}$

$$\frac{P_{atm}}{\rho g} + \frac{v_A^2}{2g} = z_M + \frac{P_{atm}}{\rho g} + \frac{v_M^2}{2g} \Rightarrow\ v_A^2 = v_M^2 + 2.g.Z_M$$

For a perfect liquid the volume flow $Q_v$ is constant:$S_A.v_A = S.v_M$

$$v_M = \frac{S_A.v_A}{S}$$

$S_A = 8 \times 10^{-4}\ m^2$
$S = 4\ m^2$

$$v_M = \frac{8 \times 10^{-4}\ .v_A}{4} \Rightarrow v_M = 2 \times 10^{-4}\ .v_A$$

$v_M^2 = 4 \times 10^{-8}\ .v_A^2 \ll v_A^2, we\ then\ neglect\ v_M^2\ compared\ to\ v_A^2$
$v_A^2 = 2.g.Z_M \Rightarrow\ v_A = \sqrt{2.g.Z_M}$

2- $Q_v\ = S_A.v_A = S_A.\sqrt{2.g.Z_M}$

3- $v_M = \frac{S_A.\sqrt{2.g.Z_M}}{S}$

4- Speed $v_M$ is expressed by:

$$v_M = -\frac{dZ_M}{dt}$$

$$-\frac{dZ_M}{dt} = \frac{S_A.\sqrt{2.g.Z_M}}{S}$$

$$\frac{dZ_M}{\sqrt{Z_M}} = -\frac{S_A.\sqrt{2.g}}{S}.dt$$

We integrate from $t = 0$ until the moment $T$ when the reservoir has been completely emptied: $Z_M = Z_A = 0\ m$

$$\int_{Z_M}^{0} \frac{dZ_M}{\sqrt{Z_M}} = -\frac{S_A.\sqrt{2.g}}{S}.\int_0^{\tau} dt$$

$$-2\sqrt{Z_M} = \frac{S_A.\sqrt{2.g}}{S}.T$$

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$$T = \frac{2\sqrt{Z_M}.S}{S_A.\sqrt{2.g}} = \frac{2\sqrt{2}\times 4}{8\times 10^{-4}\times\sqrt{2\times 9.81}} \approx 3193 \text{ s}$$

## Exercise 6.3:

The aorta is the largest artery in the body. It receives the blood that leaves the heart and distributes it to the arteries throughout the body. The heart rate for an adult is 80 beats per minute. With each beat, the heart injects a volume $v_b = 0.075\ L$ into the aorta.

1- Calculate total volume $V_t$ of blood flowing through the aorta in one minute. Deduce the volume flow $Q_v$.

2- Calculate average speed $v_{moy}$ of blood flow knowing that the diameter of the aorta is $d = 2cm$.

3- Calculate the Reynolds number $R_e$ for flow in the aorta knowing that the dynamic viscosity of the blood is$\eta = 5\times 10^{-3} Pa.s$ and its density is $\rho = 1060\ Kg.m^{-3}$. Deduce the flow regime.

4- Determine the critical speed $v_{\text{critical}}$ at which the regime becomes turbulent.

5- The blood distributed by the aorta ultimately reaches the capillaries. A blood capillary is an extremely thin blood vessel of medium radius $r_c = 5\mu m$. The blood circulates there at an average speed $v_{cap} = 0.06\ cm\ s^{-1}$ Calculate the volume flow rate $Q_{cap}$ of blood in this capillary.

6- Determine the average number $N_{cap}$ of capillaries present in the body in a human being.

## Solution

1- The total volume of blood flowing through the aorta in one minute is:

$$V_t = 80\times v_b = 80\times 0.075 = 6\ L$$

$$Q_v = 6L.minute^{-1} = \frac{6\times\ 10^{-3}m^3}{60\ s} = 10^{-4}m^3.s^{-1}$$

2- $Q_v = v_{moy}.S$

$S$ is the section of the aorta:

$$S = \pi.\left(\frac{d}{2}\right)^2 = \pi.\left(\frac{2\times\ 10^{-2}}{2}\right)^2 = 3.14\times\ 10^{-2}m^2$$

$$v_{moy} = \frac{Q_v}{S} = \frac{10^{-4}}{3.14\times\ 10^{-2}} = 0.32\ m.s^{-1}$$

3- The Reynolds number is defined by:

$$R_e = \frac{\rho.v_m.d}{\eta} = \frac{1060\times 0.32\times 2\times\ 10^{-2}}{5\times\ 10^{-3}} = 1356.8$$

$$R_e < 2000\ The\ flow\ regime\ is\ laminar$$

4- The regime becomes turbulent for $R_e > 3000$ . The critical speed from which the flow becomes turbulent is for $R_e = 3000$.

$$v_{\text{critical}} = \frac{\eta.R_e}{\rho.d} = \frac{5\times\ 10^{-3}\times 3000}{1060\times 2\times\ 10^{-2}} = 0.71m/s$$

5- $Q_{cap} = v_{cap}.S_{cap} = v_{cap}.\pi.r^2 = 6\times\ 10^{-4}\times\pi\times(5\times\ 10^{-6})^2 = 4.7\times\ 10^{-14}m^3/s.$

6- $Q_v = N_{cap}.Q_{cap}$

$$N_{cap} = \frac{Q_v}{Q_{cap}} = \frac{10^{-4}}{4.7\times\ 10^{-14}} = 2.13\times\ 10^{10}$$