Stochastic Processes: Definitions and Examples

A stochastic process with state space S is a collection of random variables $\{X_t, t \in T\}$ defined on the same probability space $(\Omega, \mathcal{F}, P)$. The set T is called its *parameter set*. If $T = \mathbb{N} = \{0, 1, 2, \ldots\}$, the process is said to be a *discrete parameter process*. If T is not countable, the process is said to have a *continuous parameter*. In the latter case the usual examples are $T = \mathbb{R}_+ = [0, \infty)$ and $T = [a, b] \subset \mathbb{R}$. The index t represents time, and then one thinks of X_t as the "state" or the "position" of the process at time t. The state space is $\mathbb{R}$ in most usual examples, and then the process is said real-valued. There will be also examples where S is $\mathbb{N}$, the set of all integers, or a finite set.

For every fixed $\omega \in \Omega$, the mapping

$$\boxed{t \longrightarrow X_t(\omega)}$$

defined on the parameter set T, is called a realization, *trajectory*, sample path or sample function of the process.

Let $\{X_t, t \in T\}$ be a real-valued stochastic process and $\{t_1 < \cdots < t_n\} \subset T$, then the probability distribution $P_{t_1,\ldots,t_n} = P \circ (X_{t_1}, \ldots, X_{t_n})^{-1}$ of the random vector

$$(X_{t_1}, \ldots, X_{t_n}) : \Omega \longrightarrow \mathbb{R}^n.$$

is called a finite-dimensional marginal distribution of the process $\{X_t, t \in T\}$.

The following theorem, due to Kolmogorov, establishes the existence of a stochastic process associated with a given family of finite-dimensional distributions satisfying the *consistence condition*:

Theorem 3 *Consider a family of probability measures*

$$\{P_{t_1,\ldots,t_n},\ t_1 < \cdots < t_n, n \geq 1, t_i \in T\}$$

such that:

1. *$P_{t_1,\ldots,t_n}$ is a probability on $\mathbb{R}^n$*
2. *(Consistence condition): If $\{t_{k_1} < \cdots < t_{k_m}\} \subset \{t_1 < \cdots < t_n\}$, then $P_{t_{k_1},\ldots,t_{k_m}}$ is the marginal of $P_{t_1,\ldots,t_n}$, corresponding to the indexes $k_1, \ldots, k_m$.*

Then, there exists a real-valued stochastic process $\{X_t, t \geq 0\}$ defined in some probability space $(\Omega, \mathcal{F}, P)$ which has the family $\{P_{t_1,\ldots,t_n}\}$ as finite-dimensional marginal distributions.

A real-valued process $\{X_t, t \geq 0\}$ is called a second order process provided $E(X_t^2) < \infty$ for all $t \in T$. The *mean* and the *covariance function* of a second order process $\{X_t, t \geq 0\}$ are defined by

$$\begin{aligned} m_X(t) &= E(X_t) \\ \Gamma_X(s,t) &= \text{Cov}(X_s, X_t) \\ &= E((X_s - m_X(s))(X_t - m_X(t)). \end{aligned}$$

The *variance* of the process $\{X_t, t \geq 0\}$ is defined by

$$\sigma_X^2(t) = \Gamma_X(t,t) = \text{Var}(X_t).$$

Example 12 Let X and Y be independent random variables. Consider the stochastic process with parameter $t \in [0, \infty)$

$$X_t = tX + Y.$$

The sample paths of this process are lines with random coefficients. The finite-dimensional marginal distributions are given by

$$P(X_{t_1} \leq x_1, \ldots, X_{t_n} \leq x_n) = \int_{\mathbb{R}} F_X \left(\min_{1 \leq i \leq n} \frac{x_i - y}{t_i} \right) P_Y(dy).$$

Example 13 Consider the stochastic process

$$X_t = A \cos(\varphi + \lambda t),$$

where A and φ are independent random variables such that $E(A) = 0$, $E(A^2) < \infty$ and φ is uniformly distributed on $[0, 2\pi]$. This is a second order process with

$$\begin{aligned} m_X(t) &= 0 \\ \Gamma_X(s,t) &= \frac{1}{2} E(A^2) \cos \lambda(t-s). \end{aligned}$$

Example 14 *Arrival process:* Consider the process of arrivals of customers at a store, and suppose the experiment is set up to measure the interarrival times. Suppose that the interarrival times are positive random variables $X_1, X_2, \ldots$. Then, for each $t \in [0, \infty)$, we put $N_t = k$ if and only if the integer k is such that

$$X_1 + \cdots + X_k \leq t < X_1 + \cdots + X_{k+1},$$

and we put $N_t = 0$ if $t < X_1$. Then N_t is the number of arrivals in the time interval $[0, t]$. Notice that for each $t \geq 0$, N_t is a random variable taking values in the set $S = \mathbb{N}$. Thus, $\{N_t, t \geq 0\}$ is a continuous time process with values in the state space $\mathbb{N}$. The sample paths of this process are non-decreasing, right continuous and they increase by jumps of size 1 at the points $X_1 + \cdots + X_k$. On the other hand, $N_t < \infty$ for all $t \geq 0$ if and only if

$$\sum_{k=1}^{\infty} X_k = \infty.$$

Example 15 Consider a discrete time stochastic process $\{X_n, n = 0, 1, 2, \ldots\}$ with a finite number of states $S = \{1, 2, 3\}$. The dynamics of the process is as follows. You move from state 1 to state 2 with probability 1. From state 3 you move either to 1 or to 2 with equal probability 1/2, and from 2 you jump to 3 with probability 1/3, otherwise stay at 2. This is an example of a *Markov chain*.

A real-valued stochastic process $\{X_t, t \in T\}$ is said to be *Gaussian or normal* if its finite-dimensional marginal distributions are multi-dimensional Gaussian laws. The mean $m_X(t)$ and the covariance function $\Gamma_X(s,t)$ of a Gaussian process determine its finite-dimensional marginal distributions. Conversely, suppose that we are given an arbitrary function $m : T \to \mathbb{R}$, and a symmetric function $\Gamma : T \times T \to \mathbb{R}$, which is nonnegative definite, that is

$$\sum_{i,j=1}^{n} \Gamma(t_i, t_j) a_i a_j \geq 0$$

for all $t_i \in T$, $a_i \in \mathbb{R}$, and $n \geq 1$. Then there exists a Gaussian process with mean m and covariance function Γ.

Example 16 Let X and Y be random variables with joint Gaussian distribution. Then the process $X_t = tX + Y$, $t \geq 0$, is Gaussian with mean and covariance functions

$$\begin{aligned} m_X(t) &= tE(X) + E(Y), \\ \Gamma_X(s,t) &= st\mathrm{Var}(X) + (s+t)\mathrm{Cov}(X,Y) + \mathrm{Var}(Y). \end{aligned}$$

Example 17 *Gaussian white noise*: Consider a stochastic process $\{X_t, t \in T\}$ such that the random variables X_t are independent and with the same law $N(0, \sigma^2)$. Then, this process is Gaussian with mean and covariance functions

$$\begin{aligned} m_X(t) &= 0 \\ \Gamma_X(s,t) &= \begin{cases} 1 & \text{if } s = t \\ 0 & \text{if } s \neq t \end{cases} \end{aligned}$$

Definition 4 *A stochastic process $\{X_t, t \in T\}$ is equivalent to another stochastic process $\{Y_t, t \in T\}$ if for each $t \in T$*

$$P\{X_t = Y_t\} = 1.$$

We also say that $\{X_t, t \in T\}$ is a version of $\{Y_t, t \in T\}$. Two equivalent processes may have quite different sample paths.

Example 18 Let ξ be a nonnegative random variable with continuous distribution function. Set $T = [0, \infty)$. The processes

$$\begin{aligned} X_t &= 0 \\ Y_t &= \begin{cases} 0 & \text{if } \xi \neq t \\ 1 & \text{if } \xi = t \end{cases} \end{aligned}$$

are equivalent but their sample paths are different.

Definition 5 *Two stochastic processes $\{X_t, t \in T\}$ and $\{Y_t, t \in T\}$ are said to be indistinguishable if $X_\cdot(\omega) = Y_\cdot(\omega)$ for all $\omega \notin N$, with $P(N) = 0$.*

Two stochastic process which have right continuous sample paths and are equivalent, then they are indistinguishable. Two discrete time stochastic processes which are equivalent, they are also indistinguishable.

Definition 6 *A real-valued stochastic process $\{X_t, t \in T\}$, where T is an interval of $\mathbb{R}$, is said to be continuous in probability if, for any $\varepsilon > 0$ and every $t \in T$*

$$\lim_{s \to t} P(|X_t - X_s| > \varepsilon) = 0.$$

Definition 7 *Fix $p \geq 1$. Let $\{X_t, t \in T\}$be a real-valued stochastic process, where T is an interval of $\mathbb{R}$, such that $E(|X_t|^p) < \infty$, for all $t \in T$. The process $\{X_t, t \geq 0\}$ is said to be continuous in mean of order p if*

$$\lim_{s \to t} E(|X_t - X_s|^p) = 0.$$

Continuity in mean of order p implies continuity in probability. However, the continuity in probability (or in mean of order p) does not necessarily implies that the sample paths of the process are continuous.

In order to show that a given stochastic process has continuous sample paths it is enough to have suitable estimations on the moments of the increments of the process. The following continuity criterion by Kolmogorov provides a sufficient condition of this type:

Proposition 8 (Kolmogorov continuity criterion) *Let $\{X_t, t \in T\}$ be a real-valued stochastic process and T is a finite interval. Suppose that there exist constants $a > 1$ and $p > 0$ such that*

$$E(|X_t - X_s|^p) \leq c_T |t - s|^\alpha \tag{1}$$

for all $s, t \in T$. Then, there exists a version of the process $\{X_t, t \in T\}$ with continuous sample paths.

Condition (1) also provides some information about the modulus of continuity of the sample paths of the process. That means, for a fixed $\omega \in \Omega$, which is the order of magnitude of $X_t(\omega) - X_s(\omega)$, in comparison $|t - s|$. More precisely, for each $\varepsilon > 0$ there exists a random variable G_ε such that, with probability one,

$$|X_t(\omega) - X_s(\omega)| \leq G_\varepsilon(\omega) |t - s|^{\frac{\alpha}{p} - \varepsilon}, \tag{2}$$

for all $s, t \in T$. Moreover, $E(G_\varepsilon^p) < \infty$.

The Poisson Process

A random variable $T : \Omega \to (0, \infty)$ has *exponential distribution of parameter* $\lambda > 0$ if

$$P(T > t) = e^{-\lambda t}$$

for all $t \geq 0$. Then T has a density function

$$f_T(t) = \lambda e^{-\lambda t} \mathbf{1}_{(0,\infty)}(t).$$

The mean of T is given by $E(T) = \frac{1}{\lambda}$, and its variance is $\mathrm{Var}(T) = \frac{1}{\lambda^2}$. The exponential distribution plays a fundamental role in continuous-time Markov processes because of the following result.