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Stochastic Process 2023/2024

Exercise series N^0 01

Exercise 01

Let $X_t, t \in \mathbb{N}$ be a random process, with $X_t = mt + b + \xi_t$, where $\xi_t, t \in \mathbb{N}$ are independent and identically distributed random variables, such that

$$E[\xi_t] = 0, \text{ et } Var[\xi_t] = \sigma^2, \forall t \in \mathbb{N}.$$

1) Is the process (X_t) stationary?
2) We consider a new process $Y_t, t \in \mathbb{N}$, by $Y_t = X_t - X_{t-1}$. Show that (Y_t) is stationary.

Exercise 02

1) Let $X_t, t \in \Im \subset \mathbb{N}$ be a random process, such that

$$P(X_t = 1) = P(X_t = -1) = 0.5, \ \forall t \in \Im.$$

Is this process stationary?
2) Let $X_t, t \in \Im \subset \mathbb{N}$ be a random process, such that

$$P(X_t = 0) = 1 - \frac{1}{t}, P\left(X_t = \sqrt{t}\right) = \frac{1}{2t}, P\left(X_t = -\sqrt{t}\right) = \frac{1}{2t}.$$

Is this process stationary?

Exercise 03

Let $Y_t, t \in \mathbb{N}$ be a random process, $Y_t = (-1)^t X_t, t \in \mathbb{N}$, where $X_t, t \in \mathbb{N}$ is a stationary process

$$E[X_t] = 0, E[X_t X_s] = 0 \forall t \neq s, \text{ et } Var[X_t] = \sigma^2, \forall t \in \mathbb{N}.$$

Consider the process $Z_t = X_t + Y_t, t \in \mathbb{N}$. Is this process stationary?

Exercise 04

Let X_n, $n \in \mathbb{N}$ be a discrete-time random process such that $E[X_n] = 0$, $\forall n \in \mathbb{N}$

$$Var[X_n] = \begin{cases} \dfrac{\sigma^2}{1-\mu^2}, \text{ si } n = 0, \\ \sigma^2, \text{ si } n \geq 1, \end{cases}$$

with $0 < \mu^2 < 1$, and $E[X_i X_j] = 0, \forall i \neq j$. Now we build a new discrete-time process

$$Z_n = \begin{cases} X_0, \text{ si } n = 0, \\ \mu Z_{n-1} + X_n, \text{ si } n \geq 1, \end{cases}$$

known as the first-order autoregressive process. Show that Z_n is stationary.

Exercise 05

Show that if the process $\{X(t), t \in \mathbb{R}\}$ is strictly stationary, such that for all $t \in \mathbb{R}$ $E[X(t)] < \infty$, then $\forall (t, s) \in \mathbb{R} \times \mathbb{R}$

$$\begin{aligned} R(s,t) &= Cov(X_t, X_s), \\ &= r(t-s). \end{aligned}$$

Exercise 06. Find the stationary processes among the following processes:
1) $X_t = Z_t$ if t is even, and $X_t = Z_t + 1$ if t is odd, with $(Z_t)_{t \in \mathbb{Z}}$ stationary;
2) $X_t = Z_1 + \cdots + Z_t$ where $(Z_t)_{t \in \mathbb{Z}}$ is white noise;
3) $X_t = Z_t + \theta Z_{t-1}$, where $(Z_t)_{t \in \mathbb{Z}}$ is white noise and $\theta \in \mathbb{R}$ a constant;

4) $X_t = Z_t Z_{t-1}$ where $(Z_t)_{t\in\mathbb{Z}}$ is white noise;
5) $Y_t = (-1)^t Z_t$ and $X_t = Y_t + Z_t$ where $(Z_t)_{t\in\mathbb{Z}}$ is white noise.

Exercise 07

1) Let $\{X(t), t \in \mathbb{R}\}$ be a stationary process whose correlation function is $r(t) = e^{-\beta|t|}$, $\beta > 0$. Calculate its spectral density.

2) Let $X(t) = \theta X(t-1) + \epsilon(t)$, $|\theta| < 1$, or $\{\epsilon(t)\}$ follows a distribution $N(0, \sigma^2)$, Show that

$$r(t) = \frac{\sigma^2}{1-\theta^2}\theta^{|t|} \text{ et } f(\lambda) = \frac{\sigma^2}{2\pi}\left|1-\theta e^{i\lambda}\right|^{-2}.$$

Exercise 08

Let $\{X(t), t \in \mathbb{R}\}$ be a random process satisfying the following conditions
1) $E[|X(t)|] < \infty$, $\forall t \in \mathbb{R}$
2) $\forall \omega \in \Omega$ it exists $X'(t) = X'(t;\omega) = \frac{d}{dt}X(t,\omega)$.
3) $|X\prime(t,\omega)| \leq Y(\omega)$, $\omega \in \Omega$, where Y is a random variable such that $E[|Y|] < \infty$.
Show that $E[X(t)]$ is differentiable with respect to t and we have $\frac{d}{dt}E[X(t)] = E[X'(t)]$.

Exercise 09

Let $\{X(t), t \in [a,b]\}$ be a random process such that
1) $|X(t,\omega)| \leq Y(\omega)$, $\omega \in \Omega$, $\forall t \in [a,b]$, where Y is an integrable random variable.
2) for all $\omega \in \Omega$ the trajectories $X(t,\omega)$ are integrable on $[a,b]$.
3) $E[X(t)]$ is Riemann integrable over $[a,b]$.
Show that

$$\int_a^b E[X(t)]\,dt = E\left[\int_a^b X(t)\,dt\right].$$

Since the process {X(t), t■ℝ} is strictly stationary, it means that for any time shift h, the joint distribution of X(t) and X(t+h) is the same as the joint distribution of X(s) and X(s+h) for any s. In other words, the joint distribution of X(t) and X(t+h) depends only on the time lag h and not on the specific time t or s.

Now, let's consider the covariance function R(s,t) = Cov(X(t),X(s)) for any (t,s)■ℝ×R.$Withoutlossofgenerality, wecanas$

R(s,t) = Cov(X(t),X(s))

= E[(X(t) - E[X(t)])(X(s) - E[X(s)])]

= E[X(t)X(s)] - E[X(t)]E[X(s)]

Since the process {X(t), t■ℝ} is strictly stationary, we can use the time shift property to write:

E[X(t)X(s)] = E[X(t+h)X(s+h)] = R(h) + E[X(t+h)]E[X(s+h)]

where R(h) = Cov(X(t+h),X(s+h)) is the autocovariance function of the process at lag h. Note that R(h) does not depend on the specific values of t and s, but only on the time lag h.

Substituting this expression into the equation for R(s,t), we get:

R(s,t) = R(h) + E[X(t+h)]E[X(s+h)] - E[X(t)]E[X(s)]

Since we are assuming that E[X(t)] and E[X(s)] are finite for all t and s, we can take the limit as h ■ 0 to get:

R(s,t) = lim_ {h■0} [R(h) + E[X(t+h)]E[X(s+h)] - E[X(t)]E[X(s)]]

= R(0)

To show that if the process {X(t), t ■ ℝ} is strictly stationary and has finite mean, then for all (t,s) ■ ℝ × R, $wehaveR(s,t) = Cov(Xt, Xs) = r(t-s), weneedtousethepropertiesofstrictstationarityandsomealgebraicmanipulation$

First, let's recall the definition of strict stationarity: a stochastic process {X(t), t ■ ℝ} is strictly stationary if its distribution is invariant to time shifts, i.e., for any s,t ■ ℝ and any k ■ ℝ,

P(X(t) ■ x) = P(X(t+k) ■ x),

where P is the probability measure.

Using this property, we can show that Cov(Xt,Xs) = r(t-s) for all s,t ■ ℝ:

Cov(Xt,Xs) = E[(Xt - E[Xt])(Xs - E[Xs])] (by definition of covariance)

= E[(Xt - E[Xt])(Xt+k - E[Xt+k])]

= E[(Xt - E[Xt])(Xt+k - E[Xt] + E[Xt] - E[Xt+k])]

= E[(Xt - E[Xt])(Xt+k - E[Xt])] + E[(E[Xt] - E[Xt+k])(Xt - E[Xt+k])]

= E[(Xt - E[Xt])(Xt+k - E[Xt])] (since E[Xt] = E[Xt+k])

= E[XtXt+k] - E[Xt]E[Xt+k] - E[Xt]E[Xt+k] + E[Xt]^2

= E[XtXt+k] - E[Xt]^2

Now, let's define r(k) = Cov(Xt,Xt+k). Using the definition of covariance, we have:

r(k) = Cov(Xt,Xt+k) = E[(Xt - E[Xt])(Xt+k - E[Xt+k])]

Substituting k = s-t, we get:

r(s-t) = Cov(Xt,Xs) = E[(Xt - E[Xt])(Xs - E[Xs])]

Comparing this with our expression for Cov(Xt,Xs) above, we get:

Cov(Xt,Xs) = r(s-t)

Therefore, if the process {X(t), t ■ ℝ} is strictly stationary and has finite mean, then for all (t, s) ■ ℝ × R, $wehaveR(s, t) = Cov(Xt, Xs) = r(t - s)$.

To find the autocorrelation function r(t) of the given process {X(t)}, we can use the formula:

r(t) = Cov(X(t), X(t-■)) / Var(X(t))

where Cov is the covariance function and Var is the variance function.

Let's start with the covariance function:

Cov(X(t), X(t-■)) = E[X(t)X(t-■)] - E[X(t)]E[X(t-■)]

Since X(t) = ■X(t-1) + ■(t), we have:

X(t)X(t-■) = (■X(t-1) + ■(t))(■X(t-1-■) + ■(t-■))

= ■^2X(t-1)X(t-1-■) + ■■(t-■)X(t-1) + ■■(t)X(t-1-■) + ■(t)■(t-■)

Taking expectations and using the fact that ■(t) and ■(t-■) are uncorrelated and have mean zero, we get:

E[X(t)X(t-■)] = ■^2 E[X(t-1)X(t-1-■)] + ■^2 ■(■)

where ■(■) is the Kronecker delta function, which equals 1 if ■ = 0 and 0 otherwise. Similarly, we have:

E[X(t)] = ■ E[X(t-1)]

E[X(t-■)] = ■^{ |■|} E[X(t)]

Substituting these expressions into the formula for Cov, we get:

Cov(X(t), X(t-■)) = ■^{ |■|} r(0) + ■^2 ■(■)

where r(0) = Var(X(t)) is the autocovariance at lag 0. Solving for r(0), we get:

r(0) = (Cov(X(t), X(t-■)) - ■^2 ■(■)) / ■^{ |■|}

Substituting ■ = 0, we get:

r(0) = Var(X(t)) = (Cov(X(t), X(t)) - ■^2) / (1 - ■^2)

Since X(t) = ■X(t-1) + ■(t), we have:

Var(X(t)) = ■^2 Var(X(t-1)) + ■^2

Substituting this expression into the formula for r(0), we get:

r(0) = ■^2 / (1 - ■^2)

Finally, substituting r(0) into the expression for Cov(X(t), X(t-■)), we get:

Cov(X(t), X(t-■)) = ■^{ |■|} ■^2 / (1 - ■^2)

Dividing both sides by Var(X(t)), we get:

r(■) = ■^{ |■|} ■^2 / (1 - ■^2)

Therefore, the autocorrelation function of the process {X(t)} is:

r(t) = ((■^2)/(1-■^2))■^{|t|}

This result shows that the autocorrelation function of the process decays geometrically with lag t, and depends on the magnitude of ■ and ■^2. When ■ = 0, the process reduces to a white noise process with constant variance, and when ■ = 1, the process reduces to a random walk with