University of Biskra
Mathematics Department
Module: Analysis 1

First year license
2023/2024

Exam

Exercise 1 ***(./06.50 pts)*** Consider the recursive sequence $(u_n)_{n\in\mathbb{N}}$ defined by

$$\begin{cases} u_0 = \sqrt{2} \\ u_{n+1} = \sqrt{2+u_n}, \quad n \in \mathbb{N}. \end{cases}$$

1. Show that for all $n \in \mathbb{N}$, we have $u_n \leq 2$.
2. Check that $(u_n)_{n\in\mathbb{N}}$ is a bounded sequence (upper and lower bounded).
3. Analyze the sign of $u_{n+1}^2 - u_n^2$ according to n, then deduce that u_n is a monotonic sequence.
4. Check that $(u_n)_{n\in\mathbb{N}}$ is a convergent sequence; then calculate its limit.

Exercise 2 ***(./09.00 pts)*** Considering the real function f defined by

$$f(x) = \begin{cases} \frac{sin(ax)}{x}, & if \quad x < 0; \\ 1, & if \quad x = 0; \\ e^{bx} - x, & if \quad x > 0, \end{cases}$$

with a and b are two real numbers.

1. Determine the values of a and b for which f is continuous and differentiable at $x = 0$.
2. Check that, if $a = b = 1$ then
   (a) the function f admits at least one extremum in the interval $]-2\pi; -\pi[$
   (b) there exists at least one point of intersection of the graph of f with the line $y = 2x$, in the interval $]0; 1[$.

Exercise 3 ***(./04.50 pts)*** Let f be the real function defined by:

$$f(x) = \log_{a(x)}(1 + sin(x)), \qquad \text{with } a(x) = 1 - x.$$

1. Determine the domain of the function f.
2. Using the Maclaurin expansion, calculate $\lim_{x\to 0} f(x)$, then check if f admits a removable discontinuity at $x = 0$.

Note:

- $\log_a(.)$ denotes the logarithmic function with base a.
- Maclaurin series of e^x, $ln(1+x)$ and $sin(x)$ are
  $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + ...$
  $ln(1+x) = \frac{x}{1} - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ...$
  $sin(x) = \frac{x}{1!} + \frac{-x^3}{3!} + \frac{x^5}{5!} + ...$

Good luck

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Exam Solution

Solution of the Exercise 1

1. To show that $\forall n \in \mathbb{N}$, we have $u_n \leq 2$, we use the induction technique.

(a) For $n = 0$, we have $u_0 = \sqrt{2} \leq 2$, so $P(0)$ is true.

(b) Assume that $P(n)$ is true i.e. $u_n \leq 2$.

(c) $P(n+1)$? ($u_n \leq 2$?)

$$\begin{aligned} \text{From (1b) we have } u_n \leq 2 \quad &\Leftrightarrow \quad u_n + 2 \leq 2 + 2 \\ &\Leftrightarrow \quad \sqrt{u_n + 2} \leq \sqrt{4} \\ &\Leftrightarrow \quad u_{n+1} \leq 2 \\ &\Leftrightarrow \quad P(n+1) \text{ is true.} \end{aligned}$$

Conclusion: From the results (1a)–(1c) we conclude that:

$$\forall n \in \mathbb{N}, \text{ we have } u_n \leq 2.$$

2. On the one hand, we have already shown, in the first question, that $(u_n)_{n\in\mathbb{N}}$ is an upper-bounded sequence. On the other hand, the expression of u_n clearly shows that $\forall n \in \mathbb{N}$, $u_n \geq 0$ ($as\ \sqrt{x} \geq 0$) (lower-bounded sequence). So,

$$\forall n \in \mathbb{N}, \text{ we have } 0 \leq u_n \leq 2.$$

3. Let put $P_2(u_n) = u_{n+1}^2 - u_n^2 = -u_n^2 + u_n + 2$

$$P_2(u_n) = 0 \Rightarrow \Delta = 9 \Rightarrow \begin{cases} u_n^{(1)} = \frac{-1-3}{-2} = 2 \\ u_n^{(2)} = \frac{-1+3}{-2} = -1 \end{cases} \Rightarrow$$

u_n		-1		0		2	
$P_2(u_n)$	-	0	+		+	0	-

As $0 \leq u_n \leq 2$, then we conclude that

$$\begin{aligned} \forall n \in \mathbb{N},\ P_2(u_n) \geq 0 \quad &\Leftrightarrow \quad \forall n \in \mathbb{N},\ \ u_{n+1}^2 - u_n^2 \geq 0 \\ &\Leftrightarrow \quad \forall n \in \mathbb{N},\ u_{n+1}^2 \geq u_n^2 \\ &\Leftrightarrow \quad \forall n \in \mathbb{N},\ u_{n+1} \geq u_n (\text{because } \sqrt{.} \text{ is an increasing function and } u_n \geq 0) \\ &\Leftrightarrow \quad (u_n)_{n\in\mathbb{N}} \text{ is an increasing sequence for all } n \text{ ie is a monotonic sequence.} \end{aligned}$$

4. From the above results we have

(a) $(u_n)_{n\in\mathbb{N}}$ is a bounded sequence

(b) $(u_n)_{n\in\mathbb{N}}$ is a monotonic sequence

therefore, the sequence $(u_n)_{n\in\mathbb{N}}$ is convergent. Assume that $\lim_{n\to\infty} u_n = l$, so $\lim_{n\to\infty} u_{n+1} = l$.

$$\lim_{n\to\infty} u_{n+1} = l \Rightarrow l = \lim_{n\to\infty} \sqrt{2+u_n} \Rightarrow l = \sqrt{2+l} \Rightarrow l^2 - l - 2 = 0$$

$$\Rightarrow \begin{cases} l_1 = 2 \\ l_2 = -1 \quad \textit{rejected, because } u_n \geq 0. \end{cases} \Rightarrow \lim_{n\to\infty} u_n = 2.$$

Solution of the Exercise 2 ***(./09.00 pts)*** Considering the real function f defined by

$$f(x) = \begin{cases} \frac{sin(ax)}{x}, & if \quad x < 0; \\ 1, & if \quad x = 0; \\ e^{bx} - x, & if \quad x > 0, \end{cases}$$

with a and b are two real numbers.

1. Determine the values of a and b for which f is continuous and differentiable at $x = 0$.

(a) f is continuous at $x = 0 \Rightarrow \lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0) = 1$

$$\begin{cases} 1 = \lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{sin(ax)}{x} = \lim_{x\to 0^-} a\frac{sin(ax)}{ax} = a \\ 1 = \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} e^{bx} - x = 1 \quad (b \neq 0) \end{cases}$$

So, f is continuous at $x = 0$ if $a = 1$ and $b \in \mathbb{R}^*$.

(b) f is differentiable at $x = 0 \Rightarrow \begin{cases} 1) f \textit{ is continuous at } x = 0 \Rightarrow a = 1 \textit{ and } b \in \mathbb{R}^* \\ 2) \lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^-} \frac{f(x)-f(0)}{x-0} = l, \ l \in \mathbb{R} \end{cases}$

$$\begin{cases} \lim_{x\to 0^-} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^-} \frac{sin(x)-x}{x^2} = \lim_{x\to 0^-} \frac{x-\frac{x^3}{3!}-x}{x^2} = \lim_{x\to 0^-} \frac{-x}{3!} = 0 \\ \lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^+} \frac{e^{bx}-x-1}{x} = \lim_{x\to 0^+} \frac{1+(bx)+\frac{(bx)^2}{2!}-x-1}{x} = \lim_{x\to 0^+} \frac{(bx)+\frac{(bx)^2}{2!}-x}{x} = b - 1 \Rightarrow b = 1 \end{cases}$$

So, f is differentiable at $x = 0$ if $a = 1$ and $b = 1$.

2. For $x \in]-2\pi, -\pi[\ \Rightarrow f(x) = \frac{sin(x)}{x}$. We note that

- ✓ $f(-2\pi) = f(-\pi) = 0$,
- ✓ f is continuous on $[-2\pi, -\pi]$
- ✓ f is differentiable on $]-2\pi, -\pi[$,

so, based on the **Roll theorem** we deduce that $\exists c \in]-2\pi, -\pi[$ such that $f'(c) = 0$ (c is an extremum).

3. To answer this question we use the intermediate value theorem. For this, let's define the new function g on]0,1[as follows:

$$g(x) = f(x) - 2x = e^x - 3x.$$

We note that,

(a) g is continuous on]0,1[,

(b) $\left\{ \begin{array}{l} g(0) = 1 \\ \\ g(1) = e^1 - 3. \end{array} \right. \Rightarrow g(0)g(1) < 0$

From (3a) and (3b) we conclude that
$\exists c \in]0,1[$ such that $g(c) = 0 \Rightarrow \exists c \in]0,1[$ such that $f(c) - 2c = 0$ hence, $\exists c \in]0,1[$ such that $f(c) = 2c$.

Exercise 4 ***(./04.50 pts)*** Note that the function f can be rewritten as follow:

$$f(x) = \frac{ln(1 + sin(x))}{ln(1 - x)}.$$

1. $D_f = \{x \in \mathbb{R},\ 1 + sin(x) > 0\ and\ 1 - x > 0\ and\ 1 - x \neq 1\}$

$$\left\{ \begin{array}{l} 1 + sin(x) > 0 \Rightarrow sin(x) > -1 \\ 1 - x > 0 \Rightarrow x < 1 \\ 1 - x \neq 1 \Rightarrow x \neq 0 \end{array} \right. \Rightarrow \left\{ \begin{array}{l} x \in D_1 = \mathbb{R}/\{-\frac{\pi}{2} + 2k\pi,\ k \in \mathbb{Z}\} \\ x \in D_2 =]-\infty, 1[\\ x \in D_3 = \mathbb{R}^* \end{array} \right.$$

hence $D_f = D_1 \cap D_2 \cap D_3 =]-\infty, 0[\cup]0, 1[/\{-\frac{\pi}{2} - 2k\pi,\ k \in \mathbb{N}\}$.

2. $ln(1 + sin(x)) = (x - x^3/3 + ...) - (x - x^3/3 + ...)^2/2 + (x - x^3/3 + ...)^3/3 - ....$
$ln(1 - x) = -x + x^2/2 - x^3/3 + ...$

$$\lim_{x \to 0} \frac{ln(1 + sin(x))}{ln(1 - x)} = \lim_{x \to 0} \frac{(x - x^3/3 + ...) - (x - x^3/3 + ...)^2/2 + (x - x^3/3 + ...)^3/3 - ....}{-x + x^2/2 - x^3/3 + ...} = -1$$

3. As f is not defined at $x = 0$ and $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x)$ we conclude that we can remove the discontinuity of f at 0. Hence,

$$\widetilde{f}(x) = \left\{ \begin{array}{ll} f(x), & if\ x \in D_f; \\ -1, & if\ x = 0. \end{array} \right.$$